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ON AN OVERDETERMINED SYSTEM OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

BY

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The paper deals with the existence of solution of the Cauchy problem for an overdetermined system of the form

$$z_{x_\nu} = f^\nu(X, Y, z, z_Y) \quad (\nu = 1, \dots, p), \tag{1}$$

where $X = (x_1, \dots, x_p)$, $Y = (y_1, \dots, y_n)$, $z_Y = (z_{y_1}, \dots, z_{y_n})$. Our considerations will be based on the following lemma which can be easily derived from a theorem due to Pliś [1]:

LEMMA. *Let the functions $g(x, Y, z, Q, \Lambda)$ and $\omega(Y)$, where $Q = (q_1, \dots, q_n)$, $\Lambda = (\lambda_1, \dots, \lambda_p)$, be of class C^2 and bounded together with their first and second derivatives in the strip*

$$|x| < \tilde{a}(\Lambda), \qquad Y, z, Q \text{ arbitrary}, \qquad \sum_{\mu=1}^{p} |\lambda_\mu| \leqslant a < +\infty,$$

$\tilde{a}(\Lambda)$ being a continuous function of Λ. Suppose the inequalities

$$|g_{y_i}| \leqslant \tilde{A}(\Lambda), \qquad |g_z| \leqslant \tilde{B}(\Lambda), \qquad |g_{q_i}| \leqslant \tilde{C}(\Lambda),$$

$$|g_{y_i y_j}|, |g_{zz}|, |g_{q_i q_j}|, |g_{y_i z}|, |g_{q_i z}|, |g_{y_i q_j}| \leqslant \tilde{L}(\Lambda),$$

$$|\omega_{y_i}| \leqslant D, \qquad |\omega_{y_i y_j}| \leqslant J$$

hold true, where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{L}$ are continuous functions of Λ. Put

$$\tilde{M}(\Lambda) = 2D + \tilde{A}(\Lambda) \min\left(4\tilde{a}(\Lambda), \left(\tilde{B}(\Lambda) + \tilde{L}(\Lambda)\right)^{-1}\right),$$

$$\tilde{\beta}(\Lambda) = \min\left(\tilde{a}(\Lambda), \left\{4n(1+nJ)\left[\left(1+\tilde{M}(\Lambda)\right)^2 \tilde{L}(\Lambda) + \tilde{B}(\Lambda)(1+2J)\right]\right\}^{-1}\right).$$

Under these assumptions the Cauchy problem for the equation with parameters

$$z_x = g(x, Y, z, z_Y, \Lambda)$$

and with initial condition

$$z(0, Y) = \omega(Y)$$

admits a unique solution $u(x, Y, \Lambda)$ *depending on parameters* Λ *which is of class* C^2 *with respect to* (x, Y) *in the strip*

$$|x| < \tilde{\beta}(\Lambda), \qquad Y \text{ arbitrary},$$

and for Λ *satisfying the inequality* $\sum_{\mu=1}^{p} |\lambda_\mu| < a$, *and has continuous derivatives* u_{λ_ν}, $u_{y_i\lambda_\nu}$. *The derivatives* u_{y_i} *satisfy the inequality* $|u_{y_i}| \leqslant \tilde{M}(\Lambda)$. *Moreover, for every finite* β_0 *such that* $0 < \beta_0 \leqslant \min_{\Lambda} \tilde{\beta}(\Lambda)$, *the following property holds true:*

(2) $u, u_{y_i}, u_{y_iy_j}, u_{\lambda_\nu}, u_{y_i\lambda_\nu}$ *are bounded in the strip*

$$|x| \leqslant \beta_0, \ Y \text{ arbitrary}, \ \sum_{\mu=1}^{p} |\lambda_\mu| < a,$$

the bounds depending only on β_0 *and on the bounds of* g *and* ω *and of their first and second derivatives.*

Now we formulate the main theorem of our paper.

THEOREM. *Let* $f^\nu(X, Y, z, Q)$, $\nu = 1, \ldots, p$, *be of class* C^1, *bounded together with their first derivatives for*

$$\sum_{\mu=1}^{p} |x_\mu - \overset{0}{x}_\mu| < a < +\infty, \qquad Y, z, Q \text{ arbitrary}, \tag{3}$$

and satisfy the inequalities

$$|f^\nu_{y_i}| \leqslant A, \qquad |f^\nu_z| \leqslant B, \qquad |f^\nu_{q_i}| \leqslant C. \tag{4}$$

Suppose, further, that the first derivatives satisfy with respect to all variables a Lipschitz condition with the constant L *and that the compatibility conditions*

$$f^\nu_{x_\mu} + f^\nu_z f^\mu + \sum_{j=1}^{n} f^\nu_{q_j}(f^\mu_{y_j} + f^\mu_z q_j) \equiv f^\mu_{x_\nu} + f^\mu_z f^\nu + \sum_{j=1}^{p} f^\mu_{q_j}(f^\nu_{y_j} + f^\nu_z q_j) \tag{5}$$

hold true in (3).

Let $\omega(Y)$ *be of class* C^1, *bounded for arbitrary* Y, *and satisfy the inequalities*

$$|\omega_{y_i}| \leqslant D. \tag{6}$$

Assume finally that ω_{y_i} *satisfy a Lipschitz condition with a constant* J. *Put*

$$M = 2D + A\min\left(4a, \ (B+L)^{-1}\right),$$

$$\beta = \min\left(a, \{4n(1+nJ)[(1+M)^2L + B(1+2J)]\}^{-1}\right).$$

Under these assumptions there exists a unique solution $z(X, Y)$ *of system* (1) *satisfying the initial condition*

$$z(\overset{0}{X}, Y) = \omega(Y) \quad \big(\overset{0}{X} = (\overset{0}{x}_1, \ldots, \overset{0}{x}_p)\big), \tag{7}$$

which is of class C^1 *in the strip*

$$\sum_{\mu=1}^{p} |x_\mu - \overset{0}{x}_\mu| < \beta, \qquad Y \text{ arbitrary.} \tag{8}$$

Proof. Uniqueness follows from a general uniqueness theorem (see [2], § 42). Now, to prove the existence, observe that there exist sequences of functions $f^{\nu r}(X, Y, z, Q)$, $\omega^r(Y)$, $r = 1, 2, \ldots$, which are of class C^2 in (3) such that

(9) $f^{\nu r}$ and ω^r converge, together with their first derivatives, to f^ν and ω and to their respective derivatives, uniformly in (3).

Moreover, the functions $f^{\nu r}$ and ω^r can be chosen so as to be uniformly bounded, together with their first and second derivatives, and to satisfy (4), (6) and

$$|f^{\nu r}_{y_i y_j}|, \ |f^{\nu r}_{zz}|, \ |f^{\nu r}_{y_i z}|, \ |f^{\nu r}_{q_i q_j}|, \ |f^{\nu r}_{q_i z}| \leqslant L, \qquad |\omega^r_{y_i y_j}| \leqslant J. \tag{10}$$

We consider the approximate system

$$z_{x_\nu} = f^{\nu r}(X, Y, z, z_Y) \qquad (\nu = 1, \ldots, p)$$

and its transformed equation, obtained by Mayer's transformation

$$\begin{aligned} X &= \overset{0}{X} + \Lambda x \qquad \big(\Lambda = (\lambda_1, \ldots, \lambda_p)\big), \\ z_x &= g^r(x, Y, z, z_Y, \Lambda), \end{aligned} \tag{11}$$

where

$$g^r(x, Y, z, Q, \Lambda) = \sum_{\mu=1}^{p} \lambda_\mu f^{\mu r}(\overset{0}{X} + \Lambda x, Y, z, Q).$$

For equation (11) consider the initial condition

$$z(0, Y) = \omega^r(Y). \tag{12}$$

Now, we easily check, by (4), (6) and (10) satisfied by $f^{\nu r}$ and ω^r, that for every fixed r and for Λ satisfying the inequality

$$\lambda = \sum_{\mu=1}^{p} |\lambda_\mu| < a,$$

g^r and ω^r satisfy all the assumptions of the Lemma with

$$\tilde{a}(\Lambda) = a\lambda^{-1}, \quad \tilde{A}(\Lambda) = \lambda A, \quad \tilde{B}(\Lambda) = \lambda B, \quad \tilde{C}(\Lambda) = \lambda C, \quad \tilde{L}(\Lambda) = \lambda L$$

Moreover, g^r, ω^r and their first and second derivatives are uniformly bounded. Now, since in our case we have

$$\tilde{M}(\Lambda) = 2D + \tilde{A}(\Lambda)\min\left(4\tilde{a}(\Lambda), \left(\tilde{B}(\Lambda) + \tilde{L}(\Lambda)\right)^{-1}\right)$$
$$= 2D + \lambda A \min\left(4\lambda^{-1}a, (B+L)^{-1}\lambda^{-1}\right)$$
$$= 2D + A\min\left(4a, (B+L)^{-1}\right) = M,$$
$$\tilde{\beta}(\Lambda) = \min\left(\tilde{a}(\Lambda), \left\{4n(1+nJ)\left[\left(1+\tilde{M}(\Lambda)\right)^2\tilde{L}(\Lambda) + \tilde{B}(\Lambda)(1+2J)\right]\right\}^{-1}\right)$$
$$= \lambda^{-1}\min\left(a, \{4n(1+nJ)[(1+M)^2L + B(1+2J)]\}^{-1}\right) = \beta\lambda^{-1},$$

we conclude, by the Lemma, that for every $r = 1, 2, \ldots$ and Λ satisfying the inequality

$$\lambda = \sum_{\mu=1}^{p} |\lambda_\mu| < \beta \tag{13}$$

there exists a unique solution $u^r(x, Y, \Lambda)$ of (11) and (12) which is of class C^2 with respect to (x, Y) in the strip

$$|x| < \beta\lambda^{-1}, \qquad Y \text{ arbitrary}, \tag{14}$$

and has continuous derivatives $u^r_{\lambda_\nu}$, $u^r_{y_i\lambda_\nu}$ [1], while the derivative $u^r_{y_i}$ satisfies the inequality $|u^r_{y_i}| \leqslant M$. Notice that, by (13), we have

$$1 < \min_{\Lambda} \beta\lambda^{-1},$$

and hence, by (2) and by the properties of functions g^r and ω^r listed above, we obtain that the functions $u^r(1, Y, \Lambda)$ and $u^r_{y_i}(1, Y, \Lambda)$ are uniformly bounded and equicontinuous in the strip

$$\sum_{\mu=1}^{p} |\lambda_\mu| < \beta, \qquad Y \text{ arbitrary}. \tag{15}$$

Therefore it follows by Arzelà's theorem that there is a subsequence r_s so that

$$\lim_{s\to\infty} u^{r_s}(1, Y, \Lambda) = u(Y, \Lambda), \qquad \lim_{s\to\infty} u^{r_s}_{y_i}(1, Y, \Lambda) = u_{y_i}(Y, \Lambda) \tag{16}$$

uniformly in (15). It follows that the derivatives u_{y_i} of the limit function $u(Y, \Lambda)$ are continuous in (15). We will show that the derivatives u_{λ_ν} are continuous too and satisfy in (15)

$$u_{\lambda_\nu}(Y, \Lambda) = f^\nu\left(\overset{0}{X} + \Lambda, Y, u(Y, \Lambda), u_Y(Y, \Lambda)\right). \tag{17}$$

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(1) The approximate functions $f^{\nu r}$ and ω^r were introduced just in order to guarantee this regularity of u^r.

Indeed, for fixed ν, r and Λ satisfying (13), put

$$\varphi(x, Y) = u^r_{\lambda_\nu}(x, Y, \Lambda) - xf^{\nu r}(R)$$

for (x, Y) in (14), where

$$R = \big(\overset{0}{X} + \Lambda x, Y, u^r(x, Y, \Lambda), u^r_Y(x, Y, \Lambda)\big).$$

If we take advantage of the fact that u^r satisfies (11) and $u^r_{y_i y_j}$, $u^r_{\lambda_\nu}$, $u^r_{y_i \lambda_\nu}$ are continuous, then a simple computation shows that

$$\varphi_x = \sum_{\mu=1}^{p} \lambda_\mu f^{\mu r}_z(R)\varphi + \sum_{\mu=1}^{p} \lambda_\mu \sum_{k=1}^{n} f^{\mu r}_{q_k}(R)\varphi_{y_k} + x \sum_{\mu=1}^{p} \lambda_\mu \eta^r_{\mu\nu}(x, Y), \tag{18}$$

where

$$\eta^r_{\mu\nu}(x, Y) = f^{\mu r}_{x_\nu}(R) + f^{\mu r}_z(R) f^{\nu r}(R) + \sum_{k=1}^{n} f^{\mu r}_{q_k}(R)[f^{\nu r}_{y_k}(R) + f^{\nu r}_z u^r_{y_k}] -$$
$$- \Big\{ f^{\nu r}_{x_\mu}(R) + f^{\nu r}_z(R) f^{\mu r}(R) + \sum_{k=1}^{n} f^{\nu r}_{q_k}(R)[f^{\mu r}_{y_k}(R) + f^{\mu r}_r u^r_{y_k}] \Big\}.$$

By the compatibility conditions (5), by (9) and by the inequality $|u^r_{y_i}| \leqslant M$, we get

$$|\eta^r_{\mu\nu}(x, Y)| \leqslant \varepsilon_r, \tag{19}$$

where ε_r is a constant so that

$$\lim_{r\to\infty} \varepsilon_r = 0. \tag{20}$$

Finally, by inequalities (4) satisfied by the functions $f^{\nu r}$, by (13) and (19), we get from (18)

$$|\varphi_x| \leqslant \beta B |\varphi| + \beta C \sum_{k=1}^{n} |\varphi_{y_k}| + \beta \varepsilon_r$$

in (14). On the other hand, since $u^r(0, Y, \Lambda) = \omega^r(Y)$, we have

$$\varphi(0, Y) = 0$$

by the definition of φ. From the last two relations it follows (see [2], § 37) that in (14) we have

$$|\varphi| \leqslant \begin{cases} \varepsilon_r B^{-1}(e^{\beta B|x|} - 1) & \text{if} \quad B \neq 0, \\ \beta \varepsilon_r |x| & \text{if} \quad B = 0. \end{cases}$$

In particular, for $x = 1$ we have

$$\big|u^r_{\lambda_\nu}(1, Y, \Lambda) - f^{\nu r}\big(\overset{0}{X} + \Lambda, Y, u^r(1, Y, \Lambda), u^r_Y(1, Y, \Lambda)\big)\big|$$
$$\leqslant \begin{cases} \varepsilon_r B^{-1}(e^{\beta B} - 1) & \text{if} \quad B \neq 0, \\ \beta \varepsilon_r & \text{if} \quad B = 0. \end{cases}$$

By (9), (16) and (20) it follows from the last inequality that the sequence $u^r_{\lambda_\nu}(1, Y, \Lambda)$ is uniformly convergent and hence we get in the limit relation (17) and the continuity of $u_{\lambda_\nu}(Y, \Lambda)$. On the other hand, since in (11)

$$g^r(x, Y, Q, 0) = 0,$$

and since $u^r(x, Y, \Lambda)$ satisfies (11) and (12), it follows that

$$u^r(x, Y, 0) = \omega^r(Y)$$

whence, by (9) and (16), we obtain in the limit

$$u(Y, 0) = \omega(Y). \tag{21}$$

Now, put

$$z(X, Y) = u(Y, X - \overset{0}{X}).$$

Since $u(Y, \Lambda)$ was of class C^1 in (15), the function $z(X, Y)$ is of class C^1 in (8) and, by (17) and (21), it satisfies (1) and (7), which completes the proof.

REFERENCES

[1] A. Pliś, *Generalization of the Cauchy problem for a system of partial differential equations*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 4 (1956), p. 741-744.

[2] J. Szarski, *Differential inequalities*, Warszawa 1965.

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