A classification of continua by certain cutting properties

by

E. S. Thomas, Jr.* (Ann Arbor, Mich.)

Introduction. By a *continuum* we mean a compact connected metric space. A *subcontinuum* of a space X is a closed connected subset of X and by $S(X)$ we mean the collection of subcontinua of X.

Let M be a continuum and U, V open subsets of M. A collection $C \subseteq S(M)$ *cuts between* U *and* V *in* M provided no member of C meets $U \cup V$ and every member of $S(M)$ which meets both U and V meets a member of C. A subcollection C of $S(M)$ *separates* M in case $M - \bigcup C$ is disconnected. Thus, in general, cutting is weaker than separating.

With each nondegenerate continuum M we associate a cardinal number n as follows: n is the smallest cardinal z such that some subcollection C of $S(M)$ with cardinality z cuts between a pair of open sets in M. We say that M *is of type* n. Thus an arc is of type 1 and a simple closed curve is of type 2.

The initial motivation for the work presented here was a conversation with F. Burton Jones in which the question arose: Is every continuum of type n, $2 \leqslant n < \aleph_0$, separated by n of its subcontinua? We show that the answer is yes (cf. Corollaries 2.2 and 3.2 below). This follows from some structure theorems for such continua. Our major results (Theorems 2.3 and 3.1) say that if M is of type n where $2 \leqslant n < \aleph_0$ then by collapsing n subcontinua of M points one obtains a continuum of type n consisting of two indecomposable continua meeting at n points. No neat analogues of the results for finite n have been found for continua of type $\aleph_0$, in fact, we give an example of a continuum M of type $\aleph_0$ every indecomposable subcontinuum of which is nowhere dense in M.

Throughout the paper we use the notation X° and $\overline{X}$ to denote, respectively, the interior of X and the closure of X in M. Relative interior and closures will be written out in words.

1. Throughout sections 1 and 2 let M denote a continuum of type n where $3 \leqslant n < \aleph_0$ and let $\{A_1, \ldots, A_n\} \subset S(M)$ cut between the open sets U_0 and V_0 of M. Let $\mathfrak{P}$ denote the collection of ordered pairs (U', V')

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where U', V' are open sets containing U_0, V_0 respectively, and $A_1, \ldots,$ cut between U' and V'. Order $\mathfrak{P}$ in the natural way: $(U', V') \leqslant (U'', V'')$ provided $U' \subseteq V''$, $V' \subseteq U''$; by Zorn's lemma there is a maximal pair in $\mathfrak{P}$. In sections 1 and 2 (U, V) denotes some fixed maximal pair.

We now list some useful facts.

LEMMA 1.1. (a) *If $K \in S(M)$ meets at least two of the A_i then K contains U or V.*

(b) *If K_1, K_2 are in $S(M)$ and K_1°, K_2° meet U and V, respectively, then $(K_1 \cup K_2)^\circ$ contains at least $n-1$ of the A_i.*

Proof. Part (a) is obvious. To prove (b) suppose $K_1, K_2 \in S(M)$ and A_1, A_2 contain points a_1, a_2, resp., not in $(K_1 \cup K_2)^\circ$. Since K_1, K_2 do not cut between any pair of sufficiently small open neighborhoods of a_1 and a_2, there is, for each positive integer i, a $C_i \in S(M)$ such that $C_i \cap (K_1 \cup K_2) = \emptyset$ and each of $\operatorname{dist}(a_1, C_i)$, $\operatorname{dist}(a_2, C_i)$ is less than $1/2$. Then $C = \bigcap_{j=1}^{\infty} \overline{\bigcup \{C_i \mid i \geqslant j\}}$ is a subcontinuum of M containing a_1 and a_2 and missing K_1° and K_2°. Since C meets A_1 and A_2, it contains U or V (by part (a)); say $U \subset C$. Thus both of K_1° and K_2° miss U.

LEMMA 1.2. *U and V are connected.*

Proof. Since no subcontinuum of M separates M, it suffices to show that $M-U$ and $M-V$ are connected. We first show that some component of $M-U$ contains all the A_i. If this is not the case, then $M-U = E+F$, separated sets, where we may assume $A_1 \subseteq E$, $A_2 \subseteq F$. We may also assume $F^\circ \cap V \neq \emptyset$. Since $A_2, \ldots, A_n$ do not cut between any pair of open sets in M, there is $K \in S(M)$ such that K joins U to $F^\circ \cap V$ and K misses each A_i which lies in F. There is a subcontinuum K' of K which meets U and $F^\circ \cap V$ and misses E. Then K' misses every A_i but joins U to V, a contradiction.

Thus there is a component A of $M-U$ containing all the A_i. By part (a) of Lemma 1.1, $V \subset A$. If $A \neq M-U$, write $M-U = E \cup F$ nonvoid and separated, where $A \subseteq E$. It is easily shown that $(U \cup F, V)$ is a member of $\mathfrak{P}$; this contradicts maximality of (U, V) in $\mathfrak{P}$. Thus $A = M-U$, i.e. $M-U$ is connected. Similarly $M-V$ is connected.

In the remainder of section 1 and throughout section 2, P and Q will denote the continua $\overline{U}$ and $\overline{V}$, respectively. We remark that since $n \geqslant 3$, $M-(P \cup Q)$ is connected, whence $M-\overline{(M-P \cup Q)} = (P \cup Q)^\circ$ is connected. In particular $P \cup Q$ is connected.

LEMMA 1.3. *One of P, Q is indecomposable.*

Proof. Suppose to the contrary that each of P and Q is the union of two proper subcontinua: $P = P_1 \cup P_2$, $Q = Q_1 \cup Q_2$. Each of P_1, Q_1 has nonvoid interior relative to M; thus by part (b) of Lemma 1.1, $P_1 \cup Q_1$ contains $n-1$, hence at least two, of the A_i. By part (a) of Lemma 1.1, $P_1 \cup Q_1$ contains U or V, say $U \subset P_1 \cup Q_1$. Since $Q \cap U = \emptyset$, we have $U \subset P_1$ which implies that $\overline{U} = P \subseteq P_1$, contradicting $P_1 \subset P$.

THEOREM 1.4. *Both P and Q are indecomposable.*

Proof. By Lemma 1.3 we may assume that P is indecomposable. Before proving the theorem we establish the following auxiliary result.

(A) *Q is indecomposable if there is a subcollection $\{A_1', \ldots, A_n'\}$ of $S(M)$ satisfying*:

(i) $A_i \subseteq A_i'$, $i = 1, \ldots, n$,

(ii) $A_i' - P^\circ = \emptyset$, $i = 1, \ldots, n$,

(iii) $U \not\subset \bigcup_{i=1}^{n} A_i'$,

(iv) $V \cap A_i' = \emptyset$, $i = 1, \ldots, n$.

For suppose such a collection exists and Q is the union of proper subcontinua Q_1 and Q_2. Denote by K the subcontinuum $\overline{M-(P \cup Q_1)}$ of M. Since Q_1 is proper in Q, $V \not\subset K$. Also $U \cap K = \emptyset$, so by (iii) and (iv) above, $K \cup A_1' \cup \ldots \cup A_n'$ contains neither U nor V. By (i) and part (a) of Lemma 1.1 we conclude that K meets at most one A_i'. Hence, reordering the A_i' if necessary, we have: $A_1' \cup \ldots \cup A_{n-1}' \subset M-K = (P \cup Q_1)^\circ \subseteq P^\circ \cup Q_1$. From (ii) it follows that Q_1 meets at least two of the A_i and since Q_1 is proper in Q, this contradicts part (a) of Lemma 1.1. Thus, statement (A) is proved.

To prove Theorem 1.4 we construct a family $A_1', \ldots, A_n'$ having properties (i)-(iv). The construction splits into two cases.

Case 1. *All the A_i miss Q.* In this case P° contains at least $n-1$ of the A_i, otherwise the continuum $\overline{M-(P \cup Q)}$ meets at least two of the A_i contradicting Lemma 1.1, part (a). Let i be an integer, $1 \leqslant i \leqslant n$. If $A_i \subset P^\circ$ construct A_i' as follows. Pick $K \in S(M)$ meeting U and V and missing $\bigcup \{A_j \mid j \neq i\}$; thus $K \cap A_i \neq \emptyset$. There is a component C of $K \cap P^\circ$ meeting A_i and we let $A_i' = A_i \cup C$. Since P contains at least two of the A_i, each A_i' constructed in the above way lies in some composant of P. If $A_i - P^\circ \neq \emptyset$ we merely set $A_i' = A_i$. Properties (i), (ii) and (iv) are easy to verify for the A_i' we have defined, and property (iii) follows from the fact that each A_i' either lies in a composant of P or is one of the original A_i (hence misses U).

Case 2. *Q meets some A_i.* We may assume $Q \cap A_1 \neq \emptyset$. If $2 \leqslant i \leqslant n$ and $A_i \subset P^\circ$ then construct A_i' as follows. Let $K \in S(M)$ join U to V missing $\bigcup \{A_j \mid j \neq i,\ j \neq n\} \cup \{u_0\}$ where u_0 is some point of U. If K misses A_n then it must meet A_i and we construct A_i' as we did in Case 1. (Here A_i' will lie in a composant of P because K does not contain u_0.)

Suppose then that K meets A_n. Now $V-K \neq \emptyset$ since Q meets A_1 and $K \cap A_1 = \emptyset$. Hence there is $L \in S(M)$ meeting $V-K$ and U and missing $\bigcup \{A_j \mid j \neq i,\ j \neq n\} \cup \{K \cup A_n\}$. Now L meets A_i and does not contain U since $K \cap U \neq \emptyset$ and $L \cap K = \emptyset$. We now define A_i' as in Case 1, using L in place of K. Again A_i' lies in a composant of P. As before, if $A_i - P^\circ \neq \emptyset$ we let $A_i' = A_i$. The A_i' satisfy (i)-(iv). This completes the proof of the theorem.

Before proceeding we briefly describe what happens if $M \neq P \cup Q$. For the next result we assume $M-(P \cup Q) \neq \emptyset$ and denote by K the continuum $\overline{M-(P \cup Q)}$. By Lemma 1.1, part (a), $(P \cup Q)^\circ$ contains at least $n-1$ of the A_i; say $A_1 \cup \ldots \cup A_{n-1} \subset (P \cup Q)^\circ$.

The main facts about K are summarized in the following theorem.

THEOREM 1.5 (a) *If K is of type m then $m \geqslant n-1$.*

(b) *K is irreducible about each of the sets $K \cap P$ and $K \cap Q$.*

(c) *If $K_1 \in S(K)$ has nonvoid interior relative to K, then K_1 meets each of the sets $Q-P$, $P-Q$ and A_n.*

Proof. (a) Suppose that X and Y are nonvoid sets open in K and that $K_1, \ldots, K_r \in S(K)$ cut X from Y in K. Then $X-(P \cup Q)$ and $Y-(P \cup Q)$ are nonvoid open subsets of M and the $r+1$ continua $K_1, \ldots, K_r$, $P \cup Q$ cut between them. Hence $r+1 \geqslant n$ and if K is of type m then $m+1 \geqslant n$.

(b) Suppose that K_1 were a proper subcontinuum of K and $K \cap P \subseteq K_1$. (We do not exclude the possibility that $K \cap P = \emptyset$.) Then the subcontinua Q and K_1 would separate M, contradicting $n \geqslant 3$.

(c) Let $K_1 \in S(K)$ have nonvoid interior in K (hence also $K_1^\circ \neq \emptyset$). Suppose $K_1 \cap (Q-P) = \emptyset$ and consider the set $D = M-(P \cup \overline{K-K_1})$. Q° is an open subset of D. It is also closed in D, for if x is a point of D which is a limit point of Q° then $x \in Q$ and $x \notin P$ so that $x \notin K_1$. Since also $x \notin K-K_1$, we have $x \notin P \cup K \cup (K-K_1) = P \cup K$, i.e., $x \in Q^\circ$. So Q° is open and closed in D. Since D contains points not in Q° (namely, points of K_1°), D is disconnected. Since $\overline{K-K_1} \in S(M)$ (by part (a) of this theorem) there is a contradiction to $n \geqslant 3$.

Finally we show $K_1 \cap A_n \neq \emptyset$. If $K_1^\circ - A_n = \emptyset$ we are done, so assume $K_1^\circ - A_n \neq \emptyset$. Since (U, V) is a maximal pair, we may choose $L_1, L_2 \in S(M)$ joining $K_1^\circ - A_n$ to U and V respectively and missing $A_1, \ldots, A_n$. Then $L_1 \cup L_2 \cup K_1$ meets some A_j; since L_1, L_2 miss all the A_i, K_1 must meet this A_j. Since $A_1 \cup \ldots \cup A_{n-1}$ lies in $(P \cup Q)^\circ$ and hence misses K_1, we must have $j = n$ as asserted.

There are several natural questions we might ask about the way in which $A_1, \ldots, A_n$ and K (if it exists) are situated in a continuum M of type n. For example: Do some (or all) of the A_i meet $P \cap Q$? Must each of P and Q meet some A_i? (i.e.: Is Case 1 in the proof of Theorem 1.4 impossible?) Is the number of components of $P \cap Q$ (or $K \cap P$ or $K \cap Q$) restricted by n? Does $A_1 \cup A_2 \cup \ldots \cup A_n \cup K$ separate M?

The following example shows that the answer in each case is no. Let $k \geqslant 2$ be an integer; we indicate how to construct a continuum M of type 3 in which A_1, A_2, A_3 may be chosen so that:

(1) $A_1 \cup A_2 \subset P^\circ$ and $A_3 \cap (P \cup Q) = \emptyset$.

(2) K is nonvoid and each of $P \cap Q$, $K \cap P$, $K \cap Q$ has k components.

(3) $M-(A_1 \cup A_2 \cup A_3 \cup K)$ is connected.

The continuum M may be obtained as the union of three indecomposable continua P, Q, K in the plane which intersect as indicated in figure 1.

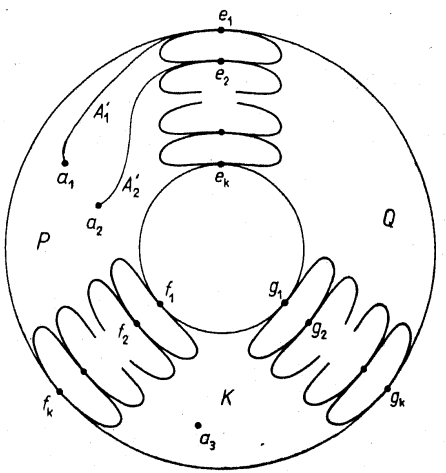

Fig. 1

P, Q and K are to be chosen so that the following conditions hold:

$$P \cap Q = \{e_1, \ldots, e_k\}; \quad K \cap P = \{f_1, \ldots, f_k\}; \quad K \cap Q = \{g_1, \ldots, g_k\};$$

a_1 and a_2 lie in $P-(Q \cup K)$, and a_3 is in $K-(P \cup Q)$. A_1' and A_2' lie in composants of P; A_1' contains a_1 and e_1, while A_2' contains $a_2, e_2, \ldots, e_k$. Let C_0 be the composant of P containing A_2'; then C_0 contains none of $e_1, f_1, \ldots, f_k$. Finally, if C is a composant of P, Q or K and $C \neq C_0$ then C contains at most one of $e_1, f_1, \ldots, f_k, g_1, \ldots, g_k$.

For $i = 1, 2, 3$ let $A_i = \{a_i\}$; then M is of type 3; the subcontinua A_1, A_2, A_3 cut between the pair $P-(A_1' \cup A_2' \cup K)$ and $Q-(P \cup K)$

and if (U, V) is a maximal pair corresponding to A_1, A_2, A_3 then (modulo a relabeling) $\overline{U} = P$ and $\overline{V} = P$. It is easy to verify (1), (2) and (3).

As is indicated by property (3) of this example some additional work is needed to show that a continuum of type n is separated by n of its subcontinua.

2. Before proving the main theorem of this section we introduce some notation and establish some basic facts.

For each point x of P [resp., Q] let $P(x)$ [$Q(x)$] denote the x-composant of P [Q]. If $x \in P \cap Q$ then there is a unique integer i, *denoted by* $i(x)$, such that $P(x) \cup Q(x)$ meets A_i. (There is at least one such integer because the A_i cut U from V, and at most one because of Lemma 1.1, part (a).)

Observe that if x and y are in $P \cap Q$ and $i(x) \neq i(y)$ then $\big(P(x) \cup Q(x)\big) \cap \big(P(y) \cup Q(y)\big) = \emptyset$, for otherwise there would be a continuum not containing U or V but joining $A_{i(x)}$ to $A_{i(y)}$. For $j = 1, \ldots, n$, denote by C_j the set of x in $P \cap Q$ such that $i(x) = j$. As before we may assume $A_1 \cup \ldots \cup A_{n-1} \subset (P \cup Q)^\circ$. It follows that each of $C_1, \ldots, C_{n-1}$ is nonvoid. (As the example of the preceding section shows, C_n may be void.) In any case, the C_i are disjoint and their union is $P \cap Q$.

THEOREM 2.1 *Assume the above notation. There exist continua* $K_1, \ldots, K_n$ *in* M *such that*:

(a) $C_i \cup A_i \subseteq K_i$ *for* $i = 1, \ldots, n-1$;

(b) $C_n \cup A_n \cup K \subseteq K_n$ (*where* C_n *or* K *may be void*);

(c) $K_i \subset M - (U \cup V)$ *for* $i = 1, \ldots, n$.

Proof. (In what follows, if $C_n = \emptyset$ the corresponding definition will be vacuous.) Fix $i \leqslant n$ and let x be a point of C_i; it is easy to verify that exactly one of the following occurs:

(p) There is a continuum $\hat{P}(x)$ lying in $P(x)$ missing U and irreducible from x to A_i.

(q) There is a continuum $\hat{Q}(x)$ lying in $Q(x)$ missing V and irreducible from x to A_i.

Correspondingly, we define $C_i(P)$ and $C_i(Q)$ to be the set of x in C_i enjoying properties (p) or (q) respectively. Thus C_i is the union of the disjoint sets $C_i(P)$ and $C_i(Q)$.

Let

$$K_i(P) = A_i \cup \bigcup \{\hat{P}(x) \mid x \in C_i(P)\}^- \quad \text{and} \quad K_i(Q) = A_i \cup \bigcup \{\hat{Q}(x) \mid x \in C_i(Q)\}^-.$$

Since each $\hat{P}(x)$ misses U and lies in P, $K_i(P)$ is a subcontinuum of M missing $U \cup V$; similarly for $K_i(Q)$. Let $K_i = K_i(P) \cup K_i(Q)$ for $i < n$ and let $K_n = K_n(P) \cup K_n(Q) \cup A_n \cup K$. Properties (a), (b) and (c) follow immediately from the construction.

COROLLARY 2.2. *Each continuum of type* n, $3 \leqslant n < \aleph_0$, *is separated by* n *of its subcontinua. More specifically, if* $K_1, \ldots, K_n$ *are the continua given in Theorem* 2.1 *then* $M - (K_1 \cup \ldots \cup K_n)$ *is the union of two connected open sets one containing* U *and the other* V.

Proof. This follows immediately from properties (a), (b) and (c) and the fact that $P \cap Q = C_1 \cup \ldots \cup C_n$.

By a *basic continuum of type* n ($3 \leqslant n < \aleph_0$) we mean a continuum M of type n for which the A_i may be chosen so that each A_i is a singleton, say $A_i = \{a_i\}$; $P \cap Q = \{a_i \mid i = 1, \ldots, n\}$ and $M = P \cup Q$. Clearly these conditions on the A_i determine U and V uniquely; $U = P - Q$, $V = Q - P$.

Intuitively, the basic continua of type n are the prototypes for arbitrary continua of type n. This idea is expressed more precisely in the next theorem.

THEOREM 2.3. *Let* M *be a continuum of type* n. *Then there is a basic continuum* N *of type* n *and a monotone continuous function* q *mapping* M *onto* N.

Proof. Using the notation of this section, let $K_1, \ldots, K_n$ be the continua given in Theorem 2.1. Let D denote the decomposition of M whose nondegenerate elements are $K_1, \ldots, K_n$. Then D is monotone and upper semi-continuous. Let N denote the quotient space of M modulo D and q the quotient map.

N is a continuum and using the fact that the K_i separate M and the monotonity of q it is easy to verify that N is of type n.

Let $U' = q(U)$ and $V' = q(V)$; then U' and V' are open in N and the points $q(K_1), \ldots, q(K_n)$ cut between (in fact, separate) these two sets in M. Since $N = U' \cup V' \cup \big(q(K_1) \cup \ldots \cup q(K_n)\big)$ the pair (U', V') is maximal. Now the closures in N of U' and V' are just $q(P)$ and $q(Q)$, respectively (where $M = P \cup Q \cup K$ in the usual way). Since q is monotone and continuous, $q(P)$ and $q(Q)$ are indecomposable. Thus $N = q(P) \cup q(Q)$ is basic.

3. Let M be a continuum of type 2. If A_1, A_2 cut between a pair of open sets then again there is a maximal pair (U, V). As in section 1 one can show that $M - U$, $M - V$, U and V are all connected, and, as before, we let $P = \overline{U}$, $Q = \overline{V}$.

Since a simple closed curve is of type 2, neither P nor Q need be indecomposable. If P and Q are indecomposable; $P \cap Q = \{a_1, a_2\}$ where $A_i = \{a_i\}$, $i = 1, 2$; and $P \cup Q = M$, then we say that M *is a basic continuum of type* 2.

In view of the wide range of possibilities for the structure of continua of type 2 the most satisfying theorem seems to be the following analogue of Theorem 2.3. We omit many details in the proof.

THEOREM 3.1. *Let M be of type 2; there is a monotone continuous image of M which is either a simple closed curve or a basic continuum of type 2.*

Proof. One of the following three possibilities holds:

(a) $M-(P\cup Q)$ is disconnected and nonvoid;

(b) $M-(P\cup Q)$ is connected and nonvoid;

(c) $M=P\cup Q$.

If (a) or (b) holds then there is a monotone continuous image of M which is of type 2 and for which (c) holds.

To see this, suppose that (a) holds. Then $M-(P\cup Q)$ has exactly two components whose closures we denote by H and K. Each of H and K meets both P and Q. Let D be the decomposition of M whose nondegenerate elements are H and K, let $\widetilde{M}$ be the quotient space of M modulo D and let q be the quotient map.

No subcontinuum of $\widetilde{M}$ cuts between a pair of open subsets of $\widetilde{M}$ since q is monotone and continuous. $\widetilde{M}$ is of type 2; indeed, let $\widetilde{A}_1=q(H)$, $\widetilde{A}_2=q(K)$; then $\widetilde{A}_1$, $\widetilde{A}_2$ cut between the open sets $q(U)$ and $q(V)$ (because no continuum separates H^0 from K^0 in M). If $(\widetilde{U},\widetilde{V})$ is the corresponding maximal pair and $\widetilde{P},\widetilde{Q}$ are the closures of $\widetilde{U}$, $\widetilde{V}$ in $\widetilde{M}$ then $\widetilde{P}=q(P)$ and $\widetilde{Q}=q(Q)$ and $\widetilde{P}\cap\widetilde{Q}$ is the pair of points $\widetilde{A}_1,\widetilde{A}_2$. Finally $\widetilde{M}=\widetilde{P}\cup\widetilde{Q}$, i.e., (c) holds for $\widetilde{M}$.

If (b) holds for the original continuum M, then a similar result is obtained using the decomposition whose nondegenerate element is $\overline{M-(P\cup Q)}$.

To sum up, we may assume at the outset that (c) holds for M. Using the process of the preceding section one constructs continua $K_1,K_2\in S(M)$ such that $A_i\subseteq K_i$, $i=1,2$, and $M-(K_1\cup K_2)=U'\cup V'$, disjoint open sets, with $U\subseteq U'$ and $V\subseteq V'$. Moreover, each of P,Q is irreducible from K_1 to K_2.

Now suppose that one of P, Q, say P, admits the unit interval $[0,1]$ as a monotone continuous image *via* a map h. Let D be the decomposition of M whose elements are Q and the point inverses $\{h^{-1}[r]\mid r\in(0,1)\}$. Then D is monotone and upper semi-continuous, the quotient space is a simple closed curve and we are done.

If neither P nor Q maps into $[0,1]$ in the above way, then by Theorem 10, section 1, of [1], each of them contains an indecomposable subcontinuum with (relative) nonvoid interior, say $\hat{P}\subseteq P$, $\hat{Q}\subseteq Q$.

If $M=\hat{P}\cup\hat{Q}$ then the quotient space modulo the decomposition whose nondegenerate elements are K_1 and K_2 is a basic continuum of type 2.

If $M-(\hat{P}\cup\hat{Q})$ is nonvoid then it has either one component, whose closure is denoted by L, or two, whose closures we denote by H and K.

In the first case L meets exactly one of K_1 and K_2, say $L\cap K_1\neq\emptyset$. We then factor M by the decomposition whose nondegenerate elements are $L\cup K_1$ and K_2. In the second case each of H and K meets exactly one of K_1 and K_2 and *vice versa*, say $H\cap K_1\neq\emptyset$ and $K\cap K_2\neq\emptyset$, and we factor by the decomposition whose nondegenerate elements are $H\cup K_1$ and $K\cup K_2$. In either case the quotient space is basic of type 2.

As in the preceding section we have:

COROLLARY 3.2. *Every continuum of type 2 is separated by a pair of its subcontinua.*

4. Much of the theory developed for continua of type n, $2\leqslant n<\aleph_0$ fails to have any counterpart in the study of continua of type $\aleph_0$. Some results, too complicated and incomplete to give here, have been obtained by modifying the idea of a maximal pair. To illustrate the differences encountered in passing from finite cardinals to $\aleph_0$ we construct an example of a continuum M of type $\aleph_0$ and a collection $\{a_i\mid i=1,\ldots\}$ of points of M whose union A separates M such that each component of $M-A$ has void interior in M. In particular every indecomposable subcontinuum of M has void interior.

The continuum M is obtained by identifying pairs of points of a Cantor set of indecomposable continua.

Let S denote an indecomposable subcontinuum of the plane; let $\{C_i\mid i=0,1,2,\ldots\}$ be a countable pairwise disjoint collection of composants of S; and let a_0 be a point of C_0. Let K denote a Cantor set in $[0,1]$; let E denote the set of endpoints of the components of $[0,1]-K$; and let $\{p_1,p_2,\ldots\}$ denote the collection of ordered pairs $E\times E$. Write $p_i=(e_i,f_i)$, $i=1,2,\ldots$

Let $T=K\times S$; we think of T and all the arcs constructed below as lying in Euclidean 3-dimensional space. Let A_0 be the arc $[0,1]\times\{a_0\}$. Having chosen $A_0,A_1,\ldots,A_{n-1}$, $n\geqslant 0$, let A_n be an arc with endpoints a_n and b_n such that $A_n\cap(T\cup A_0\cup\ldots\cup A_{n-1})=\{a_n,b_n\}$; $a_n\in\{e_n\}\times C_n$; $b_n\in\{f_n\}\times C_n$, and $\operatorname{dist}(A_n,A_0)\leqslant 1/2^n$.

Let $N=T\cup\bigcup_{n=0}^{\infty}A_n$; then N is a continuum. Let D denote the decomposition of N whose nondegenerate elements are the A_i, $i=0,1,2,\ldots$; then D is monotone and upper semi-continuous. The quotient space M is a continuum of type $\aleph_0$ and the points of M corresponding to the A_i separate M. It is not difficult to see that M has the properties stated in the opening paragraph of this section.

In addition to the general problem of getting a structure theory (and, perhaps, finding a reasonably simple class of prototypes) for continua of type $\aleph_0$ there are at least two questions which should be answered.

Does every continuum of type $\aleph_0$ contain an indecomposable subcontinuum? Is every continuum of type $\aleph_0$ separated by a countable collection of its subcontinua?

Reference

[1] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Mat. 50 (1965).

UNIVERSITY OF CALIFORNIA, Riverside,
and
UNIVERSITY OF MICHIGAN, Ann Arbor, Michigan

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Open mappings on graphs and manifolds*

by

Richard A. Duke (Seattle, Wash.)

1. Introduction. In [3], pp. 182 and 197, Whyburn has shown that the image of a finite graph under a (light) open mapping is again a finite graph and also that the image under a light, open mapping of a 2-manifold is a 2-manifold. The purpose of this paper is to investigate the conditions under which an open mapping $f(G) = H$ defined on a graph G can be extended to a light, open mapping $F(M) = N$ defined on a 2-manifold M when G and H are imbedded in M and N, respectively.

In section 3 it is shown that if $f(G) = H$ is such an open mapping, then there exist imbeddings of G and H in some orientable 2-manifolds M and N respectively, and an extension of f to a light, open map $F(M) = N$. The imbedding of H may be taken to be any orientable, 2-cell (and hence any minimal) imbedding of H. Further, any light, open map $F(M) = N$ on a closed orientable 2-manifold M can be obtained as an extension of such a map $f(G) = H$ on a graph G minimally imbedded in M. In section 4 it is shown that for each positive integer n there is an open mapping of a planar graph onto some graph of genus n, and this is used to show that the imbedding obtained for G in the above result may necessarily be non-minimal.

2. Background. The term *mapping* will be used to denote a continuous transformation. A mapping $f(X) = Y$ is said to be *open* provided that every open set in X maps onto a set open in Y. If for each y in Y, $f^{-1}(y)$ is totally disconnected, f is said to be a *light* mapping. The term *graph* will denote a finite, connected 1-complex.

If $f(G) = H$ is an open mapping on a graph G, then H is also a graph. Further, f is a light mapping and it is possible to designate certain interior points on the edges of G and H as additional vertices in such a way that f maps each edge of G topologically onto an edge in H. In light of these facts, we shall assume for the remainder that any open mapping on a graph is a simplicial transformation.

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