On convergence groups and convergence uniformities

by

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Introduction. Some non-topological convergence structures encountered in analysis exhibit properties reminiscent of uniform spaces. An investigation of such structures has been made by Cook and Fischer [2]. We give a somewhat more lattice-oriented development of the same subject.

We denote by $C'(S)$ the complete lattice of all convergence structures on a set S, and by $W(S)$ the smallest sub complete lattice of $C'(S)$ that contains all of the completely regular topologies on S. A member q of $W(S)$ is said to be "weakly uniformizable", and to each such structure there corresponds an equivalence class $[q]$ of "weak convergence uniformities" which contains both a finest and a coarsest member. We extend the notion of completeness and show that each weak convergence uniformity has a "completion". A convergence group is a special type of weakly uniformizable convergence structure. A simple characterization is given for the smallest sub complete lattices of $C'(S)$ that include, respectively, the set of all convergence groups and the set of all topological groups defined for a given Abelian group $(S, +)$.

Finally, we note that our weak convergence uniformity and the corresponding (somewhat stronger) structure used by Cook-Fischer are both too permissive, in the sense that they describe as "uniformizable" all T_2 topologies. A criterion consisting of three conditions is suggested as a measure of suitability for future efforts to define the notion of "convergence uniformity".

I. Convergence structures. A *convergence function* q on a set S is a mapping of the set $F(S)$ of all filters on S into the set of all subsets of S which is order-preserving (finer filters map into larger sets) and has the property $x \in q(\dot{x})$, all $x \in S$, where $\dot{x}$ is the ultrafilter generated by $\{x\}$. If $x \in q(\mathcal{F})$, then we say that "the filter $\mathcal{F}$ q-converges to x". The filter $\mathcal{V}_q(x)$ obtained by intersecting the collection of all filters that q-converge to x is called the *q-neighborhood filter at x*. If $\mathcal{V}_q(x)$ q-converges to x for each $x \in S$, then q is called a *pretopology*.

A partial order relation among convergence functions on the same set S can be introduced as follows: $p \leqslant q$ means that $q(\mathcal{F}) \subset p(\mathcal{F})$ for each $\mathcal{F} \in F(S)$. The set $C(S)$ of all convergence functions on S is then a complete lattice, whose greatest element is the discrete topology and whose least element is the indiscrete topology. The set $T(S)$ of all topologies on S is regarded as a subset of $C(S)$; the former is a complete lattice in its own right, but not a sub complete lattice of $C(S)$. Since a number of different lattices are considered in this paper, it will be convenient to use "$\inf_c$" and "$\sup_c$" to represent, respectively, the operations infimum and supremum in $C(S)$.

Let $q \in C(S)$. There is a finest pretopology $\pi(q)$ coarser than q, defined by $\mathcal{V}_{\pi(q)}(x) = \mathcal{V}_q(x)$, all $x \in S$. We may also associate with q the set function I_q defined for a given $A \subset S$ by $I_q(A) = \{x \in A\colon A \in \mathcal{V}_q(x)\}$. The set $\{U\colon I_q(U) = U\}$ is a topology on S which we designate $\lambda(q)$; $\lambda(q)$ is the finest topology coarser than q. The set $\{I_q(A)\colon A \subset S\}$ is a base for the topology $\varphi(q)$ on S. $\lambda(q) \leqslant \varphi(q)$; $\varphi(q)$ and q are in general not comparable; $\lambda(q) = \varphi(q)$ if and only if $\pi(q)$ is a topology.

A convergence function q on S will be called a *convergence* structure if and only if it satisfies the following additional condition: $x \in q(\mathcal{F})$ implies $x \in q(\mathcal{F} \cap \dot{x})$. The set of all convergence structures on S is denoted $C'(S)$. The two theorems that follow are not difficult to prove.

THEOREM 1.1. *A convergence function q is representable as the* $\inf_c$ *of a set of topologies if and only if $q \in C'(S)$.*

THEOREM 1.2. *$C'(S)$ is the smallest sub complete lattice of $C(S)$ that includes $T(S)$.*

If q is a convergence structure, then one can show that the associated topologies $\sigma(q)$ and $\varrho(q)$ (see [3]) coincide with $\lambda(q)$.

Henceforth, we shall restrict our attention to convergence structures rather than convergence functions. The pair (S, q), with $q \in C'(S)$, will be called a *convergence space.* The separation axioms T_1 and T_2 can be introduced into a convergence space in an obvious way. (S, q) is T_1 if, for each x in S, $\dot{x}$ q-converges only to x; (S, q) is T_2 if every filter in $F(S)$ q-converges to at most one point.

II. Weakly uniformizable convergence structures. *A uniformity* on a set S is considered in this paper to be a filter on $S \times S$ which is symmetric, envelops the diagonal Δ, and has the "square root property". Such a filter is more often called a "uniform structure", but this term might easily be confused with some of our later terminology.

We denote by $T_U(S)$ the set of all completely regular topologies on S, and by $U(S)$ the set of all uniformities on S. Both $T_U(S)$ and $U(S)$ are complete lattices in their natural orderings.

Some additional notation will be needed. Let $\dot{\Delta}$ denote the filter on $S \times S$ generated by the diagonal Δ. If $\mathcal{V}$ is a filter on $S \times S$ which is coarser than $\dot{\Delta}$, then $\mathcal{V}[x]$ designates the filter on S generated by $\{V[x]\colon V \in \mathcal{V}\}$, where $V[x] = \{y\colon (x, y) \in V\}$. If $V \in \mathcal{V}$, then $V^{-1} = \{(y, z)\colon (z, y) \in \mathcal{V}\}$, and $\mathcal{V}^{-1} = \{V^{-1}\colon V \in \mathcal{V}\}$; thus $\mathcal{V}$ is symmetric if $\mathcal{V} = \mathcal{V}^{-1}$. Finally, if $\mathcal{F}$ and $\mathcal{G}$ are in $F(S)$, then $\mathcal{F} \times \mathcal{G}$ is the filter on $S \times S$ generated by $\{F \times G\colon F \in \mathcal{F}, G \in \mathcal{G}\}$.

DEFINITION 1.1. A convergence structure q is *weakly uniformizable* if and only if there is a set Q of completely regular topologies such that $q = \inf_c Q$.

PROPOSITION 2.1. *If $\mathcal{F}$ is any filter on S, then $\dot{\Delta} \cap (\mathcal{F} \times \mathcal{F})$ is a uniformity on S.*

PROPOSITION 2.2. *Assume that $q \in C'(S)$, $\mathcal{F}$ q-converges to x, and $\mathcal{U}_{\mathcal{F},x} = \dot{\Delta} \cap ((\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}))$. Then $\mathcal{F}$ converges to x in the topology compatible with $\mathcal{U}_{\mathcal{F},x}$.*

PROPOSITION 2.3. *If $\mathcal{U} \in U(S)$ and $\mathcal{U}[x]$ converges to y in the topology t compatible with $\mathcal{U}$, then $\mathcal{U}[x] = \mathcal{U}[y]$.*

Proof. Since $x \in U[y] \in \mathcal{U}[y]$, $y \in U[x] \in \mathcal{U}[x]$ for all symmetric entourages U in $\mathcal{U}$; thus $\dot{y}$ t-converges to x. If $U \in \mathcal{U}$ is symmetric, then choose symmetric V in $\mathcal{U}$ such that $V^2 \subset U$. If $z \in V[x]$, then $y \in V[x]$ implies $(y, z) \in V^2$, and hence $z \in V^2[y] \subset U[y]$. This argument is reversible.

THEOREM 2.1. *A convergence structure q is weakly uniformizable if and only if $\mathcal{F}$ q-converges to y whenever $y \in \bigcap \mathcal{F}$ and $q(\mathcal{F}) \neq \emptyset$.*

Proof. Let $q = \inf_c P$, $P \subset T_U(S)$. With each $p \in P$, associate a uniformity $\mathcal{U}_p$ compatible with p. If $\mathcal{F}$ q-converges to x, then there is $p \in P$ such that $\mathcal{F}$ p-converges to x. If $y \in \bigcap \mathcal{F}$, then $\dot{y} \geqslant \mathcal{F}$ implies $\dot{y}$ p-converges to x. By Proposition 2.3, $\mathcal{U}_p[x] = \mathcal{U}_p[y]$, and $\mathcal{F} \geqslant \mathcal{U}_p[y]$ implies $\mathcal{F}$ p-converges to y; thus, $\mathcal{F}$ q-converges to y.

Conversely, assume the given condition and let $w = \{\mathcal{U}_{\mathcal{F},x}\colon \mathcal{F}$ q-converges to x, $x \in S\}$. We shall show that q is the $\inf_c$ of the set of those topologies compatible with some uniformity in w. By Proposition 2.2, it suffices to show that the topology p associated with an arbitrary $\mathcal{U}_{\mathcal{F},x}$ is finer than q. Assume for $x \neq y$ that there is $\mathcal{G} \in F(S)$ which p-converges to y, with $\mathcal{G} \neq \dot{y}$. Then $\mathcal{G} \geqslant \mathcal{U}_{\mathcal{F},x}[y]$ implies $y \in F \cup \{x\}$, all $F \in \mathcal{F}$. Thus $y \in \bigcap \mathcal{F}$ implies $\mathcal{F}$ q-converges to y. Since $\mathcal{U}_{\mathcal{F},x}[y] \geqslant \mathcal{F} \cap \dot{x}$, it follows that $\mathcal{G}$ q-converges to y.

Let $W(S)$ be the set of all weakly uniformizable convergence structures.

THEOREM 2.2. *$W(S)$ is the smallest sub complete lattice of $C'(S)$ that includes $T_U(S)$.*

Proof. Let $Q \subset W(S)$; $r = \sup_c Q$. If $\mathcal{F}$ r-converges to x, then $\mathcal{F}$ q-converges to x for all $q \in Q$. If $y \in \bigcap \mathcal{F}$, then $\mathcal{F}$ q-converges to y, all

$q \in Q$, and hence $\mathcal{F}$ r-converges to y; thus r is weakly uniformizable. The proof that $\inf_c Q$ is weakly uniformizable is similar. The theorem now follows from Definition 1.1 and the fact that $\sup_c$ of a set of completely regular topologies is a completely regular topology.

Since each completely regular topology is the $\sup_c$ of a set of pseudo-metrizable topologies, $W(S)$ can be regarded as the lattice-theoretic closure of the set of all pseudo-metrizable topologies in the complete lattice of all convergence structures on S.

If p is a convergence structure, let $\mathcal{V}_p = \bigcap \{\mathcal{V}_p(x) \times \mathcal{V}_p(x)\colon x \in S\}$. When p is a topology, $\mathcal{V}_p$ is the filter of "neighborhoods of the diagonal".

THEOREM 2.3. *For a pretopology p, the following statements are equivalent*: (1) *p is weakly uniformizable*; (2) *$y \in \bigcap \mathcal{V}_p(x)$ implies $\mathcal{V}_p(x) = \mathcal{V}_p(y)$*; (3) *$\mathcal{V}_p[x] = \mathcal{V}_p(x)$, all x in S.*

Proof. (1) $\Rightarrow$ (2). $y \in \bigcap \mathcal{V}_p(x)$ implies $\mathcal{V}_p(x)$ p-converges to y. Thus $\dot{x} \geqslant \mathcal{V}_p(x) \geqslant \mathcal{V}_p(y)$, implying $x \in \bigcap \mathcal{V}_p(y)$, and $\mathcal{V}_p(y) \geqslant \mathcal{V}_p(x)$.

(2) $\Rightarrow$ (3). $\mathcal{V}_p(x) \geqslant \mathcal{V}_p[x]$ in any case. Let $V \in \mathcal{V}_p(x)$. If $y \in \bigcap \mathcal{V}_p(x)$, let $V_y = V$; if y is not in $\bigcap \mathcal{V}_p(x)$, choose $V_y \in \mathcal{V}_p(y)$ such that x is not in V_y. If $W = \bigcup \{V_y \times V_y\colon y \in S\}$, then $W \in \mathcal{V}_p$, and $W[x] = V$.

(3) $\Rightarrow$ (1). Let $\mathcal{F}$ p-converge to x, and $y \in \bigcap \mathcal{F}$. Then $y \in \bigcap \mathcal{V}_p(x) = \bigcap \mathcal{V}_p[x]$. It is easy to see that $\mathcal{V}_p[y] \subset \mathcal{V}_p[x]$. Thus we have $\mathcal{F} \geqslant \mathcal{V}_p[x] \geqslant \mathcal{V}_p(y)$, and $\mathcal{F}$ p-converges to y.

It is an interesting fact that $\pi(q)$ (the finest pretopology coarser than q) may fail to be weakly uniformizable when q is weakly uniformizable. This is not the case, however, if q is a *limitierung*, i.e. if $\mathcal{F} \cap \mathcal{G}$ q-converges to x whenever both $\mathcal{F}$ and $\mathcal{G}$ q-converge to x.

THEOREM 2.4. *If $q \in W(S)$ is a limitierung, then $\pi(q)$ and $\varphi(q)$ are weakly uniformizable.*

Proof. Let $y \in \bigcap \mathcal{V}_q(x)$; then $\dot{y}$ q-converges to x. If $\mathcal{F}$ q-converges to x, then $\mathcal{F} \cap \dot{y}$ q-converges to x, and $y \in \bigcap (\mathcal{F} \cap \dot{y})$, which implies $\mathcal{F} \cap \dot{y}$ q-converges to y, and so $\mathcal{F}$ q-converges to y. Thus $\mathcal{V}_q(x) \geqslant \mathcal{V}_q(y)$. Since $x \in \bigcap \mathcal{V}_q(y)$, we can repeat the previous argument with the roles of x and y interchanged. It follows, by the previous theorem, that $\pi(q)$ is weakly uniformizable. From the fact that $\mathcal{V}_q(x) = \mathcal{V}_q(y)$, it follows easily from Theorem 3, Section II, [3] that the $\varphi(q)$-neighborhood filters for x and y coincide, and hence that $\varphi(q)$ is weakly uniformizable.

On the other hand, $\lambda(q)$ (the finest topology coarser than q) can fail to be weakly uniformizable, even when q is a weakly uniformizable pretopology.

III. Convergence groups. Let $(S, +)$ be an Abelian group with identity element 0. If $\mathcal{F}$ and $\mathcal{G}$ are filters on S, then $-\mathcal{F} = \{-F\colon F \in \mathcal{F}\}$, and $\mathcal{F} + \mathcal{G}$ is the filter generated by $\{F + G\colon F \in \mathcal{F},\ G \in \mathcal{G}\}$. The notations $\mathcal{F} - \mathcal{G}$ and $x + \mathcal{F}$ will usually replace $\mathcal{F} + (-\mathcal{G})$ and $\dot{x} + \mathcal{F}$. For a filter $\mathcal{F}$ with the property $0 \in F$, all $F \in \mathcal{F}$, it is convenient to write $n\mathcal{F}$ for $\mathcal{F} + \ldots + \mathcal{F}$ (n times); in general, if α is an ordinal number with an immediate predecessor $\alpha - 1$, $\alpha\mathcal{F} = \mathcal{F} + (\alpha - 1)\mathcal{F}$, and if α is a limit ordinal (an infinite ordinal with no immediate predecessor), $\alpha\mathcal{F} = \bigcap \{\beta\mathcal{F}\colon \beta < \alpha\}$.

DEFINITION 3.1. Let $(S, +)$ be an Abelian group and $q \in C'(S)$. Then $(S, +, q)$ is a *convergence group* if and only if, for each pair of filters $\mathcal{F}, \mathcal{G}$ on S, $\big(q(\mathcal{F}) - q(\mathcal{G})\big) \subset q(\mathcal{F} - \mathcal{G})$.

This definition is the natural one in the sense of "making the group operation continuous". The straightforward proof of the first proposition is omitted.

PROPOSITION 3.1. *If $(S, +, q)$ is a convergence group, then:*

(1) *$\mathcal{F}$ q-converges to 0 if and only if $x + \mathcal{F}$ q-converges to x;*

(2) *$x + \mathcal{V}_q(0) = \mathcal{V}_q(x)$;*

(3) *$-\mathcal{V}_q(x) = \mathcal{V}_q(-x)$.*

PROPOSITION 3.2. *A convergence group $(S, +, q)$ is a weakly uniformizable convergence structure.*

Proof. Let $\mathcal{F}$ q-converge to 0, and $y \in \bigcap \mathcal{F}$. Then $-y + \mathcal{F} \geqslant -\mathcal{F} + \mathcal{F}$, and since $-\mathcal{F} + \mathcal{F}$ q-converges to 0, so does $-y + \mathcal{F}$. But then $-y + \mathcal{F} + y = \mathcal{F}$ q-converges to y.

PROPOSITION 3.3. *Let $(S, +)$ be an Abelian group, q a pretopology. Then $(S, +, q)$ is a convergence group if and only if the following conditions are satisfied.* (1) *$\mathcal{V}_q(0) - \mathcal{V}_q(0) = \mathcal{V}_q(0)$;* (2) *$\mathcal{V}_q(x) = x + \mathcal{V}_q(0)$, all x in S.*

Proof. Let $(S, +, q)$ be a convergence group. Then (2) follows from Proposition 3.1, and (1) follows from Definition 3.1. Conversely, given (1) and (2), let $x, y \in S$, and $V(x-y) = x - y + V(0) \in \mathcal{V}_q(x-y)$, where $V(0) \in \mathcal{V}_q(0)$. Choose $V_1 \in \mathcal{V}_q(0)$ such that $V_1 - V_1 \subset V(0)$. Then $x + V_1 \in \mathcal{V}_q(x)$, $y + V_1 \in \mathcal{V}_q(y)$, and $x + V_1 - (-y - V_1) \subset V(x-y)$.

PROPOSITION 3.4. *If $(S, +, q)$ is a convergence group and q a pretopology, then $(S, +, q)$ is a topological group.*

Proof. If $V \in \mathcal{V}_q(0)$, let $V^* = \{(x, y) \in S \times S\colon x - y \in V\}$. It follows easily that $\{V^*\colon V \in \mathcal{V}_q(0)\}$ generates a uniformity $\mathcal{U}$ on S. Since $\mathcal{U}[x] = \mathcal{V}_q(x)$ for all x in S, q is a topology. The rest is clear.

PROPOSITION 3.5. *Let $Q \subset C'(S)$, $Q \neq \emptyset$, such that $q \in Q$ implies $(S, +, q)$ is a convergence group. Let $p = \sup_c Q$. Then $(S, +, p)$ is a convergence group.*

Proof. Let $x \in p(\mathcal{F})$, $y \in p(\mathcal{G})$. Then $x \in q(\mathcal{F})$, $y \in q(\mathcal{G})$ for all q in Q, and, by the given condition, $\mathcal{F} - \mathcal{G}$ q-converges to $x - y$. Thus $\mathcal{F} - \mathcal{G}$ p-converges to $x - y$.

If $(S, +, q)$ is a convergence group, then $\lambda(q)$, the finest topology coarser than q, is both homogeneous and weakly uniformizable.

For a given Abelian group $(S, +)$, let $T(S, +)$ be the set of all topologies t such that $(S, +, t)$ is a topological group, and let $C(S, +)$ be the set of all convergence structures q such that $(S, +, q)$ is a convergence group. It is easy to see that neither $T(S, +)$ nor $C(S, +)$ is closed under the operation $\inf_c$. We seek the smallest subcomplete lattices of $C'(S)$ that include $T(S, +)$ and $C(S, +)$ respectively.

DEFINITION 3.2. $(S, +, q)$ is a *weak convergence group* if and only if the following conditions are satisfied: (1) $\mathcal{F}$ q-converges to 0 if and only if $x+\mathcal{F}$ q-converges to x; (2) $\mathcal{F}$ q-converges to 0 implies $\mathcal{F}-\mathcal{F}$ q-converges to 0.

Every convergence group is a weak convergence group. Also, a weak convergence group is a weakly uniformizable convergence structure, since the proof of Proposition 3.2 requires no alteration if "convergence group" is replaced by "weak convergence group".

THEOREM 3.1. *$(S, +, q)$ is a weak convergence group if and only if q is the $\inf_c$ of the set $Q = \{p \in C(S, +)\colon p \geqslant q\}$.*

Proof. If $q = \inf_c Q$, then it is not difficult to show that q satisfies the two conditions specified in Definition 3.2; thus $(S, +, q)$ is a weak convergence group. Conversely, assume that $(S, +, q)$ is a weak convergence group. If $p = \inf_c Q$, then $p \geqslant q$ is clear. Let $\mathcal{F}$ q-converge to 0, and let $\mathcal{F}' = \mathcal{F}-\mathcal{F}$. Let r be the convergence structure defined by: (1) $\mathcal{G}$ r-converges to 0 if and only if $\mathcal{G} \geqslant n\mathcal{F}'$ for some positive integer n; (2) $\mathcal{G}$ r-converges to x if and only if $-x+\mathcal{G}$ r-converges to 0. It is easy to see that $r \in Q$, and that $\mathcal{F}$ r-converges to 0; thus $\mathcal{F}$ p-converges to 0, and $p = q$.

Let $W(S, +) = \{q \in C'(S)\colon (S, +, q)$ is a weak convergence group$\}$.

COROLLARY. *$W(S, +)$ is the smallest sub complete lattice of $C'(S)$ that includes $C(S, +)$.*

Proof. The verification that $W(S, +)$ is closed under $\sup_c$ and $\inf_c$ is straightforward. The result now follows from Theorem 1.

DEFINITION 3.3. $(S, +, q)$ is a *pseudo convergence group* if and only if it is a weak convergence group with the property that, for all ordinal numbers a, $a\mathcal{F}$ q-converges to 0 whenever $\mathcal{F}$ q-converges to 0.

Let $P(S, +) = \{q \in C'(S)\colon (S, +, q)$ is a pseudo convergence group$\}$. One can find examples of a convergence group that is not a pseudo convergence group and of a pseudo convergence group that is not a convergence group.

THEOREM 3.2. *$q \in P(S, +)$ if and only if $q = \inf_c Q$, where $Q = \{p\colon p \in T(S, +)$ and $p \geqslant q\}$.*

Proof. If $q = \inf_c Q$, then q is clearly a weak convergence group. Let $\mathcal{F}$ q-converge to 0; then there is $p \in Q$ such that $\mathcal{F} \geqslant \mathcal{V}_p(0)$. But $a\mathcal{F} \geqslant a\mathcal{V}_p(0) = \mathcal{V}_p(0)$, and hence $a\mathcal{F}$ p-converges to 0. Thus $a\mathcal{F}$ q-converges to 0, and $q \in P(S, +)$. Conversely if $q \in P(S, +)$ let $p = \inf_c Q$, and let $\mathcal{F}$ q-converge to 0. Let $\mathcal{F}' = \mathcal{F}-\mathcal{F}$. Consider the convergence structure r defined by: $\mathcal{G}$ r-converges to 0 if and only if $\mathcal{G} \geqslant a\mathcal{F}'$; where a is the least ordinal number such that $a\mathcal{F}'+a\mathcal{F}' = a\mathcal{F}'$; $\mathcal{G}$ r-converges to x if and only if $-x+\mathcal{G}$ r-converges to 0. It is easy to see that r is a topological group, and $r \in Q$. Since $x \in r(\mathcal{F})$ implies $x \in p(\mathcal{F})$, $p = q$.

It can be shown that a pseudo convergence group q is a convergence group if and only if the set Q defined in the preceding theorem is a dual ideal in the lattice $T(S, +)$.

COROLLARY. *$P(S, +)$ is the smallest sub complete lattice of $C'(S)$ that includes $T(S, +)$.*

Let $(K, +)$ be a subgroup of $(S, +)$; let $(S', +)$ be the quotient group whose elements are cosets of S modulo K; let $0'$ denote the identity element of S'. If $g\colon S \to S'$ is the canonical homomorphism and q a convergence structure on S, then the *quotient convergence structure* q' on S' is defined by: $\mathcal{G}$ q'-converges to y if and only if there is $\mathcal{F} \in F(S)$ q-converging to x such that $g(x) = y$ and $\mathcal{G} \geqslant g(\mathcal{F})$.

THEOREM 3.3. *If $(S, +, q)$ is a weak convergence group (respectively, convergence group, pseudo convergence group, topological group) and q' the quotient convergence structure corresponding to a subgroup $(K, +)$, then $(S', +, q')$ is a weak convergence group (respectively, convergence group, pseudo convergence group, topological group).*

Proof. Let $(S, +, q)$ be a weak convergence group, $S' = S/K$, and assume that $0' \in q'(\mathcal{G})$. Then there is x in K and $\mathcal{F}$ in $F(S)$ such that $\mathcal{F}$ q-converges to x and $\mathcal{G} \geqslant g(\mathcal{F})$, where g is the canonical homomorphism. But $\mathcal{F}_1 = -x+\mathcal{F}$ q-converges to 0, and $g(\mathcal{F}) = g(\mathcal{F}_1)$. Since $(\mathcal{F}_1-\mathcal{F}_1)$ q-converges to 0, and $\mathcal{G}-\mathcal{G} \geqslant g(\mathcal{F}_1)-g(\mathcal{F}_1) = g(\mathcal{F}_1-\mathcal{F}_1)$, $(\mathcal{G}-\mathcal{G})$ q'-converges to $0'$. Translations are preserved under homomorphisms, and it follows that $(S', +, q')$ is a weak convergence group.

The analogous result for convergence groups is known (see [4]). For pseudo convergence groups, the result follows from the fact that $g(a\mathcal{F}) = ag(\mathcal{F})$.

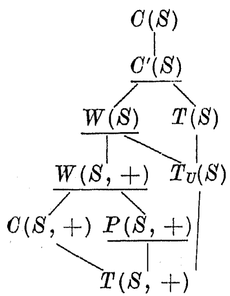
Next, let $Q \subset W(S, +)$, and $q = \inf_c Q$. Let $(S', +)$ be a quotient group of $(S, +)$ with kernel K and canonical homomorphism g. For each $p \in Q$, let $(S', +, p')$ be the quotient weak convergence group corresponding to $(S, +, p)$, and let $Q' = \{p' \in C'(S')\colon p \in Q\}$. If $q' = \inf_c Q'$, then the following conclusion can be drawn.

THEOREM 3.4. *$(S', +, q')$ is the quotient weak convergence group of $(S, +, q)$ under the canonical homomorphism g.*

Proof. If $\mathcal{F}$ q-converges to 0, then $\mathcal{F}$ p-converges to 0 for some p in Q; thus $g(\mathcal{F})$ p'-converges to $0'$, and $g(\mathcal{F})$ q'-converges to $0'$. Conversely, let $\mathcal{G}$ q'-converge to $0'$, i.e. for some p in Q, $\mathcal{G}$ p'-converges to $0'$. Then there is $\mathcal{F}$ p-converging to x in K such that $\mathcal{G} \geqslant g(\mathcal{F})$; as in the proof of the previous theorem, there is also $\mathcal{F}_1$ p-converging to 0 with $g(\mathcal{F}) = g(\mathcal{F}_1)$. But $\mathcal{F}_1$ also q-converges to 0, and the theorem is proved.

A corresponding result can be established for quotient pseudo convergence groups.

In the lattice diagram that follows, the order relation is set inclusion. Each entry is a complete lattice in the order relation defined on $C(S)$; underlined entries are sub complete lattices of $C(S)$.



IV. Convergence uniformities.

DEFINITION 4.1. A *weak convergence uniformity* w is an anti-residual set of uniformities on S; i.e., if $\mathcal{U} \in w$ and $\mathcal{V}$ is a uniformity finer than $\mathcal{U}$, then $\mathcal{V} \in w$.

For economy in writing, "weak convergence uniformity" will be shortened to "weak uniformity".

Any uniformity $\mathcal{U}$ can be regarded as a weak uniformity if we identify $\mathcal{U}$ with $w_{\mathcal{U}} = \{\mathcal{V} \in U(S)\colon \mathcal{V} \geqslant \mathcal{U}\}$.

With each weak uniformity w, there is an associated weakly uniformizable convergence structure q_w. If $\mathcal{U} \in w$, we denote by $t_{\mathcal{U}}$ the topology compatible with $\mathcal{U}$; then $q_w = \inf_c\{t_{\mathcal{U}}\colon \mathcal{U} \in w\}$.

Approaching from another direction, let q be a weakly uniformizable convergence structure, and let $[q]$ be the set of all weak uniformities compatible with q (i.e., $[q] = \{w\colon q_w = q\}$). We single out two members of $[q]$ of particular interest:

(1) $$w_q^* = \{\mathcal{U}_{\mathcal{F},x}\colon \mathcal{F}\ q\text{-converges to } x\};$$

(2) $$w_q' = \{\mathcal{U} \in U(S)\colon t_{\mathcal{U}} \geqslant q\}.$$

Remark. If a weak uniformity w is defined as a non-anti-residual set of uniformities on S, then it will be assumed without further comment that w includes those additional uniformities needed to satisfy the anti-residual property.

The following partial order relation among weak uniformities will be utilized: $w_1 \leqslant w_2$ means that for each $\mathcal{U}_2 \in w_2$ there is $\mathcal{U}_1 \in w_1$ such that $\mathcal{U}_1 \leqslant \mathcal{U}_2$. We now show that $[q]$ contains both a greatest and a least element.

THEOREM 4.1. *If $w \in [q]$, then $w_q' \leqslant w \leqslant w_q^*$.*

Proof. If $\mathcal{U} \in w$, then $\mathcal{U} \in w_q'$ is obvious, and thus $w_q' \leqslant w$. If $\mathcal{U}_q^* \in w_q^*$, then we can assume that there is a filter $\mathcal{F}$ q-converging to x such that $\mathcal{U} \geqslant \mathcal{U}_{\mathcal{F},x}$. Since $q = q_w$, there is $\mathcal{W} \in w$ such that $\mathcal{F}$ $t_{\mathcal{W}}$-converges to x. Thus, $\mathcal{F} \times \mathcal{F} \geqslant \mathcal{W}$, implying $\mathcal{U}_{\mathcal{F},x} \geqslant \mathcal{W}$, and $w \leqslant w_q^*$.

DEFINITION 4.2. Let w be a weak uniformity on S. $\mathcal{F}$ is a *w-Cauchy filter* if and only if there is $\mathcal{U}$ in w such that $\mathcal{F} \times \mathcal{F} \geqslant \mathcal{U}$.

DEFINITION 4.3. A weak uniformity w is *complete* if and only if each w-Cauchy filter q_w-converges to some point in S.

If q is in $W(S)$, then there is always at least one complete weak uniformity compatible with q, namely w_q^*. A uniformity, regarded as a convergence uniformity, is complete in the usual sense if and only if it is complete in the sense of the preceding definition.

The pair (S, w) consisting of a set S and a weak uniformity w on S will be termed a *weakly uniform space*. A definition of a completion for a weakly uniform space which generalizes the standard definition can be given in several different ways, from among which we choose the following.

DEFINITION 4.4. $(\hat{S}, \hat{w})$ is a *completion* of the weakly uniform space (S, w) if and only if there is a one-to-one function $\sigma\colon S \to \hat{S}$ with the following properties: (1) for each y in $\hat{S}$, there is a filter on $\sigma(S)$ which converges to y relative to a topology compatible with one of the uniformities in $\hat{w}$; (2) if $\mathcal{U} = \bigcap w$, and $\hat{\mathcal{U}} = \bigcap \hat{w}$, then $\sigma(\mathcal{U})$ coincides with the restriction of $\hat{\mathcal{U}}$ to $\sigma(S)$.

THEOREM 4.2. *Each weakly uniform space (S, w) has a completion.*

Proof. Let $w = \{\mathcal{U}_\alpha\colon \alpha \in \Lambda\}$, $\hat{S}$ be the set of all w-Cauchy filters, and $\tilde{S}_\alpha = \{\mathcal{F} \in \hat{S}\colon \mathcal{F} \times \mathcal{F} \geqslant \mathcal{U}_\alpha\}$. The uniformity $\tilde{\mathcal{U}}_\alpha$ on $\tilde{S}_\alpha$ is defined as follows: for each symmetric entourage U in $\mathcal{U}_\alpha$, let $\tilde{U} = \{(\mathcal{F}, \mathcal{G})\colon \mathcal{F}, \mathcal{G} \in \tilde{S}_\alpha$ and $U \in \mathcal{F} \times \mathcal{G}\}$; let $\tilde{\mathcal{U}}_\alpha$ be the uniformity generated by $\{\tilde{U}\colon U \in \mathcal{U}_\alpha\}$. For each α in Λ, $(\tilde{S}_\alpha, \tilde{\mathcal{U}}_\alpha)$ is a complete uniform space, and if $\sigma\colon S \to \tilde{S}_\alpha$ is defined by $\sigma(x) = \dot{x}$, for all α in Λ, then $\sigma(S)$ is dense in $\tilde{S}_\alpha$. (See Chapter 2, Section 3, Theorem 2, [1]). Next, let $\hat{\mathcal{U}}_\alpha = \dot{\Delta} \cap \tilde{\mathcal{U}}_\alpha$, where Δ is the diagonal of $\hat{S}$. We now regard σ as a mapping of S into $\hat{S}$. If w is generated by $\{\hat{\mathcal{U}}_\alpha\colon \alpha \in \Lambda\}$, then it is routine to verify that $(\hat{S}, \hat{w})$ is a completion of (S, w).

DEFINITION 4.5. A weak uniformity which is a dual ideal in the lattice $U(S)$ is called a *directed convergence uniformity*.

A convergence group which is also a pseudo convergence group is compatible with a directed convergence uniformity; this result is clear from the remark following Theorem 3.2. A directed convergence uniformity is a "uniform convergence structure" in the sense of Cook-Fischer [2]. Any convergence structure compatible with a directed convergence uniformity is T_2 whenever it is T_1.

From Theorem 2.1 it follows that every T_1 topology is a weakly uniformizable convergence structure. This may be compared with

THEOREM 4.3. *Every T_2 topology p is a convergence structure compatible with a directed convergence uniformity.*

Proof. Let $\mathfrak{U}_x = \dot{\Delta} \cap \big(\mathfrak{V}_p(x) \times \mathfrak{V}_p(x)\big)$, and $w = \{\mathfrak{U}_x\colon x \text{ in } S\}$. It is easily seen that w is a weak uniformity compatible with p. But the set w' of finite intersections of members of w is a directed convergence uniformity, and $w' \in [p]$.

Concluding remarks. A meaning has not yet been assigned to the term "convergence uniformity". It would seem appropriate to reserve this name for a weak convergence uniformity satisfying the following conditions: (1) A convergence group is a uniformizable convergence structure; (2) If a topology is uniformizable as a convergence structure, then it is uniformizable in the usual sense; (3) A T_2 uniform convergence space has a (unique?) Hausdorff completion. A definition that meets the first two conditions is the following: w is a convergence uniformity if and only if w is a weak uniformity and $\lambda(q_w)$ is uniformizable. I do not know of a definition that will satisfy all three conditions.

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Some relational systems and the associated topological spaces

by

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The aim of this paper is to investigate relational systems $\langle S, R\rangle$ (S is the field of a binary relation R), and associated algebras:

$$\mathcal{A}(S, R) = \langle \boldsymbol{P}(S), \cup, \cap, -, \mathrm{C}\rangle \tag{1}$$

where $\boldsymbol{P}(S)$ is the set of all subsets of S, $\langle \boldsymbol{P}(S), \cup, \cap, -\rangle$ is the Boolean algebra of subsets of S, and the operation C is defined on the elements of $\boldsymbol{P}(S)$ as follows:

$$\mathrm{C}X = \{y\colon \bigvee x\,(x \in X \wedge xRy)\}\,. \tag{2}$$

It is easy to see that if the relation R is a quasi-ordering, i.e. if it satisfies two conditions (see [1]):

$$\begin{aligned} &\text{a. } xRx && (\textit{reflexivity}), \\ &\text{b. } (xRy \wedge yRz) \rightarrow xRz && (\textit{transitivity}), \end{aligned} \tag{3}$$

then the algebra $\mathcal{A}(S, R)$ is a topological field of sets (this means that it satisfies the equalities: A. $X \subset \mathrm{C}X$; B. $\mathrm{C}(X \cup Y) = \mathrm{C}X \cup \mathrm{C}Y$; C. $\mathrm{CC}X = \mathrm{C}X$; D. $\mathrm{C}\emptyset = \emptyset$ ($\emptyset$ is the empty set)).

The purpose of these investigations is to characterize the topological fields of sets and related pseudo-Boolean algebras for some special relational systems, e.g. systems satisfying some additional equalities, having a logical meaning (cf. Theorem 1 and Corrolary 3).

This is a continuation of the well-known papers of Tarski and McKinsey [5] and Rasiowa and Sikorski [6].

1. Representation of totally distributive topological spaces. A topological space: $\langle \boldsymbol{P}(S), \cup, \cap, -, \mathrm{C}\rangle$ is *totally distributive* if and only if for every set $X \in \boldsymbol{P}(S)$

$$\mathrm{C}X = \bigcup_{x \in X} \mathrm{C}\{x\}\,. \tag{4}$$

Hence every finite topological space is totally distributive.