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A simple elementary proof of $M(x) = \sum_{n\leqslant x} \mu(n) = o(x)$

by

M. Kalecki (Warszawa)

The proof starts from a very simple deduction of

$$(1)\quad M(x)\log^2 x = -\sum_{p\leqslant x} M\left(\frac{x}{p}\right)\log^2 p + \sum_{pp'\leqslant x} M\left(\frac{x}{pp'}\right)\log p \log p' + O(x\log x).$$

On the basis of (1) and Selberg's theorem the inequality

$$(2)\quad |M(x)|\frac{\log^2 x}{2} \leqslant \int_1^x \left|M\left(\frac{x}{t}\right)\right|\log t\,dt + o(x\log^2 x)$$

is arrived at. Finally it is shown that this inequality is self-contradictory if $M(x) \neq o(x)$.

The proof of $R(x) = \psi(x) - x = o(x)$ by Selberg (as presented by Hardy and Wright [2]) passes through similar stages in relation to $R(x)$. After a deduction from Selberg's theorem of

$$R(x)\log^2 x = -\sum_{n\leqslant x} R\left(\frac{x}{n}\right)\Lambda(n)\log n + \sum_{mn\leqslant x} R\left(\frac{x}{mn}\right)\Lambda(m)\Lambda(n) + O(x\log x)$$

on the basis of this and Selberg's theorem the inequality

$$|R(x)|\frac{\log^2 x}{2} \leqslant \int_1^x \left|R\left(\frac{x}{t}\right)\right|\log t\,dt + O(x\log x)$$

is arrived at. Next it is shown (by using Selberg's theorem again) that this inequality is self-contradictory if $R(x) \neq o(x)$. Starting from (1) it is possible to apply the argument of Selberg's proof to $M(x)$ rather than to $R(x)$ because this proof is based solely on those characteristics of $R(x)$ which this function shares with $M(x)$.

Our proof, however, uses certain specific characteristics of $M(x)$ which render it much simpler(1). In this respect it shows a certain affinity to the proof of $M(x) = o(x)$ by Postnikov and Romanov (as presented by Gelfond and Linnik [1]). This proof uses Selberg's theorem in order to obtain the inequality

$$|M(x)|\log x \leqslant \sum_{n\leqslant x}\left|M\left(\frac{x}{n}\right)\right| + O(x\log\log x).$$

Next it is shown that this is self-contradictory if $M(x)\neq o(x)$. It seems, however, that our proof is somewhat simpler and more "natural". We shall now proceed with its presentation.

LEMMA 1.

(1) $$M(x)\log^2 x = -\sum_{p\leqslant x} M\left(\frac{x}{p}\right)\log^2 p + \sum_{pp'\leqslant x} M\left(\frac{x}{pp'}\right)\log p\log p' + O(x\log x).$$

Proof. We have

$$\mu(n)\log^2 n = \mu(n)\Big(\sum_{d|n}\Lambda(d)\Big)^2 = \mu(n)\sum_{d|n}\Lambda^2(n) + \mu(n)\sum_{dd'|n}\Lambda(d)\Lambda(d').$$

Hence

$$\begin{aligned}\sum_{n\leqslant x}\mu(n)\log^2 n &= -\sum_{p\leqslant x} M\left(\frac{x}{p}\right)\log^2 p + O(x)\sum_{p^2\leqslant x}\frac{\log^2 p}{p^2} + \sum_{pp'\leqslant x} M\left(\frac{x}{pp'}\right)\log p\log p' + \\ &\quad + O(x)\sum_{p^2p'\leqslant x}\frac{\log p\log p'}{p^2p'} + O(x)\sum_{pp'^2\leqslant x}\frac{\log p\log p'}{pp'^2} \\ &= \sum_{p\leqslant x} M\left(\frac{x}{p}\right)\log^2 p + \sum_{pp'\leqslant x} M\left(\frac{x}{pp'}\right)\log p\log p' + \\ &\quad + O(x) + O(x)\sum_{p'\leqslant x}\frac{\log p'}{p'}\sum_{p^2\leqslant \frac{x}{p'}}\frac{\log p}{p^2} + O(x)\sum_{p\leqslant x}\frac{\log p}{p}\sum_{p'^2\leqslant\frac{x}{p}}\frac{\log p'}{p'^2},\end{aligned}$$

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(1) After this paper had been completed I came across the paper of Nevanlinna [3], who simplified the proof of the contradiction involved in $|R(x)|\dfrac{\log^2 x}{2} \leqslant \displaystyle\int_1^x \left|R\left(\frac{x}{t}\right)\right|\log t\,dt + O(x\log x)$ if $R(x)\neq o(x)$ applying a method bearing some analogy to that applied by myself in proving the same for (2) if $M(x)\neq o(x)$.

and bearing in mind

$$\sum_{n\leqslant x}\mu(n)(\log^2 x - \log^2 n) = O(x\log x),$$

we obtain

$$M(x)\log^2 x = -\sum_{p\leqslant x} M\left(\frac{x}{p}\right)\log^2 p + \sum_{pp'\leqslant x} M\left(\frac{x}{pp'}\right)\log p\log p' + O(x\log x).$$

LEMMA 2.

(2) $$|M(x)|\frac{\log^2 x}{2} \leqslant \int_1^x \left|M\left(\frac{x}{t}\right)\right|\log t\,dt + o(x\log^2 x).$$

Proof. We obtain from (1)

$$|M(x)|\log^2 x \leqslant \sum_{p\leqslant x}\left|M\left(\frac{x}{p}\right)\right|\log^2 p + \sum_{pp'\leqslant x}\left|M\left(\frac{x}{pp'}\right)\right|\log p\log p' + O(x\log x).$$

Let us form a sequence $y_i = a(1+\varepsilon)^i\log x$ where ε is an arbitrary constant, $0\leqslant i\leqslant k$ and k and a are determined as follows

(3) $$k = \left[\frac{\log x - \log\log x}{\log(1+\varepsilon)}\right];\qquad a = \frac{x}{(1+\varepsilon)^k\log x} = O(1).$$

We have for $y_i < t \leqslant y_{i+1}$

(4) $$M\left(\frac{x}{t}\right) = M\left(\frac{x}{y_i}\right) + O\left(\varepsilon\frac{x}{y_i}\right) = M\left(\frac{x}{y_i}\right) + O\left(\varepsilon\frac{x}{t}\right).$$

Hence

$$\begin{aligned}&|M(x)|\log^2 x \\ &\leqslant \sum_{0\leqslant i<k}\left|M\left(\frac{x}{y_i}\right) + O\left(\varepsilon\frac{x}{y_i}\right)\right|\Big(\sum_{y_i<p\leqslant y_{i+1}}\log^2 p + \sum_{y_i<pp'\leqslant y_{i+1}}\log p\log p'\Big) + \\ &\qquad + O(x\log x).\end{aligned}$$

Taking into consideration the theorem of Selberg in the form

$$\sum_{p\leqslant y}\log^2 p + \sum_{pp'\leqslant y}\log p\log p' = 2y\log y + O(y),$$

we obtain

$$\sum_{y_i<p\leqslant y_{i+1}}\log^2 p + \sum_{y_i<pp'\leqslant y_{i+1}}\log p\log p' = 2\varepsilon y_i\log y_i + O(y_i)$$

and, bearing in mind that according to (3)

$$k = \frac{\log x - \log\log x - \log a}{\log(1+\varepsilon)},$$

we have

$$|M(x)|\frac{\log^2 x}{2} \leqslant \varepsilon \sum_{0\leqslant i<k}\left(y_i\left|M\left(\frac{x}{y_i}\right)\right|\log y_i + \varepsilon O(x\log y_i)\right) + O\left(\frac{x}{\varepsilon}\log x\right).$$

Applying (4) we obtain

$$\varepsilon y_i\left(\left|M\left(\frac{x}{y_i}\right)\right|\log y_i + \varepsilon O(x\log y_i)\right) = \int_{y_i}^{y_{i+1}}\left|M\left(\frac{x}{t}\right)\right|\log t\,dt + O(\varepsilon x)\int_{y_i}^{y_{i+1}}\frac{\log t}{t}\,dt$$

and, bearing in mind the arbitrariness of ε,

$$|M(x)|\frac{\log^2 x}{2} \leqslant \int_{y_0=a\log x}^{x}\left|M\left(\frac{x}{t}\right)\right|\log t\,dt + O(\varepsilon x)\int_{y_0=a\log x}^{x}\frac{\log t}{t}\,dt + O\left(\frac{x}{\varepsilon}\log x\right)$$

$$= \int_1^x\left|M\left(\frac{x}{t}\right)\right|\log t\,dt + O(\varepsilon x\log^2 x) = \int_1^x\left|M\left(\frac{x}{t}\right)\right|\log t\,dt + o(x\log^2 x).$$

LEMMA 3.

(5) $$\int_1^x\left|M\left(\frac{x}{t}\right)\right|\log t\,dt \leqslant g(1-\lambda)\frac{x\log^2 x}{2},$$

where $g = \limsup|M(x)/x|$ *and* λ *is a constant* >0 *if* $g>0$.

Proof. Denoting x/t by w we have

(6) $$\int_1^x\left|M\left(\frac{x}{t}\right)\right|\log t\,dt = x\int_1^x|M(w)|\log\frac{x}{w}\frac{dw}{w^2}.$$

We shall show that

$$\int_1^x|M(w)|\log\frac{x}{w}\frac{dw}{w^2} \leqslant g(1-\lambda)\frac{\log^2 x}{2}.$$

We can write

$$\sum_{v<n\leqslant u}\frac{\mu(n)}{n}\log\frac{x}{n} = \sum_{v<n\leqslant u}\log\frac{x}{n}\cdot\frac{M(n+1)-M(n)}{n}$$

$$= \sum_{v<n\leqslant u}\frac{M(n)}{n(n+1)}\log\frac{x}{n} + O(1) + \frac{M(u)}{u}\log\frac{x}{u} - \frac{M(v)}{v}\log\frac{x}{v},$$

where $u\leqslant x$, and taking into consideration that

$$\sum_{n\leqslant u}\frac{\mu(n)}{n}\log\frac{x}{n} = \sum_{n\leqslant u}\frac{\mu(n)}{n}\log\frac{u}{n} + \log\frac{x}{u}\sum_{n\leqslant u}\frac{\mu(n)}{n} = O(1) + O\left(\log\frac{x}{u}\right)$$

we arrive at

$$O\left(\log\frac{x}{u}\right) + O\left(\log\frac{x}{v}\right) = \sum_{v<n\leqslant u}\frac{M(n)}{n(n+1)}\log\frac{x}{n} + O\left(\log\frac{x}{u}\right) + O\left(\log\frac{x}{v}\right)$$

and thus obtain

(7) $$\left|\int_v^u M(w)\log\frac{x}{w}\frac{dw}{w^2}\right| = \left|\sum_{v<n\leqslant u}\frac{M(n)}{n(n+1)}\log\frac{x}{n} + O\left(\log\frac{x}{v}\right)\right| < a\log\frac{x}{v},$$

where a is a definite positive constant.

Let us now form a sequence $z_i = b(1+h)^i\log x$, where h is an arbitrary constant, $0\leqslant i\leqslant j$ and j and b are determined as follows:

(8) $$j = \left[\frac{\log x - \log\log x}{\log(1+h)}\right]; \quad b = \frac{x}{(1+h)^j\log x} = O(1).$$

Let us denote the maximum $|M(w)/w|$ for $z_i < w \leqslant z_{i+1}$ by m_i. If $M(w)$ does not change the sign in the interval z_i, z_{i+1}, then according to (7)

(9) $$\int_{z_i}^{z_{i+1}}|M(w)|\log\frac{x}{w}\frac{dw}{w^2} < a\log\frac{x}{z_i}.$$

If, however, $M(w)$ does change the sign in this interval, $|M(w)|$ has to pass through all integers $0\leqslant s\leqslant m_i z_i$ and

$$\int_{|M(w)|=s}|M(w)|\,dw \geqslant s.$$

Thus

$$\int_{z_i}^{z_{i+1}}|M(w)|\log\frac{x}{w}\frac{dw}{w^2} \leqslant m_i\int_{z_i}^{z_{i+1}}\log\frac{x}{w}\frac{dw}{w} - \frac{1}{z_{i+1}^2}\log\frac{x}{z_{i+1}}\sum_{s=0}^{[m_i z_i]}([m_i z_i]-s)$$

$$\leqslant m_i\int_{z_i}^{z_{i+1}}\log\frac{x}{w}\frac{dw}{w} - \frac{m_i^2 z_i^2}{2z_{i+1}^2}\log\frac{x}{z_{i+1}}(1+o(1))$$

$$= m_i\left(\log\frac{x}{z_{i+1}} + \frac{\log(1+h)}{2}\right)\log(1+h) - \frac{m_i^2}{2(1+h)^2}\log\frac{x}{z_{i+1}}(1+o(1)).$$

For $h>1$ and $m_i<1$ the expression in the last line is an increasing function of m_i because $1<(1+h)^2\log(1+h)$.

Thus we can write

$$(10)\quad \int_{z_i}^{z_{i+1}} |M(w)|\log\frac{x}{w}\frac{dw}{w^2} \leqslant \big(g+o(1)\big)\int_{z_i}^{z_{i+1}} \log\frac{x}{w}\frac{dw}{w} - \frac{g^2+o(1)}{2(1+h)^2}\log\frac{x}{z_{i+1}},$$

where $g = \limsup|M(x)/x|$.

It is now easy to show that the expression on the right-hand side exceeds $\alpha\log\frac{x}{z_i}$ if h is sufficienly large. Indeed,

$$\big(g+o(1)\big)\left(\log\frac{x}{z_{i+1}}+\frac{\log(1+h)}{2}\right)\log(1+h)-\frac{g^2+o(1)}{2(1+h)^2}\log\frac{x}{z_{i+1}}$$

$$>\log\frac{x}{z_i}\big(g+o(1)\big)\left(\left(1-\frac{\log(1+h)}{2\log\frac{x}{z_i}}\right)\log(1+h)-\frac{g+o(1)}{2(1+h)^2}\right)$$

$$\geqslant\frac{g+o(1)}{2}\left(\log(1+h)-\frac{g+o(1)}{(1+h)^2}\right)\log\frac{x}{z_i},$$

whereas if h is sufficiently large

$$\frac{g}{2}\left(\log(1+h)-\frac{g^2}{(1+h)^2}\right)>\alpha.$$

Thus according to (9) and (10) we have with a sufficiently large h

$$\int_1^x |M(w)|\log\frac{x}{w}\frac{dw}{w^2}\leqslant\big(g+o(1)\big)\int_1^x\log\frac{x}{w}\frac{dw}{w}-\frac{g^2+o(1)}{2(1+h)^2}\sum_{i=0}^{j-1}\log\frac{x}{z_{i+1}}.$$

Since according to (8)

$$j=\frac{\log x-\log\log x-\log b}{\log(1+h)},$$

we have

$$\int_1^x |M(w)|\log\frac{x}{w}\frac{dw}{w^2}\leqslant\big(g+o(1)\big)\left(\frac{\log^2 x}{2}-\frac{\big(g+o(1)\big)\log^2 x}{4(1+h)^2\log(1+h)}\right)$$

$$<g(1-\lambda)\frac{\log^2 x}{2},$$

where λ is a constant >0 if $g>0$, which in view of (6) proves the lemma.

THEOREM. *We have*

$$(11)\qquad M(x)=o(x).$$

Proof. According to (2) and (5) we have for $g>0$

$$|M(x)|\frac{\log^2 x}{2}\leqslant g(1-\lambda)\frac{x\log^2 x}{2}+o(x\log^2 x).$$

Hence

$$\left|\frac{M(x)}{x}\right|\leqslant g(1-\lambda)+o(1),$$

which involves a contradiction because $g=\limsup|M(x)/x|$. Thus $g=0$, which proves the theorem.

References

[1] А. О. Гельфонд, Ю. В. Линник, *Элементарные методы в аналитической теории чисел*, Москва 1962, pp. 76-78, 82-86.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford 1960, pp. 362-367.

[3] Veikka Nevanlinna, *Über die elementaren Beweise der Primzahlsätze und deren äquivalente Fassungen*, Annales Academiae Scientiarum Fennicae, Series A, 343 (1964).

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