COROLLARY 1. *If S is a normal Moore space which contains no dense metric subspace, then S is a counterexample of Type* D.

Proof. This follows from Theorem 1 and from Moore's metrization theorem [4].

COROLLARY 2. *If S is a normal, nonmetrizable Moore space which is not a counterexample of Type* D, *then S contains a dense, nonmetrizable subspace S' such that $S = D+M$, where D is a metrizable domain dense in S', M is the boundary in S' of D.*

Proof. Since S is not metrizable, there exists [1] a discrete collection I of mutually exclusive closed point sets not satisfying the definition of collectionwise normality with respect to any collection of domains. Denote by I' the collection to which M' belongs if and only if, for some M in I, M' is $M-\mathrm{Int}\,M$ (M minus its interior). Then I' is a discrete collection of closed sets not satisfying the definition of collectionwise normality with respect to any collection of domains, and I'^* is closed and nowhere dense. From Theorem 1 there exists a development G for S such that $C(G)\cdot(S-I'^*)$ is dense in $S-I'^*$ and thus in S. Let $M = I'^*$, $D = (S-M)\cdot(C)$, and $S' = D+M$. Then D, M, and S' have the desired properties.

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A complete, infinitary axiomatization of weak second-order logic

by

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0. Introduction. One of the reasons (amongst others) why second-order logic(s) are studied is that it is possible to characterize (up to isomorphism) many mathematical structures. However for certain structures, e.g. the natural numbers, archimidean ordered fields, a logic weaker than second-order (hence the name *weak second-order*) suffices. In the weak second-order theories the second-order variables are understood as ranging over finite, non-empty sequences of those objects to which the first-order variables refer. The notion of consequence which is customarily used in weak second-order theories is a semantical (model-theoretic) one. As remarked by Montague in [M], the methods of Gödel [G] can be used to show that, as long as proofs are required to be of finite length, no complete formalization can be obtained for weak second-order logic. In this paper we present a formalization in which the proofs are of infinite length and which is complete. Furthermore, all but one of the rules of inference have finitely many premises and the rule with infinitely many premises is similar to the ω-rule (Carnap's rule). (1)

In the last section we comment briefly on how to obtain a more constructive axiomatization.

1. The weak second-order language. The language may be briefly defined as follows:

1.1. *Symbols.*

(a) Individual variables: $\mathrm{v}_0, \ldots, \mathrm{v}_n, \ldots$.
(b) Individual constants: $\mathrm{c}_0, \ldots, \mathrm{c}_n, \ldots$.
(c) Second-order variables: $\mathrm{V}_0, \ldots, \mathrm{V}_n, \ldots$.
(d) Predicate symbol (binary): P.

(1) The author is indebted to Professor Mostowski for raising the problem of finding a complete axiomatization for weak second-order logic.

(e) Logical symbols: $\neg$ (negation), $\vee$ (disjunction), $\doteq$ (equality), $*$ (concatenation), $\precsim$ and $\succsim$.

(f) Auxiliary symbols: (and).

The restriction to a single binary predicate symbol (and no function symbols) is clearly inessential.

1.2. *Terms*. The *first-order terms* are the individual variables and the individual constants. The *second-order terms* can be recursively defined as follows:

(a) A second-order variable is a second-order term,

(b) if T and T′ are second-order terms, then so is $(\mathrm{T}*\mathrm{T}')$,

(c) if t is a first-order term, then $\precsim \mathrm{t} \succsim$ is a second-order term.

A *closed term* is a term in which no variable occurs.

An *essentially first-order term* is a second-order term in which occur no second-order variables.

The *length of an essentially first-order term* T is the natural number lh(T) such that:

(a) $\mathrm{lh}(\precsim \mathrm{t} \succsim) = 1$,

(b) $\mathrm{lh}(\mathrm{T}*\mathrm{T}') = \mathrm{lh}(\mathrm{T}) + \mathrm{lh}(\mathrm{T}')$.

The *order of the terms in an essentially first-order term* T is the finite sequence of first-order terms Ord(T) such that

(a) $\mathrm{Ord}(\precsim \mathrm{t} \succsim) = \langle \mathrm{t} \rangle$.

(b) $\mathrm{Ord}(\mathrm{T}*\mathrm{T}') = \mathrm{Ord}(\mathrm{T})^\frown \mathrm{Ord}(\mathrm{T}')$. [2]

Two essentially first-order terms T *and* T′ *are equivalent*, in symbols: $\mathrm{T} \equiv \mathrm{T}'$, iff $\mathrm{Ord}(\mathrm{T}) = \mathrm{Ord}(\mathrm{T}')$.

We shall let "$\precsim \mathrm{t}_0, \ldots, \mathrm{t}_n \succsim$" be the term $\big(\ldots((\precsim \mathrm{t}_0 \succsim * \precsim \mathrm{t}_1 \succsim) * \ldots \precsim \mathrm{t}_n \succsim\big)$.

1.3. *Atomic formulas*. An expression is an *atomic formula* iff it is of one of the following forms:

(a) Ptt′ where t and t′ are first-order terms,

(b) $\mathrm{t} \doteq \mathrm{t}'$ where t and t′ are first-order terms,

(c) $\mathrm{T} \doteq \mathrm{T}'$ where T and T′ are second-order terms.

1.4. *Formulas*. The set of formulas is the least set $\mathcal{F}$ containing all the atomic formulas and such that

(a) if θ is in $\mathcal{F}$, then so is the negation of θ, $\neg\theta$,

(b) if θ and ψ are in $\mathcal{F}$, then so is the disjunction of θ and ψ, $(\theta \vee \psi)$,

(c) if θ is in $\mathcal{F}$ and x is an individual variable, then the x-universal generalization of θ, $\bigwedge \mathrm{x}\, \theta$, is also in $\mathcal{F}$,

(d) if θ is in $\mathcal{F}$ and X is a second-order variable, then the X-universal generalization of θ, $\bigwedge \mathrm{X}\, \theta$, is also in $\mathcal{F}$.

(2) We use the symbol $^\frown$ to denote the set-theoretical concatenation of finite sequences.

We abbreviate "$\bigwedge \mathrm{x}_0 \bigwedge \mathrm{x}_1 \ldots \mathrm{x}_n \theta$" to "$\bigwedge \mathrm{x}_0 \mathrm{x}_1 \ldots \mathrm{x}_n \theta$" and correspondingly for "$\bigwedge \mathrm{X}_0 \bigwedge \mathrm{X}_1 \ldots \mathrm{X}_n \theta$".

1.5. *Other syntactical notions*. We assume as understood the following syntactical concepts:

(a) A variable, or an occurrence of a variable, being a free (bound) in a formula.

(b) $\theta(\mathrm{b}_0/\xi_0, \ldots, \mathrm{b}_n/\xi_n)$ is the proper simultaneous substitution of the terms $\xi_0, \ldots, \xi_n$ for the free variables $\mathrm{b}_0, \ldots, \mathrm{b}_n$ in the formula θ (where it is further implicitly assumed that $\mathrm{b}_0, \ldots, \mathrm{b}_n$ are distinct variables and that for each $i \leqslant n$, b_i and ξ_i are either both first-order or both second-order terms.

(c) A formula θ is a sentence, i.e. has no free variables.

2. Semantics. A *relational system* is an ω-sequence $\langle \mathrm{A}, \mathrm{R}, \mathrm{a}_0, \ldots, \mathrm{a}_n, \ldots \rangle$ such that A is a non-empty set, R a subset of $\mathrm{A} \times \mathrm{A}$ and for each n a_n an element of A. A is the *universe* of $\langle \mathrm{A}, \mathrm{R}, \mathrm{a}_n \rangle_{n<\omega}$ and the a_n's are *distinguished elements of* $\langle \mathrm{A}, \mathrm{R}, \mathrm{a}_n \rangle_{n<\omega}$. If $\mathfrak{A}$ is a relational system (and from now on german script letters shall always refer to relational systems), then an $\mathfrak{A}$-assignment is an ordered pair $\langle \mathrm{s}, \mathrm{S} \rangle$ such that s is a mapping of the set of individual variables into the universe of $\mathfrak{A}$ and S is a mapping of the set of second-order variables into the set of finite, non-empty sequences of elements of the universe of $\mathfrak{A}$.

If $\mathfrak{A} = \langle \mathrm{A}, \mathrm{R}, \mathrm{a}_n \rangle_{n<\omega}$ and $\mathcal{S} = \langle \mathrm{s}, \mathrm{S} \rangle$ is an $\mathfrak{A}$-assignment then

(a) the $\mathcal{S}$-*value of a first-order term* t, $\mathrm{t}^{\mathfrak{A},\mathcal{S}}$, is defined as follows: (i) if $\mathrm{t} = \mathrm{c}_m$, then $\mathrm{t}^{\mathfrak{A},\mathcal{S}} = \mathrm{a}_m$ and (ii) if $\mathrm{t} = \mathrm{v}_m$, then $\mathrm{t}^{\mathfrak{A},\mathcal{S}} = \mathrm{s}(\mathrm{v}_m)$,

(b) the $\mathcal{S}$-*value of a second-order term* T, $\mathrm{T}^{\mathfrak{A},\mathcal{S}}$, is defined as follows: (i) if $\mathrm{T} = \mathrm{V}_m$, then $\mathrm{T}^{\mathfrak{A},\mathcal{S}} = \mathrm{S}(\mathrm{V}_m)$, (ii) if $\mathrm{T} = (\mathrm{T}_1 * \mathrm{T}_2)$, then $\mathrm{T}^{\mathfrak{A},\mathcal{S}}$ is the concatenation $\mathrm{T}_1^{\mathfrak{A},\mathcal{S}} {}^\frown \mathrm{T}_2^{\mathfrak{A},\mathcal{S}}$ of the finite sequences $\mathrm{T}_1^{\mathfrak{A},\mathcal{S}}$ and $\mathrm{T}_2^{\mathfrak{A},\mathcal{S}}$ and (iii) if $\mathrm{T} = \langle \mathrm{t} \rangle$, then $\mathrm{T}^{\mathfrak{A},\mathcal{S}}$ is the 1-sequence $\langle \mathrm{t}^{\mathfrak{A},\mathcal{S}} \rangle$.

The concept of an $\mathfrak{A}$-assignment $\langle \mathrm{s}, \mathrm{S} \rangle$ satisfying a formula θ in $\mathfrak{A}$, in symbols: $\models_{\mathfrak{A}} \theta[\mathrm{s}, \mathrm{S}]$, is an obvious modification of the usual concept of satisfaction and hence is omitted. We shall also omit the proof of the usual properties of the satisfaction relation (e.g. that it only depends on the value assigned to the free variables, etc.). A formula θ is *valid* (or *true*) *in* $\mathfrak{A}$, $\models_{\mathfrak{A}} \theta$, just in case it is satisfied by every $\mathfrak{A}$-assignment; θ is *valid* iff it is valid in every relational system.

3. Proof-theory. We shall first give a Gentzen type proof theory. After we have shown it to be complete (i.e. that every valid formula is provable) we shall briefly consider a Hilbert type axiomatization.

The papers of Kanger [K1], Montague [M] and Rasiowa–Sikorski [R/S] were extremely helpful in determining the axiomatizations given below.

3.1. *Disjuncts.* By a *disjunct* we understand a finite (possibly empty) sequence of formulas. Capital greek letters shall be used to denote disjuncts. If $\Gamma = \langle\theta_0, \dots, \theta_k\rangle$ is a disjunct, then an $\mathfrak{A}$-assignmnet $\langle s, S\rangle$ satisfies Γ in $\mathfrak{A}$, in symbols: $\models_{\mathfrak{A}} \Gamma[s, S]$ iff $\models_{\mathfrak{A}} \theta_0 \vee \dots \vee \theta_k [s, S]$. The notions of Γ is valid (or true) in $\mathfrak{A}$ and Γ is valid are correspondingly defined.

Γ, θ, Δ is the disjunct $\Gamma^\frown\langle\theta\rangle^\frown\Delta$. Similarly for $\Gamma, \theta, \Delta, \psi$.

An *elementary formula* is either an atomic formula or the negation of an atomic formula.

An *essentially first-order formula* is a formula which does not contain any second-order variables.

An *essentially first-order equation* is an atomic formula of the form $T \doteq T'$ where T, T' are essentially first-order terms such that $\mathrm{lh}(T) = \mathrm{lh}(T')$.

An *indecomposable formula* is an elementary, essentially first-order formula which is neither an essentially first-order equation, nor the negation of an essentially first-order equation.

An *indecomposable disjunct* is a disjunct consisting of indecomposable formulas.

3.2. *Fundamental disjuncts.* A disjunct Γ is a *fundamental disjunct* just in case that it satisfies at least one of the following conditions:

(a) Γ contains a formula θ and its negation $\neg\theta$,

(b) Γ contains a set of first-order elementary formulas whose disjunction is a theorem of the first-order predicate calculus (with equality),

(c) Γ contains a formula of the form $\neg T \doteq T'$ where T, T' are essentially first-order terms such that $\mathrm{lh}(T) \neq \mathrm{lh}(T')$.

3.3. *Rules of inference.* We follow the convention that the premise(s) are placed above the conclusion in the statement of the rules of inference. Furthermore it is always assumed that the substitutions are proper and that:

(a) $x, y, z, \dots$ are first-order (i.e. individual) variables,

(b) $t, t', \dots$ are first-order terms,

(c) $X, Y, Z, \dots$ are second-order variables,

(d) $T, T', \dots$ are second-order terms.

3.3.1. *Rules of inference for the propositional connectives.*

(R.1) $$\frac{\Gamma, \theta, \Delta, \psi, \Xi}{\Gamma, (\theta\vee\psi), \Delta, \Xi}$$

(R.2) $$\frac{\Gamma, \neg\theta, \Delta \text{ and } \Gamma, \neg\psi, \Delta}{\Gamma, \neg(\theta\vee\psi), \Delta}$$

(R.3) $$\frac{\Gamma, \theta, \Delta}{\Gamma, \neg\neg\theta, \Delta}$$

3.3.2. *Rules of inference for the quantifiers (unrestricted).*

(R.4) $$\frac{\Gamma, \neg\theta(x/t), \Delta}{\Gamma, \neg\bigwedge y\, \theta(x/y), \Delta}$$

(R.5) $$\frac{\Gamma, \neg\theta(X/T), \Delta}{\Gamma, \neg\bigwedge Y\, \theta(X/Y), \Delta}$$

3.3.3. *Rules of inference for the quantifiers (restricted).*

(R.6) $$\frac{\Gamma, \theta(x/a), \Delta}{\Gamma, \bigwedge z\, \theta(x/z), \Delta}$$ provided that a is either an individual variable or constant and that it does not occur free in the conslusion

(R.7) $$\frac{\Gamma, \theta(X/Y), \Delta}{\Gamma, \bigwedge Z\, \theta(X/Z), \Delta}$$ provided that the second-order variable Y does not occur free in the conclusion.

3.3.4. *Rules of inference for equality.* For each natural number n the following are rules of inference:

(R.8) $$\frac{\{\Gamma, t_i \doteq t_i', \Delta : i \leqslant n\}}{\Gamma, T \doteq T', \Delta}$$

(R.9) $$\frac{\Gamma, \neg t_0 \doteq t_0', \dots, \neg t_n \doteq t_n', \Delta}{\Gamma, \neg T \doteq T', \Delta}$$

where T, T' are any essentially first-order terms such that $\mathrm{Ord}(T) = \langle t_0, \dots, t_n\rangle$ and $\mathrm{Ord}(T') = \langle t_0', \dots, t_n'\rangle$.

3.3.5. *Infinitary rule of inference.*

(R.10) $$\frac{\{\Gamma, \bigwedge x_0, \dots, x_n\, \theta(X/\prec x_0, \dots, x_n\succ), \Delta : n < \omega\}}{\Gamma, \bigwedge Y\, \theta(X/Y), \Delta}$$

3.3.6. *Structural rule of inference.*

$$\frac{\Gamma, \theta, \Delta, \theta, \Xi}{\Gamma, \theta, \Delta, \Xi}$$

3.4. Proofs. $\boldsymbol{FT}$ is the set of all finite sequences of natural numbers. We shall use the bold face letters $\boldsymbol{a}$ and $\boldsymbol{b}$ to denote elements of $\boldsymbol{FT}$.

$\boldsymbol{a} < \boldsymbol{b}$ iff $\boldsymbol{a}$ is an initial proper segment of $\boldsymbol{b}$. $\boldsymbol{a} \leqslant \boldsymbol{b}$ iff either $\boldsymbol{a} = \boldsymbol{b}$ or $\boldsymbol{a} < \boldsymbol{b}$. If $\boldsymbol{a} = \langle a_0, \dots, a_{n-1}\rangle$, then $\boldsymbol{a}, k = \langle a_0, \dots, a_{n-1}, k\rangle$. $\boldsymbol{0}$ is the empty sequence.

A subset $\boldsymbol{T}$ of $\boldsymbol{FT}$ is a *spread* just in case that $\boldsymbol{0} \in \boldsymbol{T}$ and if $\boldsymbol{a} \in \boldsymbol{T}$, then for all $\boldsymbol{b} < \boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{T}$.

A *branch* of a spread $\boldsymbol{T}$ is a maximal linearly ordered subset of $\boldsymbol{T}$. If the branch has a maximal element, then the maximal element is a *terminal node*.

A *tree* is a spread in which every branch is finite.

A *proof* is a function F whose domain is a tree, whose range is a set of disjuncts and such that:

(i) if a is a terminal node, then $F(a)$ is a fundamental disjunct, and

(ii) if a is not a terminal node, then $F(a)$ can be obtained from $\{F(b)\colon a < b \ \&$ for no c $(a < c < b)\}$ by an application of one of the rules of inference (**R.1**)–(**R.10**) and finitely many applications of the structural rule.

A disjunct Γ is *provable* if and only if there exists a proof F such that $F(0) = \Gamma$.

A formula θ is provable, in symbols: $\vdash \theta$, iff the disjunct $\langle\theta\rangle$ is provable.

A straightfoward induction shows that if a formula is provable then is valid.

4. Tableaux of formulas. As it usually happens with cut-free Gentzen type formalizations complications arise if a variable is allowed to occur both free and bound in a formula. Hence we shall prove the completeness of our system only for sentences (in order to prove the completeness for formulas we would need some rule of alphabetic variance). Furthermore we shall give the proof only for sentences in which occur no individual constants (it should be clear from our exposition how the proof should be modified for sentences involving individual constants or function symbols).

The method that we use to show the completeness is the standard way of associating with each sentence θ a spread S_θ and a function F_θ such that if S_θ is a tree (i.e. every branch is finite), then F_θ gives a proof of θ, while if S_θ is not a tree (i.e. it has an infinite branch) then from the range of F_θ we can define a relational system $\mathfrak{A}$ in which θ is not valid.

4.1. Definition. $\langle \mathrm{T}_n\rangle_{n<\omega}$ is an enumeration (without) repetition of all the essentially first-order terms of the form $\prec \mathrm{k}_0, \dots, \mathrm{k}_p \succ$ where p is a natural number and the k's are individual constants.

4.2. Definition. The function F_θ (where θ is a sentence in which no individual constant occurs) is inductively defined as follows (we shall omit the subscript θ for the remainder of the definition):

(i) $F(0) = \theta$,

(ii) Suppose that $F(b)$ is defined and that $F(b)$ is a disjunct in which no second-order variable occurs free. Then

Case 1. If $F(b)$ is a fundamental disjunct, then for all k, $F(b, k)$ is undefined.

Case 2. If $F(b)$ is not a fundamental disjunct nor an indecomposable disjunct. Then it must be of the form

$$\Gamma, \psi, \Delta$$

where Γ is an indecomposable disjunct (or empty) and ψ is not an indecomposable formula. The definition then preceeds by cases depending on the form of ψ.

Case 2.1. $\psi = \mathrm{T} \doteq \mathrm{T}'$ where T, T$'$ are essentially first-order terms such that $\mathrm{lh}(\mathrm{T}) = \mathrm{lh}(\mathrm{T}')$. Let then $n = \mathrm{lh}(\mathrm{T})$, $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \dots, \mathrm{t}_{n-1}\rangle$ and $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}'_0, \dots, \mathrm{t}'_{n-1}\rangle$. Then let for each $i < n$, $F(b, i) = \Gamma, \mathrm{t}_i \doteq \mathrm{t}'_i, \Delta$ and for all $i \geqslant n$, $F(b, i)$ be undefined.

Case 2.2. $\psi = \neg \mathrm{T} \doteq \mathrm{T}'$ where T, T$'$ are essentially first-order terms such that $\mathrm{lh}(\mathrm{T}) = \mathrm{lh}(\mathrm{T}')$. Let then $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \dots, \mathrm{t}_{n-1}\rangle$, $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}'_0, \dots, \mathrm{t}'_{n-1}\rangle$ and then define $F(b, 0) = \Gamma, \neg \mathrm{t}_0 \doteq \mathrm{t}'_0, \dots, \neg \mathrm{t}'_{n-1} \doteq \mathrm{t}'_{n-1}, \Delta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.3. $\psi = (\alpha \vee \beta)$. Then let $F(b, 0) = \Gamma, \alpha, \beta, \Delta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.4. $\psi = \neg(\alpha \vee \beta)$. Then let $F(b, 0) = \Gamma, \neg\alpha, \Delta$ and $F(b, 1) = \Gamma, \neg\beta, \Delta$ and $F(b, k)$ be undefined for all $k > 1$.

Case 2.5. $\psi = \neg\neg\beta$. Then let $F(b, 0) = \Gamma, \beta, \Delta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.6. $\psi = \neg \bigwedge \mathrm{y}\, \beta$. Then let n be the least natural number such that for all $a \leqslant b$, $\neg\beta(\mathrm{y}/\mathrm{c}_n)$ does not occur in $F(a)$. Then let $F(b, 0) = \Gamma, \neg\beta(\mathrm{y}/\mathrm{c}_n), \Delta, \neg\bigwedge \mathrm{y}\, \beta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.7. $\psi = \neg \bigwedge \mathrm{Y}\, \beta$. Then let n be the least natural number such that for all $a \leqslant b$, $\neg\beta(\mathrm{Y}/\mathrm{T}_n)$ does not occur in $F(a)$. Then let $F(b, 0) = \Gamma, \neg\beta(\mathrm{Y}/\mathrm{T}_n), \Delta, \neg\bigwedge \mathrm{Y}\, \beta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.8. $\psi = \bigwedge \mathrm{y}\, \beta$. Then let n be the least natural number such that c_n does not occur in $F(b)$. Then let $F(b, 0) = \Gamma, \beta(\mathrm{y}/\mathrm{c}_n), \Delta$ and let $F(b, k)$ be undefined for all $k > 0$.

Case 2.9. $\psi = \bigwedge \mathrm{Y}\, \beta$. Then let for each natural number n, $F(b, n) = \Gamma, \bigwedge \mathrm{x}_0, \dots, \mathrm{x}_n \beta(\mathrm{Y}/\prec \mathrm{x}_0, \dots, \mathrm{x}_n \succ), \Delta$ where the x's are chosen so that there are no clashes of variables.

Case 3. If $F(b)$ is an indecomposable but not a fundamental disjunct, then let $F(b, 0) = F(b)$ and let $F(b, k)$ be undefined for all $k > 0$.

Definition. If F_θ is as defined in 4.2, then S_θ is the domain of F_θ.

4.4. Lemma. *S_θ is a spread and if S_θ is a tree (i.e. every branch is finite) then F_θ is a proof of θ.*

Proof. Immediate. In fact the rules of inference and the definition of F_θ were chosen so that the above statement holds.

4.5. Lemma. *If B is an infinite branch of S_θ and $\mathfrak{B}$ is the set of formulas occurring in the range of F_θ restricted to B, then*

(a) *if* T, T' *are essentially first-order terms such that* $\mathrm{T} \doteq \mathrm{T}' \in \mathfrak{B}$, $\mathrm{lh}(\mathrm{T}) = \mathrm{lh}(\mathrm{T}')$, $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \dots, \mathrm{t}_n\rangle$ *and* $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}'_0, \dots, \mathrm{t}'_n\rangle$, *then for some* $i \leqslant n$, $\mathrm{t}_i \doteq \mathrm{t}'_i \in \mathfrak{B}$.

(b) *if* T, T' *are essentially first-order terms such that* $\neg \mathrm{T} \doteq \mathrm{T}' \in \mathfrak{B}$, $\mathrm{lh}(\mathrm{T}) = \mathrm{lh}(\mathrm{T}')$, $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \ldots, \mathrm{t}_n\rangle$ *and* $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}_0', \ldots, \mathrm{t}_n'\rangle$, *then for all* $i \leqslant n$, $\neg \mathrm{t}_i \doteq \mathrm{t}_i' \in \mathfrak{B}$.

(c) *if* $(\alpha \vee \beta) \in \mathfrak{B}$, *then* $\alpha \in \mathfrak{B}$ *and* $\beta \in \mathfrak{B}$,

(d) *if* $\neg(\alpha \vee \beta) \in \mathfrak{B}$, *then either* $\neg \alpha \in \mathfrak{B}$ *or* $\neg \beta \in \mathfrak{B}$,

(e) *if* $\neg\neg \beta \in \mathfrak{B}$, *then* $\beta \in \mathfrak{B}$,

(f) *if* $\neg \bigwedge \mathrm{y}\, \beta \in \mathfrak{B}$, *then for all* n, $\neg \beta(\mathrm{y}/\mathrm{c}_n) \in \mathfrak{B}$,

(g) *if* $\neg \bigwedge \mathrm{Y}\, \beta \in \mathfrak{B}$, *then for all* n, $\neg \beta(\mathrm{Y}/\mathrm{T}_n) \in \mathfrak{B}$,

(h) *if* $\bigwedge \mathrm{y}\, \beta \in \mathfrak{B}$, *then for some* n, $\beta(\mathrm{y}/\mathrm{c}_n) \in \mathfrak{B}$,

(i) *if* $\bigwedge \mathrm{Y}\, \beta \in \mathfrak{B}$, *then for some* n, $\beta(\mathrm{Y}/\mathrm{T}_n) \in \mathfrak{B}$,

(j) *if* $\beta \in \mathfrak{B}$, *then* $\neg \beta \notin \mathfrak{B}$,

(k) *if* $\beta_0, \ldots, \beta_n$ *are elementary first-order formulas whose disjunction is a theorem of the first-order predicate calculus, then* $\{\beta_0, \ldots, \beta_n\}$ *is not a subset of* $\mathfrak{B}$.

(l) *if* T, T' *are essentially first-order terms such that* $\mathrm{lh}(\mathrm{T}) \neq \mathrm{lh}(\mathrm{T}')$, *then* $\neg \mathrm{T} \doteq \mathrm{T}' \notin \mathfrak{B}$.

Proof. (a)–(h) are immediate from the definition of $\boldsymbol{F}_0$. To verify (i) it suffices to note that if $\bigwedge \mathrm{Y}\, \beta \in \mathfrak{B}$, then for some k, $\bigwedge \mathrm{x}_0, \ldots, \mathrm{x}_k\, \beta(\mathrm{Y}/\lessdot \mathrm{x}_0, \ldots, \mathrm{x}_k \gtrdot) \in \mathfrak{B}$. It then follows from (h) that for some n, $\beta(\mathrm{Y}/\mathrm{T}_n) \in \mathfrak{B}$. (j)–(l) hold because from the assumption that $\boldsymbol{B}$ is an infinite branch it follows that for all $\boldsymbol{b} \in \boldsymbol{B}$, $\boldsymbol{F}_0(\boldsymbol{b})$ is not a fundamental disjunct.

4.6. THEOREM. *If* $\boldsymbol{S}_\theta$ *has an infinite branch, then* θ *is not valid.*

Proof. Suppose that $\boldsymbol{B}$ is an infinite branch of $\boldsymbol{S}_\theta$ and that $\mathfrak{B}$ is the set of formulas occurring in the range of $\boldsymbol{F}_\theta$ restricted to $\boldsymbol{B}$. Then let

$$\mathrm{c}_n \sim \mathrm{c}_m \quad \text{iff either} \quad \neg \mathrm{c}_n \doteq \mathrm{c}_m \in \mathfrak{B} \quad \text{or} \quad \neg \mathrm{c}_m \doteq \mathrm{c}_n \in \mathfrak{B}\,.$$

Then define $\mathrm{c}_n \approx \mathrm{c}_m$ just in case that either $n = m$ or for some finite sequence $\mathrm{t}_0, \ldots, \mathrm{t}_p$ of individual constants we have that

$$\mathrm{c}_n = \mathrm{t}_0 \sim \mathrm{t}_1 \sim \ldots \mathrm{t}_{p+1} \sim \mathrm{t}_p = \mathrm{c}_m\,.$$

It is easily verifiable that $\approx$ is an equivalence relation on the set of individual constants. Hence we define

$$\bar{\mathrm{c}}_n = \{\mathrm{c}_m\colon\ \mathrm{c}_m \approx \mathrm{c}_n\}\,,$$

$$\mathrm{A} = \{\bar{\mathrm{c}}_n\colon\ n < \omega\}\,,$$

$$\mathrm{R} = \{\langle \bar{\mathrm{c}}_n, \bar{\mathrm{c}}_m\rangle\colon \text{ for some } \mathrm{c}_p, \mathrm{c}_q\ (\mathrm{c}_p \in \bar{\mathrm{c}}_n\ \&\ \mathrm{c}_q \in \bar{\mathrm{c}}_m\ \&\ \neg \mathrm{P}\mathrm{c}_p\mathrm{c}_q \in \mathfrak{B})\}\,,$$

$$\mathfrak{A} = \langle \mathrm{A}, \mathrm{R}, \bar{\mathrm{c}}_n\rangle_{n<\omega}\,.$$

Then we show, by induction on β, that for all $\beta \in \mathfrak{B}$, $\models_{\mathfrak{A}} \neg \beta$.

Case 1. $\beta = \mathrm{c}_n \doteq \mathrm{c}_m \in \mathfrak{B}$. Then in order to show that $\models_{\mathfrak{A}} \neg \beta$ it suffices to show that $\bar{\mathrm{c}}_n \neq \bar{\mathrm{c}}_m$, i.e. that $\mathrm{c}_n \not\approx \mathrm{c}_m$. But since we are assuming that $\mathrm{c}_n \doteq \mathrm{c}_m \in \mathfrak{B}$, if $\mathrm{c}_n \approx \mathrm{c}_m$, then 4.5(k) would be violated.

Case 2. $\beta = \neg \mathrm{c}_n \doteq \mathrm{c}_m \in \mathfrak{B}$. Then $\mathrm{c}_n \sim \mathrm{c}_m$ and hence $\bar{\mathrm{c}}_n = \bar{\mathrm{c}}_m$. Therefore $\models_{\mathfrak{A}} \neg \beta$.

Case 3. $\beta = \mathrm{T} \doteq \mathrm{T}' \in \mathfrak{B}$ where T, T' are essentially first-order terms such that $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \ldots, \mathrm{t}_n\rangle$ and $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}_0', \ldots, \mathrm{t}_n'\rangle$ (i.e. $\mathrm{lh}(\mathrm{T}) = \mathrm{lh}(\mathrm{T}')$). Then by 4.5(a) there exist $i \leqslant n$ such that $\mathrm{t}_i \doteq \mathrm{t}_i' \in \mathfrak{B}$. Hence by case 1, $\models_{\mathfrak{A}} \neg \mathrm{t}_i \doteq \mathrm{t}_i'$. Thus $\bar{\mathrm{t}}_i \neq \bar{\mathrm{t}}_i'$ and hence by the properties of the satisfaction relation $\models_{\mathfrak{A}} \neg \mathrm{T} \doteq \mathrm{T}'$, i.e. $\models_{\mathfrak{A}} \neg \beta$.

Case 4. $\beta = \neg \mathrm{T} = \mathrm{T}' \in \mathfrak{B}$, where T, T' are essentially first-order terms such that $\mathrm{Ord}(\mathrm{T}) = \langle \mathrm{t}_0, \ldots, \mathrm{t}_n\rangle$, $\mathrm{Ord}(\mathrm{T}') = \langle \mathrm{t}_0', \ldots, \mathrm{t}_n'\rangle$. Then by 4.5(b) for all $i \leqslant n$, $\neg \mathrm{t}_i \doteq \mathrm{t}_i' \in \mathfrak{B}$. Hence by case 1, $\bar{\mathrm{t}}_i = \bar{\mathrm{t}}_i'$ and then $\models_{\mathfrak{A}} \mathrm{T} \doteq \mathrm{T}'$, i.e. $\models_{\mathfrak{A}} \neg \beta$.

Case 5. $\beta = \mathrm{T} \doteq \mathrm{T}' \in \mathfrak{B}$ where T, T' are essentially first-order terms such that $\mathrm{lh}(\mathrm{T}) \neq \mathrm{lh}(\mathrm{T}')$. Then because of the definition of satisfaction we have that $\models_{\mathfrak{A}} \neg \mathrm{T} \doteq \mathrm{T}'$, i.e. $\models_{\mathfrak{A}} \neg \beta$.

Case 6. $\beta = \neg \mathrm{T} \doteq \mathrm{T}' \in \mathfrak{B}$, where T, T', are essentially first-order terms such that $\mathrm{lh}(\mathrm{T}) \neq \mathrm{lh}(\mathrm{T}')$. However, because of 4.5(l), this case cannot arise.

The remaining cases follow immediately from lemma 4.5(c)–(k).

Since $\theta \in \mathfrak{B}$, we have then shown that $\models_{\mathfrak{A}} \neg \theta$ and hence that θ is not valid.

Combining 4.4 and 4.6 we then obtain:

4.7. THEOREM. *A sentence* θ *(in which occur no individual constants) is valid if and only if it is provable.*

5. Another axiomatization for weak second-order logic.

In this section we mention a Hilbert type axiomatization of weak second-order logic.

5.1. *Axioms* [3]. We use '$\theta \Rightarrow \psi$' as an abbreviation for '$(\neg \theta \vee \psi)$'.

(A.1) *Any substitution instance of a tautology* (of the propositional calculus).

(A.2) $\bigwedge \mathrm{x}\, \theta \Rightarrow \theta(\mathrm{x}/\mathrm{t})$.

(A.3) $\bigwedge \mathrm{x}\, (\theta \vee \psi) \Rightarrow (\theta \vee \bigwedge \mathrm{x}\, \psi)$, provided x is not free in θ.

(A.4) $\bigwedge \mathrm{X}\, \theta \Rightarrow \theta(\mathrm{X}/\mathrm{T})$.

(A.5) $\bigwedge \mathrm{X}\, (\theta \vee \psi) \Rightarrow (\theta \vee \Rightarrow \bigwedge \mathrm{X}\, \psi)$, provided that X is not free in θ.

(A.6) Axioms for the equality symbol (i.e. that $\doteq$ is a congruence relation relative to both type of variables).

[3] These axioms are adapted from Montague [M].

(A.7) $\bigwedge x \bigvee X \precsim x \succsim \doteq X$ (where '$\bigvee$' is an abbreviation for '$\neg \bigwedge \neg$').

(A.8) $\bigwedge X \bigwedge Y \bigvee Z\, (X^*Y) \doteq Z$.

(A.9) $\bigwedge X \bigwedge Y \bigwedge Z\, \big((X^*Y)^*Z\big) \doteq \big(X^*(Y^*Z)\big)$.

(A.10) $\bigwedge X \bigwedge Y \bigwedge Z\, \big((X^*Y) \doteq (X^*Z) \Rightarrow Y \doteq Z\big)$.

(A.11) $\bigwedge X \bigwedge Y \bigwedge Z\, \big((Y^*X) \doteq (Z^*X) \Rightarrow Y \doteq Z\big)$.

(A.12) $\bigwedge x \bigwedge y \bigwedge X \bigwedge Y\, \big((X^* \precsim x \succsim) \doteq (Y^* \precsim y \succsim) \Rightarrow x \doteq y\big)$.

(A.13) $\bigwedge x \bigwedge y \bigwedge X \bigwedge Y\, \big((\precsim x \succsim {}^*X) \doteq (\precsim y \succsim {}^*Y) \Rightarrow x \doteq y\big)$.

5.2. *Rules of inference.* Modus Ponens (Detachment). Universal generalization with respect to both variables and the following infinitary rule:

From $\{\bigwedge x_0, \ldots, x_n\, \theta(X/\precsim x_0, \ldots, x_n \succsim)\colon n < \omega\}$.
To obtain $\bigwedge X\, \theta$.

5.3. *Completeness.* To prove the completeness of the axiomatization one can prooceed as follows:

(a) Verify that all the axioms are valid, and that the rule of inference preserve validity.

(b) Verify that if $\langle \theta_0, \ldots, \theta_k \rangle$ is a fundamental disjunct, then $\theta_0 \vee \ldots \vee \theta_k$ is provable from the above axioms and rules of inference.

(c) Use induction on the "length" of the tree to show that if Γ is a provable disjunct (in the sense of section 4), then the disjunction of the formulas in Γ is provable (in the sense of section 5).

Once (a)–(c) have been verified the completeness of the Gentzen type axiomatization proves the completeness of the above axiomatization.

6. A more constructive axiomatization. In the case of first-order number theory Shoenfield has shown that the ω-rule may be replaced by a more constructive rule whose content is: to obtain $\bigwedge x\, \theta$ from $\{\theta(\boldsymbol{n})\colon n < \omega\}$ *provided* that there exists a recursive function f such that for all n, $f(n)$ is a Gödel number of the proof of $\theta(\boldsymbol{n})$. Corresponding result holds for the axiomatization given above for weak-second order. That is, the infinitary rule can be modified to read:

(S) To obtain $\bigwedge X\, \theta$ from $\{\bigwedge x_0, \ldots, x_n\, \theta(X/\precsim x_0, \ldots, x_n \succsim)\colon n < \omega\}$ *provided* that there is a recursive function f such that for each n $f(n)$ is a Gödel number of the proof $\bigwedge x_0, \ldots, x_n\, \theta(X/\precsim x_0, \ldots, x_n \succsim)$.

That such an axiomatization is complete can be shown in the following sequence of steps.

Step 1. Gödel numbers can be assigned to proofs using the restricted rule (S). For example, as done by Shoenfield in [S].

Step 2. There exists a partial recursive function f such that if $\ulcorner\theta\urcorner$ is the Gödel number of a sentence θ, then $f(\ulcorner\theta\urcorner)$ is the Gödel number of the partial recursive function $\boldsymbol{F}_\theta$ (of section 4, definition 4.2 and where Gödel numbers instead of formulas must be used). This step can be proved using the recursion theorem of Kleene [K2] because of the effective way in which the function $\boldsymbol{F}_\theta$ is constructed.

Step 3. If θ is a valid sentence, the using again the recursion theorem we can associate with each $\boldsymbol{b}$ element of $\boldsymbol{S}_\theta$ a natural number $g(\boldsymbol{b})$ such that $g(\boldsymbol{b})$ is Gödel number of the proof (using the restricted rule (S)) of the disjunction of the formulas occurring at $\boldsymbol{F}_\theta(\boldsymbol{b})$.

The above steps are straightforward enough, but do invole a great deal of arithmetization and thus it is probably best omitted.

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