Table des matières du tome LX, fascicule 2

Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles, Topologie, Fondements de Mathématiques, Fonctions Réelles, Algèbre Abstraite.* Chaque volume paraît en 3 fascicules

Adresse de la Rédaction et de l'Échange:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, Warszawa (Pologne).

Le prix de ce fascicule est 3.00 $.

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA, Krakowskie Przedmieście 7, Warszawa (Pologne).

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Spectral structures and uniform continuity*

by

S. Leader (New Brunswick, N. J.)

0. Introduction. The class $\mathfrak{S}^*$ of all bounded, real-valued, uniform functions on a uniform space $(X, \mathfrak{U})$ need not characterize the uniform structure, but corresponds to an $\mathfrak{S}^*$-equivalence class of uniform structures on X. A characteristic invariant of the $\mathfrak{S}^*$-equivalence class is the induced proximity relation ([4], [1], [6]). Now even the class $\mathfrak{S}$ of all real-valued, uniform functions on $(X, \mathfrak{U})$ need not characterize $\mathfrak{U}$, but corresponds to an $\mathfrak{S}$-equivalence class of uniform structures [5].

The present work has its origin in the search for an intrinsic characteristic invariant of the $\mathfrak{S}$-equivalence class. Such an invariant, the "uniform spectral structure" introduced in Section 6 below, yields necessary and sufficient conditions for an $\mathfrak{S}$-equivalence class of uniform structures to have a pseudometrizable member (Theorem 13) and/or a unique member (Theorems 14 and 15). The uniform spectral spaces under spectral mappings correspond to the functionally-determined uniform spaces [5] under uniform mappings. If $\mathfrak{S} = \mathfrak{S}^*$ then the uniform spectral structure reduces trivially to the proximity structure.

The uniform spectral structures suggest the more general "spectral structures" introduced in section 2. Spectral structures form a natural setting for the Urysohn construction (Theorem 2). They yield a Stone-Weierstrass theorem (Theorem 3), a boundedness criterion (Theorem 5) generalizing Atsuji ([2], [3]), Njåstad [10], and Hejcman [7], and a concept of uniform connectedness (Theorem 6) generalizing Pervin and Mrówka [9].

1. Basic definitions and lemmas. A *spectrum* $\boldsymbol{a}$ in a set X is a sequence $\{A_n\}$ of subsets of X indexed by the integers $-\infty < n < \infty$ such that $A_n \subseteq A_{n+1}$ for all n, $\bigcap_n A_n = \emptyset$, and $\bigcup_n A_n = X$. Two points are *separated* by $\boldsymbol{a}$ if there exists n such that one point is in A_n and the other in $X - A_{n+1}$. Two points are separated by a family $\boldsymbol{P}$ of spectra if they are separated by some member of $\boldsymbol{P}$. A family $\boldsymbol{P}$ of spectra *refines*

* Research supported by the National Science Foundation (NSF GP-4413).

a family Q if every pair of points separated by Q is also separated by P. A spectrum b *splits* a spectrum a if $B_{2n} = A_n$ for all n.

A spectrum b in X is an α-*spectrum* for a pseudometric α on X if there exists $\varepsilon > 0$ such that

$$\alpha(B_n, X - B_{n+1}) \geqslant \varepsilon \quad \text{for all } n. \tag{1.1}$$

For $\mathcal{U}$ an entourage in $X \times X$ (that is, a reflexive, binary relation on X) a spectrum b in X is called a $\mathcal{U}$-*spectrum* if

$$\mathcal{U}[B_n] \subseteq B_{n+1} \quad \text{for all } n. \tag{1.2}$$

Note that (1.2) is consistent with (1.1) if $\mathcal{U} = \alpha^{-1}[0, \varepsilon)$.

For a spectrum b let $\mathcal{B}$ be the symmetric entourage consisting of all (x, y) with x and y not separated by b. Explicitly,

$$\mathcal{B} = \bigcup_n (B_{n+1} - B_{n-1})^2 \tag{1.3}$$

where the square is the cartesian product. It is easily seen that for a symmetric entourage $\mathcal{U}$,

$$b \text{ is a } \mathcal{U}\text{-spectrum if and only if } \mathcal{U} \subseteq \mathcal{B}. \tag{1.4}$$

LEMMA A. *If a splits b, then in terms of* (1.3), $\mathcal{A}^2 \subseteq \mathcal{B}$.

Proof. Given (x, z) in $\mathcal{A}^2$, there exists y with both (x, y) and (y, z) in $\mathcal{A}$. Choose n such that $A_n - A_{n-1}$ contains y. Then both x and z are in $A_{n+1} - A_{n-2}$ which since a splits b is $A_{2k+1} - B_{k-1}$ for $n = 2k$ and $B_{k+1} - A_{2k-1}$ for $n = 2k+1$. In either case $A_{n+1} - A_{n-2} \subseteq B_{k+1} - B_{k-1}$ for $k = [n/2]$. So (x, z) is in $\mathcal{B}$.

LEMMA B. *For pseudometrics α and β on X the following are equivalent:*

(i) *Every β-spectrum is an α-spectrum.*

(ii) *$\beta(A, B) = 0$ for all subsets A, B of X with $\alpha(A, B) = 0$.*

(iii) *β is uniformly continuous with respect to α.*

Proof. The equivalence of (ii) and (iii) is well known [8]. That (iii) implies (i) is trivial. To prove that (i) implies (ii) let $\beta(A, B) > 0$. Then any spectrum of the form $\{\emptyset, A, X - B, X\}$ is a β-spectrum, hence an α-spectrum by (i). So $\alpha(A, B) > 0$.

2. Spectral structures. A *spectral structure* M in a set X is a non-void family of spectra in X satisfying:

AXIOM I. *Every member of M is split by some member of M.*

AXIOM II. *Every spectrum which is refined by some member of M belongs to M.*

We call (X, M) a *spectral space*. The space is *separated* if M separates every pair of distinct points in X. A mapping f: $(X, M) \to (Y, N)$ between spectral spaces is a *spectral mapping* if $\{f^{-1}B_n\}$ belongs to M whenever $\{B_n\}$ belongs to N. An M-*entourage* is a symmetric entourage $\mathcal{U}$ such that every $\mathcal{U}$-spectrum (1.2) belongs to M.

In a pseudometric space (X, α) we let M consist of all α-spectra (1.1). Axioms I and II are readily verified.

3. Spectral functions. In the real line R we take the spectral structure induced by the metric $|x-y|$. Hereafter (X, M) will be a spectral space and $\mathfrak{S}$ the class of all its spectral functions, the spectral mappings on (X, M) into R.

THEOREM 1. *A real-valued function f on X belongs to $\mathfrak{S}$ if and only if given $\varepsilon > 0$ there exists a in M such that a separates all x, y for which $|f(x) - f(y)| \geqslant \varepsilon$.*

Proof. Given $\varepsilon > 0$ the spectrum b with $B_n = (-\infty, n\varepsilon/2)$ belongs to the spectral structure in R. So for f in $\mathfrak{S}$ the spectrum a defined by $A_n = f^{-1}B_n$ belongs to M. If x, y are not separated by a, then both $f(x)$ and $f(y)$ belong to $[(n-1)\varepsilon/2, (n+1)\varepsilon/2)$ for some n. Hence $|f(x) - f(y)| < \varepsilon$.

Conversely, given any spectrum b in R satisfying (1.1) for the Euclidean metric α we contend $f^{-1}b$ belongs to M. In view of Axiom II we need only show that $f^{-1}b$ is refined by some a in M. Such a spectrum a is offered by the hypothesis since for x, y separated by $f^{-1}b$, (1.1) implies $|f(x) - f(y)| \geqslant \varepsilon$ which implies that x and y are separated by a.

THEOREM 2. *A spectrum a belongs to M if and only if there exists f in $\mathfrak{S}$ such that*

$$fA_n \subseteq (-\infty, n] \quad \text{and} \quad f(X - A_n) \subseteq [n, \infty) \quad \text{for all } n. \tag{3.1}$$

Proof. Given f in $\mathfrak{S}$ and (3.1), the spectrum b defined by $B_n = f^{-1}(-\infty, n/2)$ belongs to M and refines a. So a belongs to M by Axiom II. The converse will be proved by a Urysohn construction.

Using Axiom I with $a_0 = a$ we choose by induction a sequence $\{a_k\}$ in M such that a_k splits a_{k-1}. Let $A(k, n)$ be the nth term of a_k. Then $A(k+1, 2n) = A(k, n)$. So $A_n = A(k, 2^k n)$ for $k = 0, 1, 2, \ldots$ Define $f(x) = \inf n 2^{-k}$ over all k, n such that $A(k, n)$ contains x. Clearly, $f(x)$ is finite for every x in X. For x in $A_n = A(0, n)$, $f(x) \leqslant n$. For x in $X - A_n$ we have x outside $A(k, 2^k n)$ for all k. So if $A(k, m)$ contains x, $m > 2^k n$ and hence $f(x) \geqslant n$. We thus have (3.1).

That f is in $\mathfrak{S}$ follows from Theorem 1 since given $\varepsilon > 0$, we can choose k with $2^{-(k-1)} < \varepsilon$. Then $|f(x) - f(y)| \geqslant \varepsilon$ implies that x and y are separated by a_k.

THEOREM 3. *A subclass $\mathfrak{T}$ of $\mathfrak{S}$ is uniformly dense in $\mathfrak{S}$ if and only if given a in M and $\varepsilon > 0$ there exists g in $\mathfrak{T}$ such that*

$$gA_n \subseteq (-\infty, (n+1)\varepsilon) \quad \text{and} \quad g(X - A_n) \subseteq ((n-1)\varepsilon, \infty). \tag{3.2}$$

Proof. Given f is $\mathfrak{S}$ and $\varepsilon > 0$, define $\boldsymbol{a}$ in $\boldsymbol{M}$ by setting $A_n = f^{-1}(-\infty, n\varepsilon)$. Choose g in $\mathfrak{T}$ subject to (3.2). Then for x in $A_n - A_{n-1}$, $(n-1)\varepsilon \leqslant f(x) < n\varepsilon$ and $(n-2)\varepsilon < g(x) < (n+1)\varepsilon$. So $|f(x) - g(x)| < 2\varepsilon$ for all x in X.

Conversely, given $\mathfrak{T}$ dense in $\mathfrak{S}$, $\boldsymbol{a}$ in $\boldsymbol{M}$, and $\varepsilon > 0$, choose f in $\mathfrak{S}$ subject to (3.1). Then εf is in $\mathfrak{S}$ by Theorem 1. So there exists g in $\mathfrak{T}$ such that

(3.3) $$|g(x) - \varepsilon f(x)| < \varepsilon \quad \text{for all } x \text{ in } X.$$

Then (3.2) follows from (3.1) and (3.3).

THEOREM 4. *Given a non-void class $\mathfrak{R}$ of real functions on a set X, let $\boldsymbol{M}$ consist of all spectra $\boldsymbol{a}$ in X for which*

(3.4) *there exist g in $\mathfrak{R}$ and $\delta > 0$ such that for all x, y separated by $\boldsymbol{a}$, $|g(x) - g(y)| > \delta$.*

Then $\boldsymbol{M}$ is a spectral structure and $\mathfrak{S}$ is the class of all real functions that are uniform with respect to $\mathfrak{R}$.

Proof. Given $\boldsymbol{a}$ subject to (3.4), define $\boldsymbol{b}$ by letting $B_{2n} = A_n$ and $B_{2n+1} = \{x\colon \alpha(gx, gA_n) < \delta/2\}$ where α is the Euclidean metric. Then $\boldsymbol{b}$ splits $\boldsymbol{a}$. Moreover, $\boldsymbol{b}$ is in $\boldsymbol{M}$ since (3.4) holds for $\boldsymbol{b}$ with $\delta/2$. Hence Axiom I.

If $\boldsymbol{b}$ is any spectrum refined by a spectrum $\boldsymbol{a}$ and (3.4) holds for $\boldsymbol{a}$, then (3.4) holds for $\boldsymbol{b}$. Hence Axiom II. That every f in $\mathfrak{S}$ is uniform with respect to $\mathfrak{R}$ follows from (3.4) and theorem 1.

Conversely, let f be uniform with respect to $\mathfrak{R}$. Given $\varepsilon > 0$ choose g in $\mathfrak{R}$ and $\delta > 0$ such that

(3.5) $$|g(x) - g(y)| > \delta \quad \text{for all } x, y \text{ with} \quad |f(x) - f(y)| \geqslant \varepsilon.$$

Construct $\boldsymbol{a}$ by defining $A_n = g^{-1}(-\infty, n\delta)$. Then

(3.6) $$\boldsymbol{a} \text{ separates } x, y \text{ if and only if} \quad |g(x) - g(y)| > \delta.$$

By (3.6) and the definition (3.4) of $\boldsymbol{M}$, $\boldsymbol{a}$ belongs to $\boldsymbol{M}$. So f belongs to $\mathfrak{S}$ by (3.5), (3.6) and Theorem 1.

COROLLARY 4(a). *A class $\mathfrak{S}$ of real functions on X is the class of all spectral functions for some spectral structure in X if and only if $\mathfrak{S}$ is non-void and contains every function uniform with respect to $\mathfrak{S}$.*

4. Bounded subsets of a spectral space.

We call a subset E of a spectral space $(X, \boldsymbol{M})$ *bounded* if for every spectrum $\boldsymbol{b}$ in $\boldsymbol{M}$ there exists n such that $B_n \supseteq E$. Boundedness is preserved under spectral mappings.

THEOREM 5. *Let $(X, \boldsymbol{M})$ be a spectral space and $\mathfrak{S}$ its class of spectral functions. Then for any subset E of X the following are equivalent:*

(i) *E is bounded.*

(ii) *Every f in $\mathfrak{S}$ is bounded on E.*

(iii) *Given any $\boldsymbol{M}$-entourage $\mathfrak{U}$ there exists a finite subset K of E and a positive integer n such that*

(4.1) $$\mathfrak{U}^n[K] \supseteq E.$$

(iv) *Given any $\boldsymbol{b}$ in $\boldsymbol{M}$ the conclusion of* (iii) *holds for the entourage* (1.3) *induced by $\boldsymbol{b}$.*

Proof. The equivalence of (i) and (ii) follows from Theorem 2. To prove that (i) implies (iii) let $\mathfrak{U}$ be any $\boldsymbol{M}$-entourage. Then $\mathfrak{U}^{n+1} \supseteq \mathfrak{U}^n$ since $\mathfrak{U}$ is reflexive. Let $\mathfrak{U}^\infty = \bigcup_n \mathfrak{U}^n$, an equivalence relation on X since $\mathfrak{U}$ is symmetric. We call the equivalence class $\mathfrak{U}^\infty[x]$ the $\mathfrak{U}$-*component* of X containing x. Clearly, a subset Q of X is a union of $\mathfrak{U}$-components if and only if

(4.2) $$\mathfrak{U}[Q] = Q.$$

We contend first that (i) implies E meets only finitely many $\mathfrak{U}$-components. For, given any sequence $\{P_k\}$ of distinct $\mathfrak{U}$-components, we can construct a spectrum $\boldsymbol{a}$ by defining

(4.3) $$A_n = \begin{cases} X - \bigcup_{k \geqslant n} P_k & \text{for} \quad n > 0, \\ \emptyset \quad \text{for} & n \leqslant 0. \end{cases}$$

Then $\mathfrak{U}[A_n] = A_n$ by (4.2) and (4.3). So $\boldsymbol{a}$ is a $\mathfrak{U}$-spectrum and thereby belongs to $\boldsymbol{M}$. By (i) some A_n contains E. So $E \cap P_k = \emptyset$ for all $k \geqslant n$.

Construct a finite set K by choosing exactly one point from $P \cap E$ for each $\mathfrak{U}$-component P which meets E. Let Q be the union of all $\mathfrak{U}$-components disjoint from E. Construct $\boldsymbol{b}$ by defining

(4.4) $$B_n = \begin{cases} \emptyset \quad \text{for} & n \leqslant 0, \\ Q \cup \mathfrak{U}^n[K] & \text{for} \quad n > 0. \end{cases}$$

Then for $n > 0$, $\mathfrak{U}[B_n] = \mathfrak{U}[Q] \cup \mathfrak{U}^{n+1}[K] = B_{n+1}$ by (4.2). Moreover $\bigcup_n B_n = Q \cup \mathfrak{U}^\infty[K] = X$. So $\boldsymbol{b}$ is a $\mathfrak{U}$-spectrum. Hence (i) implies that some B_n contains E, which implies (4.1) since E and Q are disjoint.

(iii) implies (iv) *a fortiori*.

To prove that (iv) implies (i) let $\boldsymbol{b}$ belong to $\boldsymbol{M}$. Choose K and n by (iv) so that (4.1) holds with $\mathfrak{U} = \mathfrak{B}$. Since K is finite, $K \subseteq B_m$ for some m. Thus $E \subseteq \mathfrak{U}^n[K] \subseteq \mathfrak{U}^n[B_m] \subseteq B_{m+n}$ by (4.1), (1.4), (1.2). Hence (i).

THEOREM 6. *For a subset E of a spectral space $(X, \boldsymbol{M})$ the following are equivalent:*

(i) *Given* $\boldsymbol{b}$ *in* $\boldsymbol{M}$ *and* n *such that* E *meets both* B_{n-1} *and* $X-B_n$, *then* E *meets* B_n-B_{n-1}.

(ii) $\overline{fE}$ *is connected for every* f *in* $\mathfrak{S}$.

(iii) *Every* f *in* $\mathfrak{S}$ *with a finite range is constant on* E.

(iv) *Given any* $\boldsymbol{M}$*-entourage* $\mathcal{U}$, *some* $\mathcal{U}$*-component contains* E. *That is,* $\mathcal{U}^\infty \supseteq E\times E$.

We call such a set E $\boldsymbol{M}$*-connected*.

Proof. To prove (ii) given (i) we need only show that if fE meets both $(-\infty, (n-1)\varepsilon)$ and $[n\varepsilon, \infty)$ for some $\varepsilon>0$ and some integer n, then fE also meets $[(n-1)\varepsilon, n\varepsilon)$. This follows directly from (i) if we define $B_n = f^{-1}(-\infty, n\varepsilon)$.

That (ii) implies (iii) is trivial since finite sets are closed, and a finite connected set contains at most one point.

That (iii) implies (iv) is trivial for E empty. For E non-empty consider any $\mathcal{U}$-component P which meets E. Let f be the characteristic function of P. By (iii), $fE=1$. So $E\subseteq P$.

To prove (iv) implies (i) apply (iv) to the entourage (1.3).

THEOREM 7. *A subset* E *of a spectral space* $(X, \boldsymbol{M})$ *is bounded and* $\boldsymbol{M}$*-connected if and only if for every* $\boldsymbol{M}$*-entourage* $\mathcal{U}$ *there exists* n *such that* $\mathcal{U}^n \supseteq E\times E$.

Proof. Apply Theorems 5 and 6.

5. Spectral lattices. We call a spectral structure $\boldsymbol{M}$ a *lattice* if it satisfies

(5.1) $\boldsymbol{a}\cap\boldsymbol{b}$ belongs to $\boldsymbol{M}$ for all $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{M}$,

where $\boldsymbol{a}\cap\boldsymbol{b} = \{A_n\cap B_n\}$. By Axiom II we could equivalently use union in place of intersection in (5.1) since the spectrum $\{E_n\}$ refines $\{X-E_{-n}\}$. Thus (5.1) says that $\boldsymbol{M}$ is a lattice with respect to the partial ordering $\boldsymbol{b}\leqslant\boldsymbol{c}$ defined by $B_n\subseteq C_n$ for all n.

THEOREM 8. *For a spectral space* $(X, \boldsymbol{M})$, $\boldsymbol{M}$ *is a lattice if and only if* $\mathfrak{S}$ *is a function lattice.*

Proof. Throughout this proof let $h=f\vee g$. Given f and g in $\mathfrak{S}$ and (5.1) we need only show in view of Corollary 4(a) that h belongs to $\mathfrak{S}$. By Theorem 1 it suffices to find for a given $\varepsilon>0$ some $\boldsymbol{c}$ in $\boldsymbol{M}$ such that

(5.2) $|h(x)-h(y)|<2\varepsilon$ for all (x, y) in $\mathfrak{C}$.

Define $\boldsymbol{a}$ by $A_n = f^{-1}(-\infty, n\varepsilon)$ and $\boldsymbol{b}$ by $B_n = g^{-1}(-\infty, n\varepsilon)$. Let $\boldsymbol{c}=\boldsymbol{a}\cap\boldsymbol{b}$. Then $C_n = h^{-1}(-\infty, n\varepsilon)$. Given (x, y) in $\mathfrak{C}$ there is some n such that $[(n-1)\varepsilon, (n+1)\varepsilon)$ contains both $h(x)$ and $h(y)$. Hence (5.2).

Conversely, let $\mathfrak{S}$ be a function lattice and let $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $\boldsymbol{M}$. Use Theorem 2 to choose f in $\mathfrak{S}$ satisfying (3.1) for $\boldsymbol{a}$. Similarly choose g for $\boldsymbol{b}$. Then h belongs to $\mathfrak{S}$ and satisfies (3.1) for $\boldsymbol{a}\cap\boldsymbol{b}$. So $\boldsymbol{a}\cap\boldsymbol{b}$ belongs to $\boldsymbol{M}$ by Theorem 2.

The lattice property is of interest because every spectral structure with this property induces a proximity relation and *a fortiori* a completely regular topology. Namely, call C *close* to D if no member of $\boldsymbol{M}$ separates all (c, d) in $C\times D$. Dually, $P\ll Q$ (P is *remote* from $X-Q$) if any spectrum of the form $\{\emptyset, P, Q, X\}$ belongs to $\boldsymbol{M}$. In terms of $\mathfrak{S}$, C and D are remote if there exists f in $\mathfrak{S}$ mapping C into 0 and D into 1.

6. Uniform spectral structures. A spectral structure $\boldsymbol{M}$ is called *uniform* if the following strengthening of Axiom II holds:

AXIOM II′. *Every spectrum in* X *which is refined by some finite subfamily of* $\boldsymbol{M}$ *belongs to* $\boldsymbol{M}$.

THEOREM 9. *Given a uniform structure* $\mathfrak{U}$ *on* X, *the family* $\boldsymbol{M}$ *of all* $\mathcal{U}$*-spectra* (1.2) *with* $\mathcal{U}$ *in* $\mathfrak{U}$ *is a uniform spectral structure. For every uniform spectral structure* $\boldsymbol{M}$ *in* X *there exists a minimum uniform structure* $[\boldsymbol{M}]$ *inducing* $\boldsymbol{M}$. *So uniform spectral structures correspond to equivalence classes of uniform structures, two uniform structures being equivalent if they induce the same spectral structure.*

Proof. Given $\mathcal{U}$ in $\mathfrak{U}$ and a $\mathcal{U}$-spectrum $\boldsymbol{a}$, choose $\mathcal{V}$ in $\mathfrak{U}$ with $\mathcal{V}^2\subseteq\mathcal{U}$. Define $B_{2n}=A_n$ and $B_{2n+1}=\mathcal{V}[A_n]$ to get a $\mathcal{V}$-spectrum $\boldsymbol{b}$ which splits $\boldsymbol{a}$. So Axiom I holds.

Let a spectrum $\boldsymbol{a}$ be refined by a finite family $\boldsymbol{F}$ of spectra induced by $\mathfrak{U}$. Then each member of $\boldsymbol{F}$ is a $\mathcal{U}_k$-spectrum for some $k=1, 2, \ldots, n$. Let $\mathcal{U}=\mathcal{U}_1\cap\ldots\cap\mathcal{U}_n$ in $\mathfrak{U}$. Then $\boldsymbol{a}$ is a $\mathcal{U}$-spectrum. So Axiom II′ holds.

Conversely, given a uniform spectral structure $\boldsymbol{M}$, Lemma A and Axiom I imply that the entourages $\mathcal{B}$ defined by (1.3) with $\boldsymbol{b}$ in $\boldsymbol{M}$ form a subbase for a uniform structure $[\boldsymbol{M}]$. Explicitly, $\mathcal{U}$ belongs to $[\boldsymbol{M}]$ if and only if

(6.1) $\mathcal{B}_1\cap\ldots\cap\mathcal{B}_m\subseteq\mathcal{U}$ for some $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$ in $\boldsymbol{M}$.

Equivalently, there exists a finite subfamily of $\boldsymbol{M}$ which refines every $\mathcal{U}$-spectrum. Hence Axiom II′ and (6.1) imply that every $\mathcal{U}$-spectrum belongs to $\boldsymbol{M}$. Clearly $\mathcal{B}$ belongs to $[\boldsymbol{M}]$ by (6.1) for every $\boldsymbol{b}$ in $\boldsymbol{M}$. Since $\boldsymbol{b}$ is a $\mathcal{B}$-spectrum by (1.4), $\boldsymbol{M}$ is just the spectral structure induced by $[\boldsymbol{M}]$.

Finally, if a uniform structure $\mathfrak{U}$ induces $\boldsymbol{M}$, then $\mathcal{B}$ belongs to $\mathfrak{U}$ by (1.4) for all $\boldsymbol{b}$ in $\boldsymbol{M}$. So (6.1) implies $[\boldsymbol{M}]\subseteq\mathfrak{U}$.

THEOREM 10. *Let* $(X, \mathfrak{U})$ *and* $(Y, \mathfrak{V})$ *be uniform spaces with induced spectral structures* $\boldsymbol{M}$ *and* $\boldsymbol{N}$, *respectively. Then every uniform mapping* $f\colon X\to Y$ *is spectral. The converse holds if* $\mathfrak{V}=[\boldsymbol{N}]$.

Proof. Let $\boldsymbol{b}$ belong to $\boldsymbol{N}$, $\boldsymbol{a}$ be the spectrum $f^{-1}\boldsymbol{b}$ defined by $A_n = f^{-1}B_n$, and $\mathcal{A}$ and $\mathcal{B}$ be the induced entourages (1.3). Then

(6.2) (x, y) *belongs to* $\mathcal{A}$ *if and only if* (fx, fy) *belongs to* $\mathcal{B}$.

Clearly, $\mathcal{B}$ belongs to $\mathfrak{V}$ since $\boldsymbol{b}$ belongs to $\boldsymbol{N}$ induced by $\mathfrak{V}$. Hence, if f is uniform, (6.2) implies that $\mathcal{A}$ belongs to $\mathfrak{U}$ and therefore $\boldsymbol{a}$ belongs to $\boldsymbol{M}$. So f is spectral.

Conversely, if f is spectral then $\boldsymbol{a}$ belongs to $\boldsymbol{M}$, hence $\mathcal{A}$ belongs to $\mathfrak{U}$. So (6.2) implies f is uniform for $\mathfrak{V} = [\boldsymbol{N}]$ since the entourages induced by $\boldsymbol{N}$ form a subbase for $[\boldsymbol{N}]$.

7. Simple and pseudometrizable uniform spectral spaces.

THEOREM 11. *Every pseudometrizable uniform structure* $\mathfrak{U}_a$ *on* X *is the maximum structure in its spectral equivalence class.*

Proof. Let $\mathfrak{U}$ be any uniform structure which induces the spectral structure $\boldsymbol{M}$ induced by $\mathfrak{U}_a$. Since $\boldsymbol{M}$ consists of all a-spectra (1.1), every $\mathfrak{U}$-uniform pseudometric β must satisfy (i) of Lemma B, hence (iii). So $\mathfrak{U} \subseteq \mathfrak{U}_a$.

We call a uniform spectral space $(X, \boldsymbol{M})$ *simple* if there is only one uniform structure inducing $\boldsymbol{M}$.

THEOREM 12. *The real line* R *is simple.*

Proof. By Theorem 11 we need only show that the metric uniform structure $\mathfrak{U}_a$ on R is the minimum. We must show that a is $\mathfrak{U}$-uniform for every uniform structure $\mathfrak{U}$ that is spectrally equivalent to $\mathfrak{U}_a$.

Given $\varepsilon > 0$, consider the spectrum $\boldsymbol{b}$ with $B_n = (-\infty, n\varepsilon)$. Since $\boldsymbol{b}$ is an a-spectrum, it is a $\mathcal{U}$-spectrum for some symmetric $\mathcal{U}$ in $\mathfrak{U}$. Therefore by (1.3) and (1.4), $|x-y| < 2\varepsilon$ for all (x, y) in $\mathcal{U}$.

We get the following two corollaries from Theorems 2, 9, 10, and 12.

COROLLARY (b). *Let* $\boldsymbol{M}$ *be the spectral structure induced in* X *by a uniform structure* $\mathfrak{U}$. *Then*

(7.1) $\mathfrak{S}$ *is the class of all real* $\mathfrak{U}$*-uniform functions on* X.

$[\boldsymbol{M}]$ *is the smallest uniform structure* $\mathfrak{U}$ *satisfying* (7.1).

COROLLARY (c). *A mapping* $g\colon (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *between uniform spaces is spectral relative to the induced spectral structures if and only if the composition* $f \circ g$ *is* $\mathfrak{U}$*-uniform for every real* $\mathfrak{V}$*-uniform function* f *on* Y.

We call a subfamily $\boldsymbol{N}$ of a spectral structure $\boldsymbol{M}$ *uniform* if every spectrum refined by $\boldsymbol{N}$ belongs to $\boldsymbol{M}$. A sequence $\{\boldsymbol{M}_i\}$ of subfamilies of $\boldsymbol{M}$ is called a *splitting* sequence if each spectrum in $\boldsymbol{M}_i$ is split by some spectrum in $\boldsymbol{M}_{i+1}$. A spectral space $(X, \boldsymbol{M})$ is *pseudometrizable* if $\boldsymbol{M}$ is the family of all a-spectra (1.1) for some pseudometric a.

THEOREM 13. $(X, \boldsymbol{M})$ *is pseudometrizable if and only if* $\boldsymbol{M}$ *is the union of a splitting sequence of uniform subfamilies* $\boldsymbol{M}_i$.

Proof. Given a such that $\boldsymbol{M}$ is the family of all a-spectra, let $\boldsymbol{M}_i$ be the family of all spectra $\boldsymbol{b}$ such that

(7.2) $$a(B_n, X - B_{n+1}) \geqslant 2^{-i} \quad \text{for all } n.$$

It is easy to verify that $\boldsymbol{M}_i$ has the required properties.

Conversely, let $\boldsymbol{M}$ be the union of a splitting uniform sequence $\{\boldsymbol{M}_i\}$. Let $\mathcal{U}_i$ consist of all (x, y) not separated by $\boldsymbol{M}_i$. That is,

(7.3) $$\mathcal{U}_i = \bigcap \{\mathcal{A}\colon \boldsymbol{a} \in \boldsymbol{M}_i\}.$$

Now every $\mathcal{U}_i$-spectrum is refined by the uniform family $\boldsymbol{M}_i$ and thereby belongs to $\boldsymbol{M}$. Conversely, every spectrum in $\boldsymbol{M}$ belongs to some $\boldsymbol{M}_i$ and is therefore a $\mathcal{U}_i$-spectrum.

Given $\boldsymbol{b}$ in $\boldsymbol{M}_i$ there exists $\boldsymbol{a}$ in $\boldsymbol{M}_{i+1}$ splitting $\boldsymbol{b}$. By Lemma A, $\mathcal{A}^2 \subseteq \mathcal{B}$ which together with (7.3) gives

(7.4) $$\mathcal{U}_{i+1}^2 \subseteq \mathcal{U}_i \quad \text{for all } i.$$

Hence $\{\mathcal{U}_i\}$ is a base for a pseudometrizable uniform structure which induces $\boldsymbol{M}$.

Given a uniform spectral space $(X, \boldsymbol{M})$ we call a subfamily $\boldsymbol{N}$ of $\boldsymbol{M}$ *admissible* if $\boldsymbol{N}$ belongs to a splitting sequence $\{\boldsymbol{M}_i\}$ such that

(7.5) *For every finite subfamily* $\boldsymbol{P}$ *of* $\boldsymbol{M}$ *and every* i, $\boldsymbol{M}_i \cup \boldsymbol{P}$ *is uniform.*

LEMMA C. $\boldsymbol{N}$ *is admissible if and only if the entourage* $\mathcal{U}$ *consisting of all* (x, y) *not separated by* $\boldsymbol{N}$ *belongs to some uniform structure* $\mathfrak{U}$ *which induces* $\boldsymbol{M}$. *In particular, if* $\mathcal{W}$ *belongs to* $\mathfrak{U}$ *and* $\mathfrak{U}$ *induces* $\boldsymbol{M}$, *then the family of all* $\mathcal{W}$*-spectra is admissible.*

Proof. Given $\boldsymbol{N}$ admissible choose a splitting sequence $\{\boldsymbol{M}_i\}$ satisfying (7.5) with first term $\boldsymbol{N}$. Define $\mathcal{U}_i$ by (7.3). Then (7.4) and Lemma A imply that these $\mathcal{U}_i$ together with $\mathcal{A}$ for all $\boldsymbol{a}$ in $\boldsymbol{M}$ form a subbase for a uniform structure $\mathfrak{W}$ containing $[\boldsymbol{M}]$. For $\mathcal{W}$ in $\mathfrak{W}$ there exist i and a finite subfamily $\boldsymbol{P} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ of $\boldsymbol{M}$ with

(7.6) $$\mathcal{U}_i \cap \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_k \subseteq \mathcal{W}.$$

Thus every $\mathcal{W}$-spectrum is refined by $\boldsymbol{M}_i \cup \boldsymbol{P}$ and by (7.5) belongs to $\boldsymbol{M}$. So $\mathfrak{W}$ induces $\boldsymbol{M}$ since $\mathfrak{W}$ contains $[\boldsymbol{M}]$. Moreover $\mathcal{U} = \mathcal{U}_1$ and so belongs to $\mathfrak{W}$.

Conversely, if $\mathcal{U}$ belongs to $\mathfrak{W}$ and $\mathfrak{W}$ induces $\boldsymbol{M}$, choose $\{\mathcal{U}_i\}$ in $\mathfrak{W}$ with $\mathcal{U} = \mathcal{U}_1$ and (7.4). Let $\boldsymbol{M}_i$ be the family of all $\mathcal{U}_i$-spectra. Since every $\mathcal{U}^2$-spectrum is split by a $\mathcal{U}$-spectrum, $\{\boldsymbol{M}_i\}$ is a splitting sequence. Since the entourage (7.6) belongs to $\mathfrak{W}$, (7.5) holds.

THEOREM 14. *A uniform spectral space* $(X, \boldsymbol{M})$ *is simple if and only if every admissible subfamily* $\boldsymbol{N}$ *of* $\boldsymbol{M}$ *is refined by some finite subfamily of* $\boldsymbol{M}$.

Proof. According to Lemma C the latter condition of the theorem means that every entourage $\mathcal{U}$ belonging to a uniform structure $\mathfrak{U}$ inducing $\boldsymbol{M}$ contains a basic entourage of $[\boldsymbol{M}]$. That is, $\mathfrak{U} = [\boldsymbol{M}]$.

THEOREM 15. $(X, \boldsymbol{M})$ *is simple and pseudometrizable if and only if there exists a countable subfamily* $\boldsymbol{P}$ *of* $\boldsymbol{M}$ *such that every admissible subfamily of* $\boldsymbol{M}$ *is refined by some finite subfamily of* $\boldsymbol{P}$.

Proof. Given the former condition, Theorem 13 implies that $\boldsymbol{M}$ is the union of the splitting sequence of uniform families $\boldsymbol{M}_i$ defined by (7.2). Each $\boldsymbol{M}_i$ is clearly admissible (7.5). By Theorem 14 we can choose a finite $\boldsymbol{P}_i$ which refines $\boldsymbol{M}_i$. Let $\boldsymbol{P}$ be the union over i of these $\boldsymbol{P}_i$. Since the space is simple, Lemma C implies that every admissible family is contained in some $\boldsymbol{M}_i$ and is thereby refined by $\boldsymbol{P}_i$.

Conversely, given $\boldsymbol{P} = \{\boldsymbol{p}_k\}$ satisfying the latter condition of Theorem 15, the space is simple by Theorem 14. Using Axiom I choose $\boldsymbol{q}_k$ in $\boldsymbol{M}$ splitting $\boldsymbol{p}_k$. Let $k_1 = 1$. Having chosen k_i choose $k_{i+1} > k_i$ such that the family $\boldsymbol{Q}_i$ of $\boldsymbol{q}_k$ with $k = 1, \ldots, k_i$ is refined by the family $\boldsymbol{P}_{i+1}$ of $\boldsymbol{p}_k$ with $k = 1, \ldots, k_{i+1}$. This is possible because finite subfamilies of $\boldsymbol{M}$ are admissible by Axioms I and II'. Let $\boldsymbol{M}_i$ consist of all spectra refined by $\boldsymbol{P}_i$. Then each $\boldsymbol{M}_i$ is uniform since every spectrum refined by $\boldsymbol{M}_i$ belongs to $\boldsymbol{M}_i$. Let $\mathcal{U}_i$ be the entourage (7.3) associated with $\boldsymbol{M}_i$, that is, with $\boldsymbol{P}_i$. Let $\mathcal{V}_i$ be the entourage associated with $\boldsymbol{Q}_i$. Since $\boldsymbol{q}_k$ splits $\boldsymbol{p}_k$, $\mathcal{V}_i^2 \subseteq \mathcal{U}_i$ by Lemma A. Now any $\boldsymbol{a}$ in $\boldsymbol{M}_i$ is a $\mathcal{U}_i$-spectrum and is therefore split by some $\mathcal{V}_i$-spectrum $\boldsymbol{b}$. Since $\mathcal{V}_i \supseteq \mathcal{U}_{i+1}$, $\boldsymbol{b}$ is a $\mathcal{U}_{i+1}$-spectrum and hence belongs to $\boldsymbol{M}_{i+1}$. So $\{\boldsymbol{M}_i\}$ is a splitting sequence. By hypothesis, since finite subfamilies of $\boldsymbol{M}$ are admissible, every member of $\boldsymbol{M}$ belongs to some $\boldsymbol{M}_i$. So the space is pseudometrizable by Theorem 13.

References

[1] E. M. Alfsen and J. E. Fenstad, *On the equivalence between proximity structures and totally bounded uniform structures*, Math. Scand. 7 (1959), pp. 353-360.

[2] M. Atsuji, *Uniform continuity of continuous functions on metric spaces*, Pac. J. Math. 8 (1958), pp. 11-16.

[3] — *Uniform continuity of continuous functions on uniform spaces*, Can. J. Math. 13 (1961), pp. 657-663.

[4] V. A. Efremovich, *The geometry of proximity I*, Mat. Sborn. N. S. 31 (73) (1952), pp. 189-200.

[5] J. E. Fenstad, *On l-groups of uniformly continuous functions, II. Representation theory*, Math. Zeit. 83 (1964), pp. 45-56.

[6] I. S. Gàl, *Proximity relations and precompact structures*, Nederl. Akad. Wetensch. Proc. Ser. A 62 (1959), pp. 304-326.

[7] J. Hejcman, *Boundedness in uniform spaces and topological spaces*, Czech. Math. Jour. 9 (1959), pp. 544-562.

[8] S. Leader, *On completion of proximity spaces by local clusters*, Fund. Math. 48 (1960), pp. 201-216.

[9] S. G. Mrówka and W. J. Pervin, *On uniform connectedness*, Proc. A. M. S. 15 (1964), pp. 446-449.

[10] O. Njåstad, *On uniform spaces where all uniformly continuous functions are bounded*, Monatshef. Math. 69 (1965), pp. 167-176.

RUTGERS — THE STATE UNIVERSITY

Reçu par la Rédaction le 4. 8. 1965