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Rectangular arrays and plane partitions*

by

L. CARLITZ (Durham, N. C.)

1. Introduction. Let $n_1, n_2, \ldots, n_k$ be nonnegative integers such that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k$ and let $\pi_r(n_1, n_2, \ldots, n_k)$ denote the number of $k \times r$ arrays of integers

$$\begin{array}{|ccccc|} n_1 & n_{11} & n_{12} & \ldots & n_{1,r-1} \\ n_2 & n_{21} & n_{22} & \ldots & n_{2,r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n_k & n_{k1} & n_{k2} & \ldots & n_{k,r-1} \end{array} \tag{1.1}$$

such that

$$\begin{cases} n_i \geqslant n_{i1} \geqslant \ldots \geqslant n_{i,r-1} \geqslant 0 & (i = 1, \ldots, k), \\ n_{1j} \geqslant n_{2j} \geqslant \ldots \geqslant n_{kj} & (j = 1, \ldots, r-1). \end{cases} \tag{1.2}$$

We show that

$$\pi_r(n_1, n_2, \ldots, n_k) = \left| \binom{n_j + r - 1}{r - i + j - 1} \right| \qquad (i, j = 1, \ldots, k), \tag{1.3}$$

where the right member is a determinant of order k. When $k = 2$ we are also able to obtain a fairly simple generating function, namely

$$x\big((1-x)(1-y)-z\big)\Phi(x, y, z) = \tfrac{1}{2}(1+x)z - \tfrac{1}{2}(1-x)(1-xy) + \tfrac{1}{2}(1-x)\{1-2(xy+z)+(xy-z)^2\}^{\frac{1}{2}}, \tag{1.4}$$

where

$$\Phi(x, y, z) = \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \pi_r(n, m) x^n y^m z^r.$$

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In this connection we remark that the enumeration of two-line arrays of the type

$$\text{(1.5)}\qquad \boxed{\begin{matrix} n_1 & n_2 & n_3 & \ldots & n_s & \ldots & n_r \\ m_1 & m_2 & m_3 & \ldots & m_s & & \end{matrix}}$$

where

$$n_i > m_i, \quad n_i > n_{i+1} \geqslant 0, \quad m_i > m_{i+1} \geqslant 0, \quad r \geqslant s,$$

has been discussed in [2]; see also [3].

In the next place we consider the function

$$\text{(1.6)}\qquad \pi_r(n_1, \ldots, n_k; a) = \sum a^\sigma,$$

where

$$\sigma = \sum_{i=1}^{k} \sum_{j=1}^{r-1} n_{ij} + \sum_{i=1}^{k} n_i$$

and the summation in (1.5) is over all n_{ij} that satisfy (1.2).

We show that

$$\text{(1.7)}\qquad \pi_r(n_1, \ldots, n_k; a) = a^{n_1+\ldots+n_k}\left| a^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n_j+r-1 \\ r-i+j-1 \end{bmatrix} \right| \quad (i, j = 1, \ldots, k),$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-a^n)(1-a^{n-1})\ldots(1-a^{n-k+1})}{(1-a)(1-a^2)\ldots(1-a^k)}.$$

If we put

$$\pi_{k,r}(n; a) = \sum_{n_k \leqslant \ldots \leqslant n_1 \leqslant n} \pi_r(n_1, \ldots, n_k; a)$$

it follows that

$$\text{(1.8)}\qquad \pi_{k,r}(n; a) = a^{-kn}\pi_{r+1}(n, \ldots, n; a) = \left| a^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n-r \\ r-i+j \end{bmatrix} \right| \quad (i, j = 1, \ldots, k).$$

Now let $p_{k,r}(m, n)$ denote the number of arrays

$$(n_{ij}) \quad (i = 1, \ldots, k;\ j = 1, \ldots, r)$$

that satisfy

$$\begin{cases} n_{i1} \geqslant n_{i2} \geqslant \ldots \geqslant n_{ir} \geqslant 0 & (i = 1, \ldots, k), \\ n_{1j} \geqslant n_{2j} \geqslant \ldots \geqslant n_{kj} \geqslant 0 & (j = 1, \ldots, r) \end{cases}$$

and

$$\sum_{i=1}^{k} \sum_{j=1}^{r} n_{ij} = m, \quad n_{11} \leqslant n.$$

It follows that

$$\text{(1.9)}\qquad \pi_{k,r}(n; a) = \sum_{m=0}^{\infty} p_{k,r}(m, n) a^m.$$

Moreover the determinant in (1.8) can be evaluated and (1.9) becomes

$$\text{(1.10)}\qquad \sum_{m=0}^{\infty} p_{k,r}(m, n) a^m = \frac{(a)_1(a)_2\ldots(a)_{k-1}}{(a)_r(a)_{r+1}\ldots(a)_{r+k-1}} \cdot \frac{(a)_{n+r}(a)_{n+r+1}\ldots(a)_{n+r+k-1}}{(a)_n(a)_{n+1}\ldots(a)_{n+k-1}},$$

where

$$(a)_k = (1-a)(1-a^2)\ldots(1-a^k).$$

The right member of (1.10) is evidently symmetric in k and r, as is to be expected.

Formula (1.10) is due to MacMahon ([5], p. 243); in MacMahon's notation the left member of (1.10) is denoted by $\mathrm{GF}(k; r; n)$. As MacMahon pointed out, (1.10) includes as special cases the generating functions for k-line and plane partitions; Chaundy [4] has given simpler proofs of these results.

For a combinatorial interpretation of (1.7), see § 8 below.

2. Two-line arrays. Let $\pi_r(n)$ denote the number of one-line arrays

$$\boxed{n \;\; n_1 \;\; n_2 \;\; \ldots \;\; n_{r-1}} \quad (n \geqslant n_1 \geqslant n_2 \geqslant \ldots \geqslant n_{r-1} \geqslant 0).$$

It is easily verified that

$$\text{(2.1)}\qquad \pi_r(n) = \binom{n+r-1}{r-1} \quad (r \geqslant 1).$$

Now let $\pi_r(n, m)$ denote the number of two-line arrays

$$\text{(2.2)}\qquad \boxed{\begin{matrix} n & n_1 & n_2 & \ldots & n_{r-1} \\ m & m_1 & m_2 & \ldots & m_{r-1} \end{matrix}},$$

where

$$n \geqslant n_1 \geqslant n_2 \geqslant \ldots \geqslant n_{r-1} \geqslant 0,$$
$$m \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{r-1} \geqslant 0,$$
$$n \geqslant m, \quad n_j \geqslant m_j \quad (1 \leqslant j < r).$$

Then it is evident that

$$\pi_{r+1}(n, m) = \sum_{m_1=0}^{m} \sum_{n_1=m_1}^{n} \pi_r(n_1, m_1). \tag{2.3}$$

This implies

$$\pi_{r+1}(n, m)-\pi_{r+1}(n, m-1) = \sum_{n_1=m}^{n} \pi_r(n_1, m). \tag{2.4}$$

If we define $\pi_{r+1}(n, -1) = 0$ then (2.4) holds for all $m \geqslant 0$.

Clearly

$$\pi_1(n, m) = 1 \qquad (n \geqslant m).$$

Thus (2.4) gives

$$\pi_2(n, m)-\pi_2(n, m-1) = \sum_{n_1=m}^{n} 1 = n-m+1,$$

so that

$$\pi_2(n, m)-\pi_2(n, m-1) = nm-\binom{m}{2}.$$

Since $\pi_2(n, 0) = n+1$, it follows that

$$\pi_2(n, m) = (n+1)(m+1)-\binom{m+1}{2}. \tag{2.5}$$

In the next place, taking $r = 2$ in (2.4) we get

$$\begin{aligned}
\pi_3(n, m)-\pi_3(n, m-1) &= \sum_{n_1=m}^{n} \pi_2(n_1, m)\\
&= \sum_{n_1=m}^{n} \left\{(n_1+1)(m+1)-\binom{m+1}{2}\right\}\\
&= \left\{\binom{n+2}{2}-\binom{m+1}{2}\right\}(m+1)-(n-m+1)\binom{m+1}{2}\\
&= \binom{n+2}{2}(m+1)-(n+2)\binom{m+1}{2}.
\end{aligned}$$

Hence

$$\pi_3(n, m)-\pi_3(n, 0) = \binom{n+2}{2}\left\{\binom{m+2}{2}-1\right\}-(n+2)\binom{m+2}{3}.$$

By (2.1) we have

$$\pi_r(n, 0) = \pi_r(n) = \binom{n+r-1}{r-1} \tag{2.6}$$

and therefore

$$\pi_3(n, m) = \binom{n+2}{2}\binom{m+2}{2}-(n+2)\binom{m+2}{3}. \tag{2.7}$$

At the next step we get

$$\pi_4(n, m) = \binom{n+3}{3}\binom{m+3}{3}-\binom{n+3}{2}\binom{m+3}{4}. \tag{2.8}$$

This suggests the general formula

$$\pi_r(n, m) = \binom{n+r-1}{r-1}\binom{m+r-1}{r-1}-\binom{n+r-1}{r-2}\binom{m+r-1}{r} \qquad (n \geqslant m). \tag{2.9}$$

Assuming that (2.9) holds up to and including the value r, we have, by (2.4),

$$\begin{aligned}
&\pi_{r+1}(n, m)-\pi_{r+1}(n, m-1)\\
&= \sum_{n_1=m}^{n} \left\{\binom{n_1+r-1}{r-1}\binom{m+r-1}{r-1}-\binom{n_1+r-1}{r-2}\binom{m+r-1}{r}\right\}\\
&= \left\{\binom{n+r}{r}-\binom{m+r-1}{r}\right\}\binom{m+r-1}{r-1}-\left\{\binom{n+r}{r-1}-\binom{m+r-1}{r-1}\right\}\binom{m+r-1}{r}\\
&= \binom{n+r}{r}\binom{m+r-1}{r-1}-\binom{n+r}{r-1}\binom{m+r-1}{r}.
\end{aligned}$$

It follows that

$$\pi_{r+1}(n, m)-\pi_{r+1}(n, 0) = \binom{n+1}{r}\left\{\binom{m+r}{r}-1\right\}-\binom{n+r}{r-1}\binom{m+r}{r+1}.$$

In view of (2.6) this reduces to

$$\pi_{r+1}(n, m) = \binom{n+r}{r}\binom{m+r}{r}-\binom{n+r}{r-1}\binom{m+r}{r+1}.$$

This completes the proof of (2.9).

It follows from (2.9) that

$$\sum_{m=0}^{n} \pi_r(n, m) = \binom{n+r-1}{r-1}\binom{n+r}{r}-\binom{n+r-1}{r-2}\binom{n+r}{r+1} \tag{2.10}$$

and

$$\sum_{n=0}^{p} \sum_{m=0}^{n} \pi_r(n, m) = \binom{p+r}{r}^2-\binom{p+r}{r+1}\binom{p+r}{r-1}\frac{(p+r)!(p+r+1)!}{r!(r+1)!p!(p+1)!}. \tag{2.11}$$

3. Generating functions. Put

$$\Phi_r(x, y) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \pi_r(n, m) x^n y^m. \tag{3.1}$$

It follows from (2.4) that

$$(1-y)\Phi_{r+1}(x,y)$$

$$=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\big(\pi_{r+1}(n,m)-\pi_{r+1}(n,m-1)\big)x^n y^m - y\sum_{m=0}^{\infty}\pi_{r+1}(m,m)(xy)^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{n_1=m}^{n}\pi_r(n_1,m)x^n y^m - y\sum_{m=0}^{\infty}\pi_{r+1}(m,m)(xy)^m$$

$$=(1-x)^{-1}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\pi_r(n,m)x^n y^m - y\sum_{m=0}^{\infty}\pi_{r+1}(m,m)(xy)^m,$$

so that

$$(1-x)(1-y)\Phi_{r+1}(x,y) = \Phi_r(x,y)-(1-x)y\sum_{m=0}^{\infty}\pi_{r+1}(m,m)(xy)^m \qquad (r\geqslant 1). \tag{3.2}$$

If we put

$$\Phi(x,y,z)=\sum_{r=1}^{\infty}z^r\Phi_r(x,y), \tag{3.3}$$

then by (3.2)

$$(1-x)(1-y)\Phi(x,y,z)$$

$$=(1-x)(1-y)z\Phi_1(x,y)+z\sum_{r=1}^{\infty}z^r\Phi_r(x,y)-$$

$$-(1-x)y\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\pi_{r+1}(m,m)(xy)^m z^{r+1}.$$

Since

$$\Phi_1(x,y)=(1-x)^{-1}(1-xy)^{-1},$$

this reduces to

$$\big((1-x)(1-y)-z\big)\Phi(x,y,z)=z-(1-x)y\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\pi_r(m,m)(xy)^m z^r. \tag{3.4}$$

Now by (2.9)

$$\pi_r(m,m)=\binom{m+r-1}{r-1}^2-\binom{m+r-1}{r-2}\binom{m+r-1}{r}$$

$$=\frac{(m+r)!(m+r-1)!}{(m+1)!m!r!(r-1)!}=\frac{1}{m+r}\binom{m+r}{r}\binom{m+r}{r-1},$$

for all $r\geqslant 1$. Thus

(3.5)

$$\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\pi_r(m,m)(xy)^m z^r = z\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m+r+1}\binom{m+r+1}{r}\binom{m+r+1}{r+1}(xy)^m z^r.$$

On the other hand

$$1-u-v-\{1-2(u+v)+(u-v)^2\}^{\frac12}$$

$$=1-u-v-\{(1-u-v)^2-4uv\}^{\frac12}$$

$$=(1-u-v)-(1-u-v)\left\{1-\frac{4uv}{1-u-v}\right\}^{\frac12}$$

$$=(1-u-v)-\sum_{r=0}^{\infty}\frac{(-\frac12)_r}{r!}\cdot\frac{(4uv)^r}{(1-u-v)^{2r-1}}$$

$$=2\sum_{r=0}^{\infty}\frac{(2r)!}{r!(r+1)!}\cdot\frac{(uv)^{r+1}}{(1-u-v)^{2r+1}}$$

$$=2\sum_{r=0}^{\infty}\frac{(2r)!}{r!(r+1)!}(uv)^{r+1}\sum_{s=0}^{\infty}\binom{2r+s}{s}(u+v)^s$$

$$=2\sum_{r=0}^{\infty}\frac{(2r)!}{r!(r+1)!}(uv)^{r+1}\sum_{s,j=0}^{\infty}\binom{2r+s+j}{s+j}\binom{s+j}{s}u^s v^j$$

$$=2\sum_{m,n=0}^{\infty}u^{m+1}v^{n+1}\sum_{r=0}^{\min(m,n)}\frac{(2r)!}{r!(r+1)!}\binom{m+n}{2r}\binom{m+n-2r}{m-r}$$

$$=2\sum_{m,n=0}^{\infty}\frac{(m+n)!}{m!(n+1)!}u^{m+1}v^{n+1}\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n+1}{r+1}$$

$$=2\sum_{m,n=0}^{\infty}\frac{(m+n)!}{m!(n+1)!}\binom{m+n+1}{m+1}u^{m+1}v^{n+1}.$$

Comparison with (3.5) yields

$$2xy\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\pi_r(m,m)(xy)^m z^r = 1-xy-z-\{1-2(xy+z)+(xy-z)^2\}^{\frac12}. \tag{3.6}$$

Hence (3.4) becomes

$$x\big((1-x)(1-y)-z\big)\Phi(x,y,z) = \tfrac12(1+x)z-\tfrac12(1-x)(1-xy)+\tfrac12(1-x)\{1-2(xy+z)+(xy-z)^2\}^{\frac12}. \tag{3.7}$$

For example, when $y=0$, (3.7) reduces to

$$\Phi(x,y,z)=\frac{z}{1-x-z},$$

which agrees with (2.6).

When $y=1$, (3.7) reduces to

$$(3.8)\qquad xz\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\pi_r(n,m)x^nz^r$$
$$=\tfrac{1}{2}(1-x)^2-\tfrac{1}{2}(1+x)z-\tfrac{1}{2}(1+x)\{1-2(x+z)+(x-z)^2\}^{\frac{1}{2}}.$$

It can be verified, by expansion of the right member, that (3.8) is in agreement with (2.10).

4. Three-line arrays. Let $\pi_r(n,m,k)$ denote the number of arrays

$$(4.1)\qquad \boxed{\begin{matrix} n & n_1 & n_2 & \dots & n_{r-1}\\ m & m_1 & m_2 & \dots & m_{r-1}\\ k & k_1 & k_2 & \dots & k_{r-1}\end{matrix}}$$

such that

$$n\geqslant n_1\geqslant n_2\geqslant\dots\geqslant n_{r-1}\geqslant 0,$$
$$m\geqslant m_1\geqslant m_2\geqslant\dots\geqslant m_{r-1}\geqslant 0,$$
$$k\geqslant k_1\geqslant k_2\geqslant\dots\geqslant k_{r-1}\geqslant 0,$$
$$n\geqslant m\geqslant k,\qquad n_j\geqslant m_j\geqslant k_j\qquad(1\leqslant j\leqslant r-1).$$

Then clearly

$$(4.2)\qquad \pi_1(n,m,k)=1\qquad(n\geqslant m\geqslant k),$$

$$(4.3)\qquad \pi_r(n,m,0)=\pi_r(n,m)$$

and

$$(4.4)\qquad \pi_{r+}(n,m,k)=\sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m}\sum_{n_1=m_1}^{n}\pi_r(n_1,m_1,k_1).$$

If we take $r=1$ in (4.4) we get

$$\pi_2(n,m,k)=\sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m}\sum_{n_1=m_1}^{n}1=\sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m}(n-m_1+1)$$
$$=\sum_{k_1=0}^{k}\left\{(m-k_1+1)(n+1)-\binom{m+1}{2}+\binom{k_1}{2}\right\}$$
$$=(k+1)(m+1)(n+1)-\binom{k+1}{2}(n+1)-(k+1)\binom{m+1}{2}+\binom{k+1}{3}.$$

This can be written as follows:

$$(4.5)\qquad \pi_2(n,m,k)=\begin{vmatrix}\binom{n+1}{1} & \binom{m+1}{2} & \binom{k+1}{3}\\ 1 & \binom{m+1}{1} & \binom{k+1}{2}\\ \cdot & 1 & \binom{k+1}{1}\end{vmatrix}\qquad(n\geqslant m\geqslant k).$$

Taking $r=2$ in (4.4), we get after a little manipulation

$$(4.6)\qquad \pi_3(n,m,k)=\begin{vmatrix}\binom{n+2}{2} & \binom{m+2}{3} & \binom{k+2}{4}\\ \binom{n+2}{1} & \binom{m+2}{2} & \binom{k+2}{3}\\ 1 & \binom{m+2}{1} & \binom{k+2}{2}\end{vmatrix}\qquad(n\geqslant m\geqslant k).$$

This suggests the general formula

$$(4.7)\qquad \pi_r(n,m,k)=\begin{vmatrix}\binom{n+r-1}{r-1} & \binom{m+r-1}{r} & \binom{k+r-1}{r+1}\\ \binom{n+r-1}{r-2} & \binom{m+r-1}{r-1} & \binom{k+r-1}{r}\\ \binom{n+r-1}{r-3} & \binom{m+r-1}{r-2} & \binom{k+r-1}{r-1}\end{vmatrix}\qquad(n\geqslant m\geqslant k).$$

Assuming that (4.7) holds and making use of (4.4), we get

$$\pi_{r+1}(n,m,k)=\sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m}\begin{vmatrix}\binom{n+r}{r}-\binom{m_1+r-1}{r} & \binom{m_1+r-1}{r} & \binom{k_1+r-1}{r+1}\\ \binom{n+r}{r-1}-\binom{m_1+r-1}{r-1} & \binom{m_1+r-1}{r-1} & \binom{k_1+r-1}{r}\\ \binom{n+r}{r-2}-\binom{m_1+r-1}{r-2} & \binom{m_1+r-1}{r-2} & \binom{k_1+r-1}{r-1}\end{vmatrix}$$

$$=\sum_{k_1=0}^{k}\begin{vmatrix}\binom{n+r}{r} & \binom{m+r}{r+1}-\binom{k_1+r-1}{r+1} & \binom{k_1+r-1}{r+1}\\ \binom{n+r}{r-1} & \binom{m+r}{r}-\binom{k_1+r-1}{r} & \binom{k_1+r-1}{r}\\ \binom{n+r}{r-2} & \binom{m+r}{r-1}-\binom{k_1+r-1}{r-1} & \binom{k_1+r-1}{r-1}\end{vmatrix}=\begin{vmatrix}\binom{n+r}{r} & \binom{m+r}{r+1} & \binom{k+r}{r+2}\\ \binom{n+r}{r-1} & \binom{m+r}{r} & \binom{k+r}{r+1}\\ \binom{n+r}{r-2} & \binom{m+r}{r-1} & \binom{k+r}{r}\end{vmatrix}.$$

This evidently completes the induction.

It follows from (4.7) that

$$(4.8)\qquad \sum_{k=0}^{n}\sum_{m=k}^{n}\pi_r(n,m,k)=\begin{vmatrix}\binom{n+r-1}{r-1} & \binom{n+r}{r+1} & \binom{n+r}{r+2}\\ \binom{n+r-1}{r-2} & \binom{n+r}{r} & \binom{n+r}{r+1}\\ \binom{n+r-1}{r-3} & \binom{n+r}{r-1} & \binom{n+r}{r}\end{vmatrix}.$$

In proving (4.8) it is convenient to sum first with respect to m.

A more symmetrical result can be btained for the function

$$(4.9)\qquad \pi_{3,r}(p)=\sum_{k=0}^{p}\sum_{m=k}^{p}\sum_{n=m}\pi_r(n,m,k).$$

We find that

(4.10) $$\pi_{3,r}(p) = \begin{vmatrix} \binom{p+r}{r} & \binom{p+r}{r+1} & \binom{p+r}{r+2} \\ \binom{p+r}{r-1} & \binom{p+r}{r} & \binom{p+r}{r+1} \\ \binom{p+r}{r-2} & \binom{p+r}{r-1} & \binom{p+r}{r} \end{vmatrix}.$$

Indeed it is evident from (4.4) and (4.9) that

$$\pi_{3,r}(p) = \pi_{r+1}(p, p, p).$$

5. The general case. Let $\pi_r(n_1, n_2, \ldots, n_k)$ denote the number of arrays

(3.1) $$\begin{array}{|ccccc|} n_1 & n_{11} & n_{12} & \ldots & n_{1,r-1} \\ n_2 & n_{21} & n_{22} & \ldots & n_{2,r-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ n_k & n_{k1} & n_{k2} & \ldots & n_{k,r-1} \end{array}$$

such that

$$n_j \geqslant n_{j1} \geqslant n_{j2} \geqslant \ldots \geqslant n_{j,r-1} \geqslant 0 \quad (j = 1, 2, \ldots, k),$$
$$n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k, \quad n_{1s} \geqslant n_{2s} \geqslant \ldots \geqslant n_{ks} \quad (s = 1, 2, \ldots, r-1).$$

Then we have the recurrence

(5.2) $$\pi_{r+1}(n_1, n_2, \ldots, n_k) = \sum_{m_k=0}^{n_k} \sum_{m_{k-1}=m_k}^{n_{k-1}} \ldots \sum_{m_1=m_2}^{n_1} \pi_r(m_1, m_2, \ldots, m_k).$$

Also it is evident that

(5.3) $$\pi_1(n_1, n_2, \ldots, n_k) = 1 \quad (n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k).$$

We may write (5.3) in the following form:

(5.4) $$\pi_1(n_1, n_2, \ldots, n_k) = \left|\binom{n_j}{j-i}\right| \quad (i, j = 1, 2, \ldots, k).$$

This is the case $r = 1$ of the general result

(5.5) $$\pi_r(n_1, n_2, \ldots, n_k) = \left|\binom{n_j+r-1}{r-i+j-1}\right| \quad (i, j = 1, 2, \ldots, k),$$

where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k$.

We assume that (5.5) holds up to and including the value r and apply (5.2). Then exactly as in the case $k = 3$, we first sum with respect to m_1. Thus the first column of the determinant becomes

$$\binom{n_1+r}{r-i} - \binom{m_2+r-1}{r-i} \quad (i = 1, 2, \ldots, k).$$

Adding the elements of the second column to those of the first, the latter becomes

$$\binom{n_1+r}{r-i} \quad (i = 1, 2, \ldots, k).$$

Next summing with respect to m_2, the second column becomes

$$\binom{n_2+r}{r-i+1} - \binom{m_3+r-1}{r-i+1} \quad (i = 1, 2, \ldots, k).$$

Adding the elements of the third column to those of the second, the latter becomes

$$\binom{n_2+r}{r-i+1} \quad (i = 1, 2, \ldots, k).$$

Continuing in this way we ultimately get

$$\pi_{r+1}(n_1, n_2, \ldots, n_k) = \left|\binom{n_j+r}{r-i+j}\right| \quad (i, j = 1, 2, \ldots, k).$$

Thus completing the induction.

If we define

(5.6) $$\pi_{k,r}(n) = \sum_{m_1=0}^{n} \sum_{m_2=0}^{m_1} \ldots \sum_{m_k=0}^{m_{k-1}} \pi_r(m_1, m_2, \ldots, m_k),$$

it is evident from (5.2) that

(5.7) $$\pi_{k,r}(n) = \pi_{r+1}(n, n, \ldots, n).$$

Since ([6], p. 257)

$$\left|\binom{n+r}{r-i+j}\right|_{i,j=1,\ldots,k} = \frac{1!2!\ldots(k-1)!}{r!(r+1)!\ldots(r+k-1)!} \cdot \frac{(n+r)!(n+r+1)!\ldots(n+r+k-1)!}{n!(n+1)!\ldots(n+k-1)!}$$

it follows that

(5.8) $$\pi_{k,r}(n) = \frac{1!2!\ldots(k-1)!}{r!(r+1)!\ldots(r+k-1)!} \cdot \frac{(n+r)!(n+r+1)!\ldots(n+r+k-1)!}{n!(n+1)!\ldots(n+k-1)!}.$$

The right member is evidently symmetric in k and r.

6. The weighted case. Put

(6.1) $$\pi_r(n_1, \ldots, n_k; a) = \sum a^\sigma,$$

where

$$\sigma = \sum_{i=1}^{k} \sum_{j=1}^{r-1} n_{ij} + \sum_{i=1}^{k} n_i,$$

and the summation in (6.1) is over all n_{ij} in the array

$$(6.2)\qquad \begin{array}{|ccccc|} \hline n_1 & n_{11} & n_{12} & \ldots & n_{1,r-1} \\ n_2 & n_{21} & n_{22} & \ldots & n_{2,r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n_k & n_{k1} & n_{k2} & \ldots & n_{k,r-1} \\ \hline \end{array}$$

such that

$$n_i \geqslant n_{i1} \geqslant n_{i2} \geqslant \ldots \geqslant n_{i,r-1} \geqslant 0 \qquad (i = 1, \ldots, k)$$

and

$$n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k; \qquad n_{1j} \geqslant n_{2j} \geqslant \ldots \geqslant n_{kj} \qquad (j = 1, \ldots, r-1).$$

The parameter a is arbitrary.

It is evident from the definition that

$$(6.3)\qquad \pi_1(n_1, \ldots, n_k; a) = a^{n_1+\ldots+n_k} \qquad (n_1 \geqslant \ldots \geqslant n_k).$$

Also it is easily seen that

$$(6.4)\qquad \pi_{r+1}(n_1, \ldots, n_k; a) = a^{n_1+\ldots+n_k} \sum_{m_k=0}^{n_k} \sum_{m_{k-1}=m_k}^{n_{k-1}} \ldots \sum_{m_1=m_2}^{n_1} \pi_r(m_1, \ldots, m_k; a).$$

To begin with we take $k = 2$. Then, by (6.4), we have

$$(6.5)\qquad \pi_{r+1}(n, m; a) = a^{n+m} \sum_{m_1=0}^{m} \sum_{n_1=m_1}^{n} \pi_r(n_1, m_1; a).$$

In particular we have, by (6.3),

$$\begin{aligned} \pi_2(n, m; a) &= a^{n+m} \sum_{m_1=0}^{m} \sum_{n_1=m_1}^{n} a^{n_1+m_1} = a^{n+m} \sum_{m_1=0}^{m} a^{m_1} \frac{a^{m_1} - a^{n+1}}{1-a} \\ &= \frac{a^{n+m}}{1-a} \left(\frac{1-a^{2m+2}}{1-a^2} - a^{n+1} \frac{1-a^{m+1}}{1-a} \right) \\ &= a^{n+m} \left\{ \frac{1-a^{n+1}}{1-a} \cdot \frac{1-a^{m+1}}{1-a} - a \frac{(1-a^{m+1})(1-a^m)}{(1-a)(1-a^2)} \right\} \\ &= a^{n+m} \left\{ \begin{bmatrix} n+1 \\ 1 \end{bmatrix} \begin{bmatrix} m+1 \\ 1 \end{bmatrix} - a \begin{bmatrix} m+1 \\ 2 \end{bmatrix} \right\}, \end{aligned}$$

where

$$(6.6)\qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-a^n)(1-a^{n-1})\ldots(1-a^{n-k+1})}{(1-a)(1-a^2)\ldots(1-a^k)}.$$

In the next place

$$\pi_3(n, m; a) = a^{n+m} \sum_{m_1=0}^{m} \sum_{n_1=m_1}^{n} a^{n_1+m_1} \left\{ \begin{bmatrix} n_1+1 \\ 1 \end{bmatrix} \begin{bmatrix} m_1+1 \\ 1 \end{bmatrix} - a \begin{bmatrix} m_1+1 \\ 2 \end{bmatrix} \right\}.$$

It is easily verified that

$$(6.7)\qquad \sum_{j=k}^{n} a^{j-k} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \pi_3(n, m; a) &= a^{n+m} \sum_{m_1=0}^{m} a^{m_1} \left\{ \left(\begin{bmatrix} n+2 \\ 2 \end{bmatrix} - \begin{bmatrix} m_1+1 \\ 2 \end{bmatrix} \right) \begin{bmatrix} m_1+1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} n+2 \\ 1 \end{bmatrix} - \begin{bmatrix} m_1+1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} m_1+1 \\ 2 \end{bmatrix} \right\} \\ &= a^{n+m} \sum_{m_1=0}^{m} a^{m_1} \left\{ \begin{bmatrix} n+2 \\ 2 \end{bmatrix} \begin{bmatrix} m_1+1 \\ 1 \end{bmatrix} - \begin{bmatrix} n+2 \\ 1 \end{bmatrix} \begin{bmatrix} m_1+1 \\ 2 \end{bmatrix} \right\} \\ &= a^{n+m} \left\{ \begin{bmatrix} n+2 \\ 2 \end{bmatrix} \begin{bmatrix} m+2 \\ 2 \end{bmatrix} - a \begin{bmatrix} n+2 \\ 1 \end{bmatrix} \begin{bmatrix} m+2 \\ 3 \end{bmatrix} \right\}. \end{aligned}$$

The general formula is

$$(6.8)\qquad \pi_r(n, m; a) = a^{n+m} \begin{vmatrix} \begin{bmatrix} n+r-1 \\ r-1 \end{bmatrix} & a \begin{bmatrix} m+r-1 \\ r \end{bmatrix} \\ \begin{bmatrix} n+r-1 \\ r-2 \end{bmatrix} & \begin{bmatrix} m+r-1 \\ r-1 \end{bmatrix} \end{vmatrix}.$$

Indeed, assuming the truth of (6.8), we have

$$\begin{aligned} \pi_{r+1}(n, m; a) &= a^{n+m} \sum_{m_1=0}^{m} \sum_{n_1=m_1}^{n} a^{n_1+m_1} \begin{vmatrix} \begin{bmatrix} n_1+r-1 \\ r-1 \end{bmatrix} & a \begin{bmatrix} m_1+r-1 \\ r \end{bmatrix} \\ \begin{bmatrix} n_1+r-1 \\ r-2 \end{bmatrix} & \begin{bmatrix} m_1+r-1 \\ r-1 \end{bmatrix} \end{vmatrix} \\ &= a^{n+m} \sum_{m_1=0}^{m} a^{m_1} \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} - \begin{bmatrix} m_1+r-1 \\ r \end{bmatrix} & a \begin{bmatrix} m_1+r-1 \\ r \end{bmatrix} \\ a^{-1} \begin{bmatrix} n+r \\ r-1 \end{bmatrix} - a^{-1} \begin{bmatrix} m_1+r-1 \\ r-1 \end{bmatrix} & \begin{bmatrix} m_1+r-1 \\ r-1 \end{bmatrix} \end{vmatrix} \\ &= a^{n+m} \sum_{m_1=0}^{m} a^{m_1} \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} & a \begin{bmatrix} m_1+r-1 \\ r \end{bmatrix} \\ a^{-1} \begin{bmatrix} n+r \\ r-1 \end{bmatrix} & \begin{bmatrix} m_1+r-1 \\ r-1 \end{bmatrix} \end{vmatrix} \\ &= a^{n+m} \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} & a^2 \begin{bmatrix} m+r \\ r+1 \end{bmatrix} \\ a^{-1} \begin{bmatrix} n+r \\ r-1 \end{bmatrix} & \begin{bmatrix} m+r \\ r \end{bmatrix} \end{vmatrix} = a^{n+m} \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} & a \begin{bmatrix} m+r \\ r+1 \end{bmatrix} \\ \begin{bmatrix} n+r \\ r-1 \end{bmatrix} & \begin{bmatrix} m+r \\ r \end{bmatrix} \end{vmatrix}. \end{aligned}$$

In the next place, for the three-line case, we have first of all

$$(6.9)\qquad \pi_1(n, m, k; a) = a^{n+m+k} \qquad (n \geqslant m \geqslant k).$$

Then

$$
\begin{aligned}
&a^{-n-m-k}\pi_2(n,m,k;a)\\
&= \sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m}\sum_{n_1=m_1}^{n} a^{n_1+m_1+k_1} = \sum_{k_1=0}^{k}\sum_{m_1=k_1}^{m} a^{m_1+k_1}\left(\genfrac{[}{]}{0pt}{}{n+1}{1}-\genfrac{[}{]}{0pt}{}{m_1}{1}\right)\\
&= \sum_{k_1=0}^{k} a^{k_1}\left\{\left(\genfrac{[}{]}{0pt}{}{m+1}{1}-\genfrac{[}{]}{0pt}{}{k_1}{1}\right)\genfrac{[}{]}{0pt}{}{n+1}{1}-a\left(\genfrac{[}{]}{0pt}{}{m+1}{2}-\genfrac{[}{]}{0pt}{}{k_1}{2}\right)\right\}\\
&= \genfrac{[}{]}{0pt}{}{k+1}{1}\genfrac{[}{]}{0pt}{}{m+1}{1}\genfrac{[}{]}{0pt}{}{n+1}{1}-a\genfrac{[}{]}{0pt}{}{k+1}{2}\genfrac{[}{]}{0pt}{}{n+1}{1}-a\genfrac{[}{]}{0pt}{}{k+1}{1}\genfrac{[}{]}{0pt}{}{m+1}{2}-a^3\genfrac{[}{]}{0pt}{}{k+1}{3},
\end{aligned}
$$

so that

$$
\pi_2(n,m,k;a) = a^{n+m+k}\begin{vmatrix} \genfrac{[}{]}{0pt}{}{n+1}{1} & a\genfrac{[}{]}{0pt}{}{m+1}{2} & a^3\genfrac{[}{]}{0pt}{}{k+1}{3} \\ 1 & \genfrac{[}{]}{0pt}{}{m+1}{1} & a\genfrac{[}{]}{0pt}{}{k+1}{2} \\ . & 1 & \genfrac{[}{]}{0pt}{}{k+1}{1} \end{vmatrix}. \tag{6.10}
$$

Continuing in this way we get

$$
\pi_r(n,m,k;a) = a^{n+m+k}\begin{vmatrix} \genfrac{[}{]}{0pt}{}{n+r-1}{r-1} & a\genfrac{[}{]}{0pt}{}{m+r-1}{r} & a^3\genfrac{[}{]}{0pt}{}{k+r-1}{r+1} \\ \genfrac{[}{]}{0pt}{}{n+r-1}{r-2} & \genfrac{[}{]}{0pt}{}{m+r-1}{r-1} & a\genfrac{[}{]}{0pt}{}{k+r-1}{r} \\ a\genfrac{[}{]}{0pt}{}{n+r-1}{r-3} & \genfrac{[}{]}{0pt}{}{m+r-1}{r-2} & \genfrac{[}{]}{0pt}{}{k+r-1}{r-1} \end{vmatrix}. \tag{6.11}
$$

We shall omit the proof of (6.11).

Comparison of (6.8) with (6.11) suggests the following general result:

$$
\pi_r(n_1,\ldots,n_k;a) = a^{n_1+\ldots+n_k}\left|a^{\frac12(i-j)(i-j-1)}\genfrac{[}{]}{0pt}{}{n_j+r-1}{r-i+j-1}\right| \quad (i,j=1,2,\ldots,k), \tag{6.12}
$$

where as usual we assume that $n_1 \geqslant \ldots \geqslant n_k$.

For $r=1$, (6.12) evidently agrees with (6.3). Assuming that (6.12) holds up to and including the value r, we have, by (6.4),

$$
\begin{aligned}
&a^{-n_1-\ldots-n_k}\pi_{r+1}(n_1,\ldots,n_k;a)\\
&= \sum_{m_k=0}^{n_k}\sum_{m_{k-1}=m_k}^{n_{k-1}}\cdots\sum_{m_1=m_2}^{n_1} a^{m_1+\ldots+m_k}\left|a^{\frac12(i-j)(i-j-1)}\genfrac{[}{]}{0pt}{}{m_j+r-1}{r-i+j-1}\right|.
\end{aligned} \tag{6.13}
$$

Performing the summation with respect to m_1, the first column becomes, by (6.7),

$$
a^{\frac12(i-1)(i-4)}\left(\genfrac{[}{]}{0pt}{}{n_1+r}{r-i+1}-\genfrac{[}{]}{0pt}{}{m_2+r-1}{r-i+1}\right) \quad (i=1,2,\ldots,k).
$$

The second column is

$$
a^{\frac12(i-2)(i-3)}\genfrac{[}{]}{0pt}{}{m_2+r-1}{r-i+1} \quad (i=1,2,\ldots,k).
$$

Thus multiplying the second column by a^{-1} and adding to the first, the latter becomes

$$
a^{\frac12(i-1)(i-4)}\genfrac{[}{]}{0pt}{}{n_1+r}{r-i+1} \quad (i=1,2,\ldots,k).
$$

We now perform the summation with respect to m_2; the second column becomes

$$
a^{\frac12(i-2)(i-5)}\left(\genfrac{[}{]}{0pt}{}{n_2+r}{r-i+2}-\genfrac{[}{]}{0pt}{}{m_3+r-1}{r-i+2}\right) \quad (i=1,2,\ldots,k),
$$

which finally gives

$$
a^{\frac12(i-2)(i-5)}\genfrac{[}{]}{0pt}{}{n_2+r}{r-i+2} \quad (i=1,2,\ldots,k).
$$

Proceeding in this way we find that the jth column becomes

$$
a^{\frac12(i-j)(i-j-3)}\genfrac{[}{]}{0pt}{}{n_j+r}{r-i+j} \quad (i=1,2,\ldots,k),
$$

so that (6.13) reduces to

$$
\begin{aligned}
&a^{-n_1-\ldots-n_k}\pi_{r+1}(n_1,\ldots,n_k;a)\\
&= \left|a^{\frac12(i-j)(i-j-3)}\genfrac{[}{]}{0pt}{}{n_j+r}{r-i+j}\right| \quad (i,j=1,2,\ldots,k).
\end{aligned} \tag{6.14}
$$

Now multiply the ith row of the determinant on the right of (6.14) by a^{i-1} and divide the jth column by a^{j-1}, so that the value of the determinant is unchanged. The exponent of a becomes

$$
\tfrac12(i-j)(i-j-3)+(i-1)-(j-1) = \tfrac12(i-j)(i-j-1).
$$

Therefore (6.14) becomes

$$
\pi_{r+1}(n_1,\ldots,n_k;a) = a^{n_1+\ldots+n_k}\left|a^{\frac12(i-j)(i-j-1)}\genfrac{[}{]}{0pt}{}{n_j+r}{r-i+j}\right| \quad (i,j=1,2,\ldots,k)
$$

and the induction is complete.

If we define

$$
\pi_{k,r}(n;a) = \sum_{n_k\leqslant\ldots\leqslant n_1\leqslant n}\pi_r(n_1,\ldots,n_k;a), \tag{6.15}
$$

then it is evident from (6.4) that

$$
a^{kn}\pi_{k,r}(n;a) = \pi_{r+1}(n,\ldots,n;a). \tag{6.16}
$$

Hence $\pi_{k,r}(n;a)$ is evaluated by means of (6.12):

$$\pi_{k,r}(n;a) = \left| a^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n+r \\ r-i+j \end{bmatrix} \right| \quad (i,j = 1, 2, \ldots, k). \tag{6.17}$$

We remark that the right member is a polynomial in a.

7. Applications to partitions. Let $p_{k,r}(m)$ denote the number of $k\times r$ arrays of integers

$$(n_{ij}) \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, r), \tag{7.1}$$

such that

$$\begin{cases} n_{i1} \geqslant n_{i2} \geqslant \ldots \geqslant n_{ir} \geqslant 0 & (i = 1, 2, \ldots, k), \\ n_{1j} \geqslant n_{2j} \geqslant \ldots \geqslant n_{kj} \geqslant 0 & (j = 1, 2, \ldots, r) \end{cases} \tag{7.2}$$

and

$$\sum_{i=1}^{k}\sum_{j=1}^{r} n_{ij} = m. \tag{7.3}$$

Also let $p_{k,r}(m,n)$ denote the number of arrays (7.1) that satisfy (7.2) and (7.3) and in addition $n_{11} \leqslant n$. Thus $p_{k,r}(m,n)$ enumerates partitions of m in which the parts do not exceed n, while $p_{k,r}(m)$ enumerates partitions in which the size of the parts is unrestricted.

Comparison with (6.1) and (6.15) immediately yields

$$\pi_{k,r}(n;a) = \sum_{m=0}^{\infty} p_{k,r}(m,n)\, a^m. \tag{7.4}$$

Thus (6.17) furnishes the generating function for $p_{k,r}(n,m)$. Since $\pi_{k,r}(n,a)$ is a polynomial in a, there is no question of convergence.

If $k=1$, we have

$$\sum_{m=0}^{\infty} p_{1,r}(n,m)\, a^m = \begin{bmatrix} n+r \\ r \end{bmatrix}.$$

This is of course a familiar result. When $k=2$ we get

$$\sum_{m=0}^{\infty} p_{2,r}(n,m)\, a^m = \begin{vmatrix} \begin{bmatrix} n+r \\ r \end{bmatrix} & a\begin{bmatrix} n+r \\ r+1 \end{bmatrix} \\ \begin{bmatrix} n+r \\ r-1 \end{bmatrix} & \begin{bmatrix} n+r \\ r \end{bmatrix} \end{vmatrix} = \begin{bmatrix} n+r \\ r \end{bmatrix}\begin{bmatrix} n+r \\ r-1 \end{bmatrix} \frac{(1-a)(1-a^{n+r+1})}{(1-a^r)(1-a^{r+1})}$$

$$= \frac{1-a}{1-a^{n+r+1}} \begin{bmatrix} n+r+1 \\ r \end{bmatrix}\begin{bmatrix} n+r+1 \\ r+1 \end{bmatrix}.$$

When $k=3$ we find after a little manipulation that

$$\sum_{m=0}^{\infty} p_{3,r}(m,n)\, a^m = \frac{(1-a)^2(1-a^2)}{(1-a^{r+1})^2(1-a^{r+2})} \begin{bmatrix} n+r \\ r \end{bmatrix}\begin{bmatrix} n+r+1 \\ r \end{bmatrix}\begin{bmatrix} n+r+2 \\ r \end{bmatrix}.$$

This suggests the general formula

$$\sum_{m=0}^{\infty} p_{k,r}(m,n)\, a^m \tag{7.5}$$

$$= \frac{(1-a)^{k-1}(1-a^2)^{k-2}\ldots(1-a^{k-1})}{(1-a^{r+1})^{k-1}(1-a^{r+2})^{k-2}\ldots(1-a^{r+k-1})} \begin{bmatrix} n+r \\ r \end{bmatrix}\begin{bmatrix} n+r+1 \\ r \end{bmatrix}\ldots\begin{bmatrix} n+r+k-1 \\ r \end{bmatrix}.$$

If we put

$$(a)_n = (1-a)(1-a^2)\ldots(1-a^n), \qquad (a)_0 = 1,$$

(7.5) becomes

$$\sum_{m=0}^{\infty} p_{k,r}(m,n)\, a^m \tag{7.6}$$

$$= \frac{(a)_1(a)_2\ldots(a)_{k-1}}{(a)_r(a)_{r+1}\ldots(a)_{r+k-1}} \cdot \frac{(a)_{n+r}(a)_{n+r+1}\ldots(a)_{n+r+k-1}}{(a)_n(a)_{n+1}\ldots(a)_{n+k-1}}.$$

The determinant occurring in (6.17) is a special case of a determinant evaluated by the writer in [1]. We shall however give a direct proof of the identity

$$D(k,r,n) = \frac{(a)_1(a)_2\ldots(a)_{k-1}}{(a)_r(a)_{r+1}\ldots(a)_{r+k-1}} \cdot \frac{(a)_{n+r}(a)_{n+r+1}\ldots(a)_{n+r+k-1}}{(a)_n(a)_{n+1}\ldots(a)_{n+k-1}}, \tag{7.7}$$

where

$$D(k,r,n) = \left| a^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n+r \\ r-i+j \end{bmatrix} \right| \quad (i,j = 0, 1, \ldots, k-1).$$

Clearly $D(k,r,n)$ is identical with determinant in (6.17).

If we put

$$W(k,r) = \left| a^{ri+\frac{1}{2}i(i-1)} \begin{bmatrix} j \\ i \end{bmatrix} \right| \quad (i,j = 0, 1, \ldots, k-1),$$

we get

$$D(k,r,n)W(k,r) = |c_{ij}| \quad (i,j = 0, 1, \ldots, k-1),$$

where

$$c_{ij} = \sum_{s=0}^{j} a^{\frac{1}{2}s(s-1)} \begin{bmatrix} j \\ s \end{bmatrix} a^{rs+\frac{1}{2}(i-s)(i-s-1)} \begin{bmatrix} n+r \\ n+i-j \end{bmatrix}.$$

This reduces to

$$c_{ij} = a^{\frac{1}{2}i(i-1)} \begin{bmatrix} n+r+j \\ n+i \end{bmatrix}.$$

Thus

$$D(k,r,n)W(k,r) = \left| a^{\frac{1}{2}i(i-1)} \begin{bmatrix} n+r+j \\ n+i \end{bmatrix} \right| \quad (i,j = 0, 1, \ldots, k-1)$$

$$= \frac{(a)_{n+r}(a)_{n+r+1}\ldots(a)_{n+r+k-1}}{(a)_n(a)_{n+1}\ldots(a)_{n+k-1}} \left| \frac{a^{\frac{1}{2}i(i-1)}}{(a)_{r-i+j}} \right|,$$

so that

$$D(k, r, n) = \frac{(a)_{n+r}(a)_{n+r+1}\cdots(a)_{n+r+k-1}}{(a)_n(a)_{n+1}\cdots(a)_{n+k-1}} C(k, r),$$

where $C(k, r)$ is independent of n. Taking $n = 0$ we get

$$C(k, r) = \frac{(a)_1(a)_2(a)_{k-1}}{(a)_r(a)_{r+1}\cdots(a)_{r+k-1}}.$$

and (7.7) follows at once.

It is evident from the definition that

$$p_{k,r}(m, n) = p_{r,k}(m, n). \tag{7.8}$$

It is easily verified that the right member of (7.6) is also symmetric in k, r.

Up to this point the parameter a has been arbitrary. We now assume that $|a| < 1$ and let $r \to \infty$. Then (7.6) reduces to

$$\sum_{m=0}^{\infty} p_{k,\infty}(m, n) a^m = \frac{(a)_1(a)_2\cdots(a)_{k-1}}{(a)_n(a)_{n+1}\cdots(a)_{n+k-1}}. \tag{7.9}$$

Clearly $p_{k,\infty}(n, m)$ is the number of k-line partitions of m in which the largest part does not exceed n.

If we let $n \to \infty$, (7.9) becomes

$$\sum_{m=0}^{\infty} p_{k,\infty}(m) a^m = \prod_{j=1}^{\infty} (1-a^j)^{-\min(j,k)}. \tag{7.10}$$

Finally, when $k \to \infty$, we get

$$\sum_{m=0}^{\infty} p_{\infty,\infty}(m) a^m = \prod_{j=1}^{\infty} (1-a^j)^{-j}. \tag{7.11}$$

Clearly $p_{k,\infty}(m)$ is the number of k-line partitions of m and $p_{\infty,\infty}(m)$ is the number of unrestricted plane partitions of m.

8. We can also give a combinatorial interpretation of $\pi_r(n_1, \ldots, n_k; a)$, namely

$$\pi_r(n_1, \ldots, n_k; a) = a^{n_1+\ldots+n_k} \sum_{m=0}^{\infty} p_r(m; n_1, \ldots, n_k) a^m, \tag{8.1}$$

where $p_r(m; n_1, \ldots, n_k)$ is the number of $k \times r$ arrays (6.2) satisfying

$$\begin{cases} n_i \geqslant n_{i1} \geqslant \ldots \geqslant n_{i,r-1} & (i = 1, 2, \ldots, j), \\ n_1 \geqslant \ldots \geqslant n_k, \quad n_{1j} \geqslant \ldots \geqslant n_{kj} & (j = 1, 2, \ldots, r-1) \end{cases} \tag{8.2}$$

together with

$$\sum_{i=1}^{k} \sum_{j=1}^{r-1} n_{ij} = m. \tag{8.3}$$

Thus (6.12) furnishes the generating function

$$\sum_{m=0}^{\infty} p_r(m; n_1, \ldots, n_k) a^m = \left| a^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n_j + r - 1 \\ r - i + j - 1 \end{bmatrix} \right| \quad (i, j = 1, 2, \ldots, k). \tag{8.4}$$

Put

$$p(m; n_1, \ldots, n_k) = p_\infty(m; n_1, \ldots, n_k),$$

so that $p(m; n_1, \ldots, n_k)$ is the number of arrays satisfying (8.2) and (8.3) with r infinite. Then (8.4) reduces to

$$\sum_{m=0}^{\infty} p(m; n_1, \ldots, n_k) a^m = \left| \frac{a^{\frac{1}{2}(i-j)(i-j-1)}}{(a)_{n_j+i-j}} \right| \quad (i, j = 1, 2, \ldots, k). \tag{8.5}$$

For example, when $k = 2$, we have

$$\sum_{m=0}^{\infty} p(m; n_1, n_2) a^m = \frac{1 - a - a^{n_1+1} + a^{n_2+1}}{(a)_{n_1+1}(a)_{n_2}}.$$

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