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ORDERED SETS AND COMPACT SPACES

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In this note* we treat a set X with a total (linear) order $\leqslant$, provided with its order topology. Our main interest is the possibility of extending the order in X to compact spaces containing X.

Our usage concerning compactifications is basically that of Kelley [2], p. 151, with the proviso that all spaces be Hausdorff and the exception that a compactification and "its space" are not always distinguished. If X_1 is a compact space containing X as a dense subset, X_1 provides an *orderable* compactification of X if the order of X can be so extended to X_1 as to yield the given topology of X_1 as the order topology. The last condition means precisely that the closed (open) intervals of the order be topologically closed (open). A compactification is *suborderable* if it is dominated ([2], p. 151) by an orderable compactification; these compactifications have a neat characterization, as will appear.

We denote by $F(X)$ the set of continuous, increasing functions f on X to the unit interval $I = [0, 1]$ for which $\inf f = 0$, $\sup f = 1$. We assume as known that F separates X; a proof may be modeled on Urysohn's Lemma as presented in [2], pp. 114-115.

1. LEMMA. *If $E \subseteq F$ and E separates X, the weak topology in X determined by E coincides with the order topology.*

Proof. Let $a \in X$ and $Y = \{x : x \leqslant a\}$, $Z = \{x : f(x) \leqslant f(a)$ for all $f \in E\}$. Clearly $Y \subseteq Z$ and if $x \in Z - Y$, then $f(x) = f(a)$ for all $f \in E$, a contradiction. Thus Y is closed in the weak topology and similarly the set $\{x : x \geqslant a\}$ is closed.

2. THEOREM. *The closure, X_∞, of X in $I^{F(X)}$ is orderable and dominates every orderable compactification of X.*

Proof. Define a partial order in X_∞ by declaring $z_1 \leqslant z_2$ when $z_1(f) \leqslant z_2(f)$ for each $f \in F(X)$. If $z_1 \neq z_2$ we can suppose $z_1(f) < a < z_2(f)$ for some $f \in F(X)$ and real number a. Set $A = \{z \in X_\infty : z(f) < a\}$ and

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$B = \{z \in X_\infty : a < z(f)\}$. Then $z_1 \in (A \cap X)^-$, $x_2 \in (B \cap X)^-$. But if $x_1 \in A \cap X$ and $x_2 \in B \cap X$, then $f'(x_1) \leqslant f'(x_2)$ for every $f' \in F$, so $z_1 \leqslant z_2$. This proves the first part.

For the second, let $X \subseteq X_2$, an orderable compactification, and let E be the restriction to X of $F(X_2)$. Then from the embedding $X_2 \subseteq I^E$ it is clear that X_2 is dominated by X_∞ as a compactification.

3. COROLLARY. *If $E \subseteq F$ and E separates X, the closure of X in I^E furnishes an orderable compactification; all orderable compactifications are so determined.*

In contrast to the general situation in which no minimal compactifications need exist, there is a minimal orderable compactification which we now construct. We define $\mathscr{L}$ to be the class of closed subsets A of X such that if $x \leqslant y$, $y \in A$, then always $x \in A$. (A might be called a "closed ideal.") $\mathscr{L}$ is totally ordered by defining $A \leqslant B$ when $A \subseteq B$. If $P = \{F\}$ is a non-empty subfamily of $\mathscr{L}$, $C = [\cup \{F \in P\}]^-$ is the least upper bound of P in $\mathscr{L}$, and $D = \cap \{F \in P\}$ the greatest lower bound. Thus $\mathscr{L}$ is compact in its order topology.

4. THEOREM. *Let $X \subseteq X_2$, an orderable compactification of X. For a in X_2, let $\varphi(a) = \{x \in X : x \leqslant a\}$. Then φ is a continuous function on X_2 into $\mathscr{L}$ and a homeomorphism of X into $\mathscr{L}$.*

Proof. Let $A \in \mathscr{L}$ and $A' = \{x \in X_2 : \varphi(x) \leqslant A\}$. To show that A' is closed, it is enough to show that it contains its supremum x_0. But if $x_0 \notin A'$, A' is open and then x_0 is in the closure of $(A' \cap X) \subseteq X_2$. If $x_0 \in X$, then $x_0 \in A'$ inasmuch as A is a closed subset of X. But, if $x_0 \notin X$, $\varphi(x_0) = \{x \in X : x < x_0\} \subseteq A$ since x_0 is the supremum of A'. Similarly the set $\{x_2 \in X_2 : \varphi(x_2) \geqslant A\}$ is closed and φ is continuous. Clearly φ is an open mapping of X onto $\varphi(X)$ and is thus a homeomorphism.

To obtain the minimal orderable compactification of X, we take the closure of $\varphi(X)$ in $\mathscr{L}$ and call this space X_0; its minimal character has just been demonstrated. $X_0 = \mathscr{L}$ unless X has a first element.

5. THEOREM. *X_0 is connected if and only if no element of X is an immediate successor of another.*

Proof. If b is an immediate successor of a in X, the open set $(\varphi(a), \varphi(b))$ of X_0 is void since it has void intersection with X. On the other hand, a compact ordered space is connected if it contains no immediate successors; let A_2 be an immediate successor of A_1 in X_0. Then the difference $A_2 - A_1$ is a singleton $\{x_0\} \subseteq X$ and $A_1 = \{x : x < x_0\}$. We observe that unless A_1 contains its supremum it is not closed, and this supremum precedes x_0 immediately.

6. COROLLARY. *If an orderable compactification is larger than X_0, it is disconnected.*

Proof. Suppose Y is connected and $y_1 < y_2$ in Y. Then (y_1, y_2) intersects X so that $\varphi(y_1) < \varphi(y_2)$ in Theorem 4.

7. LEMMA. *Let W, Y, Z be compactifications of X, $W \leqslant Y \leqslant Z$. If W and Z are orderable, so also is Y.*

Proof. Let g_1, g_2, g_3 be the homeomorphisms of X into W, Y, Z respectively, and f_1, f_2 be the covering maps of Z onto Y, and Y onto W, respectively; then $g_2 = f_1 \circ g_1$ and $g_3 = f_2 \circ g_2$. We claim that if $f_1(z_1) = f_1(z_2)$ for $z_i \epsilon Z$, then (if, say, $z_1 < z_2$) z_2 is an immediate successor of z_1. For if (z_1, z_2) is not void it intersects $g_1(X)$ in a dense subset of (z_1, z_2). If $x \epsilon X$ and $z_1 \leqslant g_1(x) \leqslant z_2$,

$$f_2 \circ f_1(z_1) \leqslant f_2 \circ f_1 \circ g_1(x) \leqslant f_2 \circ f_1(z_2).$$

(Here we have used the fact that $f_2 \circ f_1$ preserves order.) Thus $[z_1, z_2] \frown X$ is a singleton $\{g_1(x_0)\}$, and neither end-point of (z_1, z_2) can be equal to $g_1(x_0)$, since $f_2 \circ f_1$ is a homeomorphism on $g_1(X)$. Since z_1 has an immediate successor it is in the closure of $g_1(X) \frown \{z: z < z_1\}$ and this last set has no supremum in $g_1(X)$. But, since $(z_1, z_2) = \{g_1(x_0)\}$, the set $g_1(X) \frown \{z: z < z_1\}$ contains $g_1(x_0)$ in its closure. But g_1 is a homeomorphism; this contradiction proves the claim. Now $Y = f_1(Z)$ is ordered by setting $y_1 \leqslant y_2$ when $f_1^{-1}(y_1) \leqslant f_1^{-1}(y_2)$, or, equivalently, in such a manner that the images of closed intervals in Z are the closed intervals of Y.

We resort now to the use of uniformities to avoid certain well-known logical difficulties. A compactification of X is identified with a totally-bounded uniformity for X which yields the order topology as its uniform topology. Observe that for any set (this really *is* a set) of uniformities $\{U_a: a \epsilon A\}$ there is a strongest uniformity contained in each of the U_a and a weakest containing each of these; if each U_a is totally-bounded, so also is the sup. If also each U_a yields an orderable compactification, it is bounded above and below by the maximal and minimal uniformities of this type, by Theorems 2 and 4. But then by Lemma 7 both the inf and sup of any collection yield the order topology in X and have compact, orderable completions. Less precisely, the orderable compactifications form a complete lattice, in which inf and sup are also inf and sup among all compactifications.

8. THEOREM. *The lattice of orderable compactifications of X is isomorphic to the lattice 2^S of subsets of a certain set S.*

Proof. The set S emerges from the map φ of X_∞ onto X_0, as in Theorem 4. As in Corollary 6 and Lemma 7, it is seen that if $y_1 \neq y_2$ and $\varphi(y_1) = \varphi(y_2)$, then $y_i \notin X$ and (y_1, y_2) is void. Moreover, this is a sufficient condition that $\varphi(y_1) = \varphi(y_2)$. Evidently what is needed to complete the proof is the fact that for *any* subset T of these doubletons, there is a map of X_∞ onto an orderable compactification which collapses the doubletons in T but identifies no other points; by Lemma 7 it is enough to show that these points can be identified so that the quotient space is *Hausdorff*.

More generally suppose an ordered space Y is partitioned into a collection $\mathscr{D}$ of disjoint closed intervals $\{D_i\}$ and we say $D_1 \leqslant D_2$ if $d_1 \leqslant d_2$ for some $d_i \in D_i$ $(i = 1, 2)$. The quotient topology is Hausdorff if it is stronger than the order topology. Now if $D_0 \in \mathscr{D}$ we have to show $\{D \in \mathscr{D}: D < D_0\}$ is quotient-open and $\{D \in \mathscr{D}: D \leqslant D_0\}$ is quotient-closed. The first set contains exactly the members of $\mathscr{D}$ which include an element $< \inf D_0$; the second contains exactly those which include an element $\leqslant \sup D_0$.

9. THEOREM. *A compactification of X is suborderable if each monotone sequence in X converges in the compactification.*

Proof. Let f be a homeomorphism of X onto a dense subset of the space X_1, and g the homeomorphism of X into X_∞, as in Theorem 2. It is enough to show that if $\{x_a\}$ is a net in X such that $g(x_a)$ converges in X_∞, then $f(x_a)$ converges in X_1. Also, it can be supposed that the limit y of $g(x_a) \notin g(X)$. We claim that the sets $\{g(x): g(x) < y\}$ and $\{g(x): g(x) > y\}$ do not both contain y in their closure. For if neither set is empty, set $h(x) = 0$ for $g(x) < y$, $h(x) = 1$ for $g(x) > y$; then $h \in F(X)$ and so can be extended continuously from $g(X)$ to X_∞. For this reason we can suppose, say, that $g(x_a) < y$ for each index a and prove that $\{f(x_a)\}$ has at most one cluster-point in X_1.

If on the contrary there are distinct cluster-points z_1 and z_2 in X_1, these have neighborhoods U_i with $U_1^- \cap U_2^-$ void. On the other hand, inasmuch as $g(x_a) \to y$ and $g(x_a) < y$, there is a sequence of indices $a_1 < a_2 < a_3 < \ldots$ such that $x_{a1} < x_{a2} < x_{a3} \ldots$ and $f(x_{a_i}) \in U_1$ for i odd, $f(x_{a_i}) \in U_2$ for i even. Then $\{f(x_{a_i})\}$ must converge, which is a contradiction.

10. COROLLARY. *The Stone-Čech compactification of X, $\beta(X)$, is orderable if and only if every sequence in X contains a subsequence converging in X.*

Proof. Clearly if the stated condition on sequences holds, monotone sequences in X converge in $\beta(X)$. In the reverse direction, we use the fact that every sequence in X contains a monotone subsequence. Thus in fact what has to be proved is that if, for example, $x_1 < x_2 < x_3 < \ldots$ in X, the set $\{x_n: n = 1, 2, 3, \ldots\}$ has a least upper bound in X. In any case, writing $I = \{x: x < x_n \text{ for some } n\}$, I is open and the supposed least upper bound is the only possible boundary point of I.

There exist continuous functions f_n, $0 \leqslant f_n \leqslant 1$, in the intervals $I_n = \{x_n \leqslant x \leqslant x_{n+1}\}$ such that whenever $f_i(x_j)$ is defined $f_i(x_j) = \frac{1}{2}\big(1 + (-1)^j\big)$. Define f^* in I to coincide with f_n on I_n and $f^*(x) = 0$ if $x \leqslant x_1$. If the sequence $x_1 < x_2 < x_3 < \ldots$ does not converge in X, I is then open and closed, and f^* can be extended to all of X. But since $f^*(x_i)$ does not converge, the indicated sequence does not converge in $\beta(X)$ and then clearly β is not (dominated by) an orderable compactification.

11. Corollary. *A function in $C(X)$ is in the closure of the space of functions of bounded variation in X if and only if it transforms monotone sequences in X to converging numerical sequences.*

Proof. The subset M of functions which transform monotone sequences to convergent sequences evidently forms a closed subalgebra which includes all functions of bounded variation. The subalgebra M is $C(Y)$ for some Y, and Y is a compactification of X by Lemma 1, and by Theorem 9 an orderable compactification. By Theorem 2, however, Y furnishes a maximal orderable compactification so that $Y \cong X_\infty$. But then the functions of bounded X_∞-variation are dense in $C(X_\infty)$, i.e. in M. This completes the proof.

We observe that we have proved that $\beta(X)$ is orderable if and only if every continuous function on X is bounded.

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Addendum. After this work was done, we discovered that Banaschewski [1] treated a restricted case of the problem considered, but by somewhat different methods.

REFERENCES

[1] B. Banaschewski, *Orderable spaces*, Fundamenta Mathematicae 50 (1961), p. 21-34.

[2] J. L. Kelley, *General topology*, Princeton 1955.

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