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On the Grothendieck group of compact polyhedra

by

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1. Introduction. In an earlier note [3] we constructed a set of examples of the following phenomenon: X_1 and X_2 are compact connected polyhedra with isomorphic homology and homotopy groups but of different homotopy types. The demonstration fell into two parts. First it was shown that it is possible to construct polyhedra X_1, X_2 of different stable homotopy types such that $X_1+S\simeq X_2+S$, where S is a suitable sphere and $+$ denotes the disjoint union with base points identified. Secondly it was shown that if $X_1+A\simeq X_2+A$ for a suitable compact connected polyhedron A, then the suspensions of X_1 and X_2 have isomorphic homotopy groups, $\pi_i\Sigma X_1\cong\pi_i\Sigma X_2$.

In this paper we make a more systematic study of both parts of the argument and considerably strengthen the relevant statements. In section 2 we deal with the second part of the argument. We find it unnecessary to pass to the suspensions of X_1 and X_2 provided X_1, X_2, A are themselves already suspensions of connected polyhedra. This effects considerable improvement when it comes to finding examples. We also show that the homotopy groups kill the torsion in the Grothendieck group of suspensions of connected polyhedra. That is to say we may interpret the statement

$$\pi_i X_1\cong\pi_i X_2 \quad \text{if} \quad X_1+A\simeq X_2+A$$

as saying that π_i may be regarded as being defined on those elements of the Grothendieck group $G(\Sigma\mathcal{T}^1)$ of homotopy classes of suspensions of compact connected polyhedra which are represented by polyhedra; call this subset $G^+(\Sigma\mathcal{T}^1)$. Then we actually prove the statement

$$\pi_i X_1\cong\pi_i X_2 \quad \text{if} \quad tX_1+A\simeq tX_2+A \quad \text{for some integer } t>0;$$

that is, if X_1 and X_2 represent the same element of $G(\Sigma\mathcal{T}^1)$ modulo its torsion subgroup. Although this improvement is, at this stage, purely theoretical, it fits better into the general algebraic picture. For π_i maps $\Sigma\mathcal{T}^1$ to $\mathcal{A}b_0$, interpreted as the collection of isomorphism classes of finitely generated abelian groups. If we form the Grothendieck group of $\mathcal{A}b_0$

with respect to the direct sum operation then $G^+(\mathcal{A}b_0) = \mathcal{A}b_0$ and there is no torsion in $G(\mathcal{A}b_0)$; that is, if

$$tM_1 \oplus N \cong tM_2 \oplus N$$

for finitely generated abelian groups M, M_2, N, then $M_1 \cong M_2$ (Proposition 2.10). On the other hand

$$\pi_i(A+B) \not\cong \pi_i(A) \oplus \pi_i(B)\,, \tag{1.1}$$

in general, so our main theorem is not a *direct* consequence of the well-known property of $\mathcal{A}b_0$. Also we do not define a map

$$G\mathcal{F}^1 \to G(\mathcal{A}b_0)$$

extending π_i in view of (1.1). We are content to define

$$\pi_i\colon\ \bar{G}^+(\Sigma\mathcal{F}^1) \to \mathcal{A}b_0\,, \tag{1.2}$$

where $\bar{G}^+$ denotes the image of G^+ under the projection $G \to \bar{G} = G/T$, T being the torsion subgroup. We repeat that the improvement achieved in replacing G^+ by $\bar{G}^+$ in (1.2) is purely theoretical, since we do not even know whether they differ; we construct in section 3 examples where $X_1 \not\simeq X_2$ but $tX_1 \simeq tX_2$, $t>1$, but in our examples X_1 and X_2 represent the same element of $G(\Sigma\mathcal{F}^1)$.

We introduce a notational innovation in section 2. We are much concerned with the connectivity of the polyhedra entering into our discussion, but it is a source of numerical awkwardness that, for example, the n-sphere is $(n-1)$-connected and that every polyhedron is (-1)-connected. Moreover, if we take the smashed product of a k-connected and a l-connected polyhedron the result is $(k+l+1)$-connected. For these and other reasons we introduce the notion of *n-essential* polyhedra. A polyhedron is n-essential if all its non-vanishing homotopy groups are in dimensions $\geqslant n$. Thus every polyhedron is 0-essential and the ∞-essential polyhedra are the contractible polyhedra. Further we say that the *essentiality* of X is n, written $\operatorname{Ess} X = n$, where $0 \leqslant n \leqslant \infty$, if

$$n = \max\{m \mid X \text{ is } m\text{-essential}\}\,.$$

Of course, this is simply a device to raise the connectivity index by unity: X is n-essential if and only if it is $(n-1)$-connected.

In section 3 we refine and analyse the procedure for obtaining examples of polyhedra X_1 and X_2 such that $X_1+S \simeq X_2+S$ for a suitable sphere S. Freyd considered this phenomenon in [2]. He was there concerned with stable homotopy, whereas we are free to consider unstable phenomena. Nevertheless we base ourselves essentially on Freyd's construction procedure but, of course, use none of the arguments of [2].

Freyd observed that his example had the three properties (see last page of [2])

$$\begin{aligned} X_1+S^9 &\sim X_2+S^9\,,\\ X_1+S^5 &\sim X_2+S^5\,,\\ 2X_1 &\sim 2X_2\,, \end{aligned} \tag{1.3}$$

where $\sim$ is the stable homotopy relation. In fact, of course, they have the stronger properties (1) obtained by replacing $\sim$ by the homotopy relation $\simeq$. We show that the three properties (1.3) of Freyd's example reflect increasingly special features of that example. Specifically, Freyd takes $\alpha \in \pi_8(S^5)$ of order 8, $\beta = 3\alpha$, $X_1 = S^5 \cup_\alpha e^9$, $X_2 = S^5 \cup_\beta e^9$. Then the first property in (1.3) is just due to the fact that 3 is prime to 8, the second also uses the fact that α is a suspension element, and the third further exploits the relation $3^2 \equiv 1 \bmod 8$. Our theorems in section 3 are concerned with hypotheses on $\alpha \in \pi_{m-1}(S^n)$ and $\beta = l\alpha$ validating conclusions which generalize each of the three properties in (1.3). We are thus able to provide an 8-dimensional example of two suspension polyhedra X_1 and X_2 such that

$$\begin{aligned} X_1+S^8 &\simeq X_2+S^8\,,\\ X_1+S^4 &\simeq X_2+S^4\,,\\ 2X_1 &\simeq 2X_2\,, \quad X_1 \not\simeq X_2\,, \end{aligned} \tag{1.4}$$

whence, by our main theorem, $\pi_i X_1 \cong \pi_i X_2$, all i.

Moreover, no example of lower dimension possessing such properties can be constructed by the procedure of this section. We can construct, however, an example in dimension 7 having all the properties of example (1.4) except that it is no longer concerned with two *suspension* polyhedra. In fact, we take desuspensions of the polyhedra X_1, X_2 of (1.4). Precisely let $\alpha \in \pi_6(S^3)$ be the generator and let $\beta = 5\alpha$, $Y_1 = S^3 \cup_\alpha e^7$, $Y_2 = S^3 \cup_\beta e^7$. Then one proves by a refinement of the arguments in section 3 (see Remark (1) at the end of that section) that

$$\begin{aligned} Y_1+S^7 &\simeq Y_2+S^7\,,\\ Y_1+S^3 &\simeq Y_2+S^3\,,\\ 2Y_1 &\simeq 2Y_2\,, \quad Y_1 \not\simeq Y_2; \end{aligned} \tag{1.5}$$

and the polyhedra X_1, X_2 of (1.4) are the suspensions $X_1 = \Sigma Y_1$, $X_2 = \Sigma Y_2$. However, we cannot deduce from (1.5) that $\pi_i Y_1 \cong \pi_i Y_2$ since Y_1 and Y_2 are not suspensions. It would indeed be interesting to know whether Y_1

(1) This strengthening did not concern Freyd in [2], being quite irrelevant to the topic of that paper.

and Y_2 do have isomorphic homotopy groups; one may prove by classical homotopy arguments that their homotopy groups are indeed isomorphic up to and including dimension 10.

In section 3 we are much concerned with maps of the form

$$\underbrace{S^p+\ldots+S^p}_{t\text{ copies}} \to \underbrace{S^q+\ldots+S^q}_{t\text{ copies}}$$

or, as we may write it,

$$f\colon tS^p \to tS^q .$$

Moreover, these maps will involve no "cross-terms" but will be completely described by t^2 maps γ_{ij} specifying how the ith sphere in the domain is mapped to the jth sphere in the range. It is then natural to express the map (or its homotopy class) by a $(t\times t)$-matrix $\Gamma=(\gamma_{ij})$ with entries in $\pi_p(S^q)$. In particular, if $p=q$ then Γ is a matrix over the integers. It is then a very convenient feature of the matrix notation that if $g\colon tS^q\to tS^r$ is represented by the matrix $\Delta=(\delta_{ij})$ of elements in $\pi_q(S^r)$, then the composite map from tS^p to tS^r is represented by the product matrix $\Gamma\Delta$ if the γ_{ij} are suspensions, *provided we interpret $\gamma\delta$ to mean "first γ, then δ". We will therefore adopt this convention in section* 3. Maps will be written "on the right" so that we may say, in the discussion above, that fg is represented by $\Gamma\Delta$. We will adopt the same convention even if the γ_{ij} are not suspensions but then we may only understand by $\Gamma\circ\Delta$ the composition of homotopy classes represented by Γ, Δ and we may not compute $\Gamma\circ\Delta$ as the matrix product $\Gamma\Delta$. We remark that this convention has the consequence that, for any integer l, $l\circ\alpha$ is always $l\alpha$

$$S^p \xrightarrow{l} S^p \xrightarrow{\alpha} S^q$$

but $\alpha\circ l$ is not always $l\alpha$,

$$S^p \xrightarrow{\alpha} S^q \xrightarrow{l} S^q .$$

This seems much preferable to the situation which arises with the opposite convention.

We should also draw attention to the conventions adopted in the interests of brevity. Thus in section 2 we write $X\in\mathfrak{I}$ to mean that X is a compact polyhedron, although, in fact, $\mathfrak{I}$ is the collection of homotopy classes of compact polyhedra. Similarly, in Section 3 (and in the previous paragraph) we blur the distinction between maps and their homotopy classes so that, for example, we talk of the polyhedron $S^n\cup_\alpha e^m$ where $\alpha\in\pi_{m-1}(S^n)$. We believe this abuse of language will lead to no confusion and, in attempting to avoid it, we would be led into unnecessarily fussy formulations.

2. The main theorem. Let $\mathfrak{I}_{\mathrm{top}}$ be the collection of based homeomorphism classes of based compact polyhedra, and let $\mathfrak{I}^n_{\mathrm{top}}$ be the subcollection of n-essential polyhedra, $0\leqslant n\leqslant\infty$. We may introduce into $\mathfrak{I}_{\mathrm{top}}$ the structure of an additive commutative semigroup with zero by means of the wedge operation

$$P+Q=P\vee Q; \tag{2.1}$$

the zero is, of course, just the one-point space. Then

$$\mathrm{Ess}(P+Q)=\min(\mathrm{Ess}\,P,\mathrm{Ess}\,Q)\,, \tag{2.2}$$

so that $\mathfrak{I}^n_{\mathrm{top}}$ is a subsemigroup of $\mathfrak{I}_{\mathrm{top}}$; in fact, we have a filtration

$$\mathfrak{I}^\infty_{\mathrm{top}}\subseteq\ldots\subseteq\mathfrak{I}^n_{\mathrm{top}}\subseteq\mathfrak{I}^{n-1}_{\mathrm{top}}\subseteq\ldots\subseteq\mathfrak{I}^1_{\mathrm{top}}\subseteq\mathfrak{I}^0_{\mathrm{top}}=\mathfrak{I}_{\mathrm{top}}\,. \tag{2.3}$$

Let $\mathfrak{I}$ denote the collection of based homotopy classes ([2]) of based compact polyhedra. The addition (2.1) passes from $\mathfrak{I}_{\mathrm{top}}$ to $\mathfrak{I}$ so that $\mathfrak{I}$ acquires the structure of an additive commutative quotient semigroup of $\mathfrak{I}_{\mathrm{top}}$. Moreover, n-essentiality is a homotopy invariant so that we get a natural quotient filtration of (2.3)

$$0=\mathfrak{I}^\infty\subseteq\ldots\subseteq\mathfrak{I}^n\subseteq\mathfrak{I}^{n-1}\subseteq\ldots\subseteq\mathfrak{I}^1\subseteq\mathfrak{I}^0=\mathfrak{I}\,. \tag{2.4}$$

Let Σ denote the reduced suspension operation. Then Σ may be regarded as operating on $\mathfrak{I}_{\mathrm{top}}$ or $\mathfrak{I}$. We observe

PROPOSITION 2.5.

(i) *Σ_{top} is an endomorphism of the semigroup $\mathfrak{I}_{\mathrm{top}}$ and $\Sigma(\mathfrak{I}^n_{\mathrm{top}})\subseteq\mathfrak{I}^{n+1}_{\mathrm{top}}$, $n\geqslant 0$.*

(ii) *Σ is an endomorphism of the semigroup $\mathfrak{I}$ and $\Sigma(\mathfrak{I}^n)\subseteq\mathfrak{I}^{n+1}$, $n\geqslant 0$.*

Let $G(\mathfrak{I})$ be the Grothendieck group of $\mathfrak{I}$ with respect to the addition (2.1); we remark that this addition is, in fact, the coproduct (= sum) in the appropriate topological or homotopical category. Let

$$\varrho\colon\mathfrak{I}\to G(\mathfrak{I}) \tag{2.6}$$

be the natural map; we showed in [3] and will show again in section 3 of this paper that ϱ is not one-one. Let $T(\mathfrak{I})$ be the torsion subgroup of $G(\mathfrak{I})$ and let $\bar G(\mathfrak{I})$ be the quotient group

$$\bar G(\mathfrak{I})=G(\mathfrak{I})/T(\mathfrak{I})\,. \tag{2.7}$$

Then ϱ induces, by composition with the natural projection, the map

$$\bar\varrho\colon\mathfrak{I}\to\bar G(\mathfrak{I})\,.$$

Let

$$\bar G^+(\mathfrak{I})=\bar\varrho\mathfrak{I}\,. \tag{2.8}$$

([2]) For connected polyhedra, based homotopy type coincides with homotopy type.

We may, of course, and will apply definitions (2.6)-(2.8) to any subsemigroup of $\mathcal{T}$. Let $\mathcal{A}b$ denote the collection of isomorphism classes of abelian groups, and let $\mathcal{A}b_0$ be the subcollection consisting of finitely generated groups. Then $\mathcal{A}b$ is an additive commutative semigroup with zero under the direct sum operation

$$A+B = A\oplus B \tag{2.9}$$

and $\mathcal{A}b_0$ is a subsemigroup. We may form the Grothendieck groups $G(\mathcal{A}b)$, $G(\mathcal{A}b_0)$ and we remark, using the analogous definitions to (2.6)-(2.8),

PROPOSITION 2.10. $\bar{\varrho}\colon \mathcal{A}b_0 = \bar{G}^+(\mathcal{A}b_0)$.

In other words, if A_1, A_2, B are finitely generated abelian groups and if there exists an integer $t>0$ such that $tA_1+B \cong tA_2+B$, then $A_1\cong A_2$.

We now state our main theorem. Let π_i be the ith homotopy group functor [3], $i\geqslant 1$.

THEOREM 2.11. *The map* $\pi_i\colon \Sigma\mathcal{T}^1\to\mathcal{A}b$ *factors through* $\bar{\varrho}\colon \Sigma\mathcal{T}^1\to\bar{G}^+(\Sigma\mathcal{T}^1)$.

We remark that π_i is not a homomorphism of semigroups (that is, an additive functor). Were it so, then Theorem 2.11 would be a trivial consequence of Proposition 2.10, and the fact that π_i maps $\Sigma\mathcal{T}^1$ into $\mathcal{A}b_0$. Indeed, in that case, π_i would factor through $\bar{\varrho}\colon \Sigma\mathcal{T}^1\to\bar{G}(\Sigma\mathcal{T}^1)$ which is more than the theorem claims.

Naturally, the theorem admits an interpretation along the lines of Proposition 2.10: if [4] $X_1, X_2, P\in\Sigma\mathcal{T}^1$ and if there exists an integer $t>0$ such that the polyhedra tX_1+P, tX_2+P are of the same homotopy type, then

$$\pi_i(X_1)\cong\pi_i(X_2)\,,\quad \text{all } i\,. \tag{2.12}$$

We prepare the ground for the proof of the main theorem by introducing a ring structure into the Grothendieck group $G(\mathcal{T})$. This we do by taking the product operation in $\mathcal{T}_{\mathrm{top}}$ to be the so-called smashed product,

$$PQ = P\,\text{※}\,Q = P\times Q/P+Q\,. \tag{2.13}$$

Then

$$PQ = QP;\ Po=o\,,\ \text{where } o \text{ is the one-point space};\ PS^0 = P;$$
$$(PQ)R = P(QR); \tag{2.14}$$
$$P(Q+R) = PQ+PR\,.$$

[3] We may include the case $i=1$ as a trivial case; for $\Sigma\mathcal{T}^1\subset\mathcal{T}^2$, so that $\pi_1X=0$ for any $X\in\Sigma\mathcal{T}^1$.

[4] We allow ourselves to write $X\in\mathcal{T}$ if X is a based compact polyhedron.

Moreover, the smashed product is compatible with the homotopy relation, so that (2.14) holds in $\mathcal{T}$. It follows that this multiplication in $\mathcal{T}$ induces a unique ring structure in $G(\mathcal{T})$ such that ϱ is a multiplicative homomorphism. Thus $G(\mathcal{T})$ may henceforth be regarded as a ring; $\mathcal{T}$ itself we will call a semiring. Notice that $\mathcal{T}_{\mathrm{top}}$ and $G(\mathcal{T})$ contain a unity element 1 which is the zero-sphere. We next prove

PROPOSITION 2.15. $\mathrm{Ess}(PQ)\geqslant\mathrm{Ess}P+\mathrm{Ess}Q$.

Proof. The assertion is trivial if $\mathrm{Ess}P=0$, $\mathrm{Ess}Q=0$. Suppose $\mathrm{Ess}P>0$, $\mathrm{Ess}Q=0$. Then Q is the union of connected components $Q=Q_0\cup Q_1\cup\ldots\cup Q_n$, where Q_0 contains the base point, and so

$$PQ = PQ_0+P\times Q_1/o\times Q_1+\ldots+P\times Q_n/o\times Q_n\,.$$

It is easy to see that $\mathrm{Ess}(P\times Q_i)/(o\times Q_i)=\mathrm{Ess}P$, so that, in the light of (2.2), it only remains to establish the proposition when $\mathrm{Ess}P>0$, $\mathrm{Ess}Q>0$. The assertion is then trivial if P or Q is contractible so let us assume that $\mathrm{Ess}P=p<\infty$, $\mathrm{Ess}Q=q<\infty$; thus the first non-vanishing homotopy group of P is in dimension p, that of Q in dimension q. Then P is equivalent to a complex P_1 whose cells of lowest positive dimension are p-cells and Q is equivalent to a complex Q_1 whose cells of lowest positive dimension are q-cells, so that PQ is equivalent to the complex P_1Q_1 whose cells of lowest positive dimension are $(p+q)$-cells. Thus

$$\mathrm{Ess}PQ = \mathrm{Ess}P_1Q_1\geqslant p+q = \mathrm{Ess}P+\mathrm{Ess}Q\,.$$

Remark. We note that equality holds if $\mathrm{Ess}Q=0$, that is, $\mathrm{Ess}PQ=\mathrm{Ess}P$. On the other hand, equality need not hold if $\mathrm{Ess}P>0$, $\mathrm{Ess}Q>0$. For example, let P be a Moore space $K'(Z_m,p)$ and Q a Moore space $K'(Z_n,q)$, so that $\mathrm{Ess}P=p$, $\mathrm{Ess}Q=q$. Suppose that m, n are mutually prime. Then it is plain from the Künneth formula that the inclusion $P+Q\to P\times Q$ induces homology isomorphisms. By van Kampen's Theorem the quotient PQ is simply-connected and we have shown it has vanishing homology. Thus PQ is contractible. Thus there are, in fact, divisors of zero in $\mathcal{T}$.

COROLLARY 2.16. $\mathrm{Ess}\,\Sigma X\geqslant\mathrm{Ess}X+1$ (*equality can only fail if* $\mathrm{Ess}X=1$).

For $\Sigma X = SX$ where S is the circle.

COROLLARY 2.17. $\mathcal{T}^n$ *is an ideal in* $\mathcal{T}$, *for each* n, $0\leqslant n\leqslant\infty$.

(It is plain what is meant by an ideal in a semiring.)

COROLLARY 2.18. $G(\mathcal{T}^n)$ *is an ideal in* $G(\mathcal{T})$, *for each* n, $0\leqslant n\leqslant\infty$.

We now introduce a new equivalence relation into the collection of based compact polyhedra, broader than the homotopy relation. If Y_1, Y_2 are based compact polyhedra we write

$$Y_1\,\bar{\wedge}\,Y_2 \tag{2.19}$$

to means $\Sigma Y_1 \simeq \Sigma Y_2$, i.e., ΣY_1 and ΣY_2 represent the same element in $\mathfrak{I}$. With respect to this relation we may restate the main theorem as follows.

THEOREM 2.20. *Let Y_1, Y_2, Q be based compact connected polyhedra and let $t>0$ be an integer such that $tY_1+Q \barwedge tY_2+Q$. Then $\pi_i \Sigma Y_1 \cong \pi_i \Sigma Y_2$, all i.*

We will prove the main theorem in this form. We first enunciate some preliminary lemmas.

LEMMA 2.21. *If $Y_1 \barwedge Y_2$, $Z_1 \barwedge Z_2$, then $Y_1+Z_1 \barwedge Y_2+Z_2$, $Y_1Z_1 \barwedge Y_2Z_2$.*

Proof. It is plainly sufficient to take $Z_1 = Z_2 = Z$. Obviously $\Sigma Y_1 + \Sigma Z \simeq \Sigma Y_2 + \Sigma Z$ if $\Sigma Y_1 \simeq \Sigma Y_2$. Also $\Sigma(Y_1 Z) = (\Sigma Y_1)Z$ so $\Sigma(Y_1 Z) \simeq \Sigma(Y_2 Z)$ if $\Sigma Y_1 \simeq \Sigma Y_2$.

LEMMA 2.22. *Let $P_m(Y_1, Y_2) = 1$ if $m=1$; $P_m(Y_1, Y_2) = Y_1^{m-1} + Y_1^{m-2}Y_2 + \ldots + Y_2^{m-1}$, $m>1$. Then, if $tY_1+Q \barwedge tY_2+Q$,*

$$tY_1^m + QP_m(Y_1, Y_2) \barwedge tY_2^m + QP_m(Y_1, Y_2), \quad m \geqslant 1. \tag{2.23}$$

Proof. Evidently

$$P_{m+1}(Y_1, Y_2) = Y_1 P_m(Y_1, Y_2) + Y_2^m, \quad m \geqslant 1. \tag{2.24}$$

Thus we may prove (2.23) by induction on m, the case $m=1$ being just the hypothesis $tY_1+Q \barwedge tY_2+Q$. For if (2.23) holds for a particular value of m, then $tY_1^{m+1} + QP_{m+1}(Y_1, Y_2) = tY_1^{m+1} + Y_1 QP_m(Y_1, Y_2) + QY_2^m \barwedge tY_1 Y_m^2 + QY_2^m + Y_1 QP_m(Y_1, Y_2) \barwedge tY_2^{m+1} + QY_2^m + Y_1 QP_m(Y_1, Y_2) = tY_2^{m+1} + QP_{m+1}(Y_1, Y_2)$.

LEMMA 2.25. *Let $Q_j \in \mathfrak{I}^1$, $j=1, \ldots, s$. Then*

$$\pi_i \Sigma(Q_1 + \ldots + Q_s) \cong \bigoplus_{j=1}^{s} \pi_i \Sigma Q_j \oplus \bigoplus_k \pi_i \Sigma T_k,$$

where each T_k is of the form

$$T_k = P_1^{m_{k1}} P_2^{m_{k2}} \ldots P_s^{m_{ks}}, \tag{2.26}$$

with at least two exponents strictly positive.

This is the so-called Hilton–Milnor formula (see Milnor [5]). The number of terms T_k of a particular form is given by the Witt number [5] but we need not make that explicit here.

Proof of Theorem 2.20. We may certainly assume $i \geqslant 2$ since for $i=1$ the assertion is trivial. Thus we choose an integer $i \geqslant 2$, which is arbitrary but which will remain fixed throughout the proof.

We first suppose that $Q = o$, so that $tY_1 \barwedge tY_2$. If $\operatorname{Ess} Y_1 \geqslant i$, then $\operatorname{Ess} tY_1 \geqslant i$, $\operatorname{Ess} \Sigma tY_1 \geqslant i+1$, $\operatorname{Ess} \Sigma tY_2 \geqslant i+1$, $\operatorname{Ess} \Sigma Y_2 \geqslant i+1$. Thus $\pi_i \Sigma Y_1 = \pi_i \Sigma Y_2 = 0$. We may therefore make a downward induction on $\operatorname{Ess} Y_1$.

By Lemma 2.22, if $tY_1 \barwedge tY_2$ then $tY_1^m \barwedge tY_2^m$, $m \geqslant 1$; but $\operatorname{Ess} Y_1^m > \operatorname{Ess} Y_1$ if $m \geqslant 2$, since $\operatorname{Ess} Y_1 \geqslant 1$, so, by our inductive hypothesis,

$$\pi_i \Sigma Y_1^m \cong \pi_i \Sigma Y_2^m, \quad m \geqslant 2. \tag{2.27}$$

We now invoke Lemma 2.25, with $s=t$ and all the Q_j equal. Then

$$\pi_i \Sigma t Y_1 \cong t\pi_i \Sigma Y_1 \oplus \bigoplus_k \pi_i \Sigma T_{1k}, \tag{2.28}$$

where T_{1k} is of the form

$$T_{1k} = Y_1^{m_k}, \quad m_k \geqslant 2.$$

Similarly

$$\pi_i \Sigma t Y_2 \cong t\pi_i \Sigma Y_2 \oplus \bigoplus_k \pi_i \Sigma T_{2k}, \tag{2.29}$$

with

$$T_{2k} = Y_2^{m_k}, \quad m_k \geqslant 2.$$

Since π_i maps $\Sigma\mathfrak{I}^1$ into $\mathcal{A}b_0$, the theorem follows in this case from (2.27), (2.28), (2.29) and Proposition 2.10.

We now revert to the general case $tY_1+Q \barwedge tY_2+Q$ but suppose $\operatorname{Ess} Q \geqslant i$. Since $t>0$, we have $t(Y_1+Q) \barwedge t(Y_2+Q)$ so that, by what we have already proved, $\pi_i \Sigma(Y_1+Q) \cong \pi_i \Sigma(Y_2+Q)$. But $\operatorname{Ess} \Sigma Q \geqslant i+1$, so that

$$\pi_i \Sigma Y_1 \cong \pi_i \Sigma(Y_1+Q) \cong \pi_i \Sigma(Y_2+Q) \cong \pi_i \Sigma Y_2.$$

We may therefore make a downward induction on $\operatorname{Ess} Q$. By (2.23),

$$tY_1^m Q^n + Q^{n+1} P_m(Y_1, Y_2) \barwedge tY_2^m Q^n + Q^{n+1} P_m(Y_1, Y_2), \quad m \geqslant 1, \quad n \geqslant 0,$$

so that, by our inductive hypothesis and Proposition 2.15,

$$\pi_i \Sigma Y_1^m Q^n \cong \pi_i \Sigma Y_2^m Q^n, \quad m \geqslant 1, \ n \geqslant 0, \ m+n \geqslant 2. \tag{2.30}$$

However, by Lemma 2.25,

$$\pi_i \Sigma(tY_1+Q) \cong t\pi_i \Sigma Y_1 \oplus \pi_i \Sigma Q \oplus \bigoplus_k \pi_i \Sigma T_{1k}, \tag{2.31}$$

where T_{1k} is of the form

$$T_{1k} = Y_1^{m_k} Q^{n_k}, \quad m_k \geqslant 1, \ n_k \geqslant 0, \ m_k + n_k \geqslant 2.$$

Similarly

$$\pi_i \Sigma(tY_2+Q) \cong t\pi_i \Sigma Y_2 \oplus \pi_i \Sigma Q \oplus \bigoplus_k \pi_i \Sigma T_{2k}, \tag{2.32}$$

with

$$T_{2k} = Y_2^{m_k} Q^{n_k}, \quad m_k \geqslant 1, \ n_k \geqslant 0, \ m_k + n_k \geqslant 2.$$

Since π_i maps $\Sigma\mathfrak{T}^1$ into $\mathcal{A}b_0$, the theorem now follows from (2.30), (2.31), (2.32) and Proposition 2.10.

COROLLARY 2.33. *Let $f(Z)$ be any polynomial in the indeterminate Z with coefficients in $\mathfrak{T}$ and constant term in $\mathfrak{T}^1$. Then if $tY_1+Q\barwedge tY_2+Q$, with $Y_1, Y_2, Q \in \mathfrak{T}^1$,*

$$\pi_i \Sigma f(Y_1) \cong \pi_i \Sigma f(Y_2)\,.$$

Proof. Write $f'(Y_1, Y_2)$ for the formal quotient $\dfrac{f(Y_1)-f(Y_2)}{Y_1-Y_2}$. Then it follows immediately from Lemma 2.22 that

$$tf(Y_1)+Qf'(Y_1, Y_2)\barwedge tf(Y_2)+Qf'(Y_1, Y_2)\,.$$

Moreover, $f(Y_1), f(Y_2) \in \mathfrak{T}^1$, since $Y_1, Y_2 \in \mathfrak{T}^1$ and the constant term in f is in $\mathfrak{T}^1$; and $Qf'(Y_1, Y_2)$ is in $\mathfrak{T}^1$ since Q is in $\mathfrak{T}^1$. Thus the corollary follows from the main theorem.

Indeed it is plain that more is true. If $f(Z_1, Z_2)$ is a polynomial in the indeterminates Z_1, Z_2 with coefficients in $\mathfrak{T}$ and constant term in $\mathfrak{T}^1$ and if $tY_1+Q\barwedge tY_2+Q$ with $Y_1, Y_2, Q \in \mathfrak{T}^1$, then

$$\text{(2.34)} \qquad \pi_i \Sigma f(Y_1, Y_2) \cong \pi_i \Sigma f(Y_2, Y_1)\,, \quad \text{all } i\,.$$

We have only to define, as in Corollary 2.33, a suitable $f'(Y_1, Y_2)$. This is done by linearity once we have defined it for $f(Z_1, Z_2) = Z_1^{m_1} Z_2^{m_2}$. Then

$$f'(Z_1, Z_2) = \begin{cases} 0 & \text{if} \quad m_1 = m_2\,, \\ Z_1^{m_1} Z_2^{m_1}(Z_2^s - Z_1^s)/(Z_2 - Z_1) & \text{if} \quad m_2 = m_1 + s\,,\ s \geqslant 1\,, \\ Z_1^{m_2} Z_2^{m_2}(Z_1^t - Z_2^t)/(Z_1 - Z_2) & \text{if} \quad m_1 = m_2 + t\,,\ t \geqslant 1\,. \end{cases}$$

THEOREM 2.35. *Let $Y_1, Y_2, Q \in \mathfrak{T}^1$ with $tY_1+Q\barwedge tY_2+Q$ for some $t > 0$ and let Y_1, Y_2 be H-spaces. Then*

$$\pi_i Y_1 \cong \pi_i Y_2\,, \quad \textit{all } i\,.$$

Proof. If Y is an H-space the join $Y*Y$ fibres over ΣY with fibre Y. Moreover, the fibre is contractible in $Y*Y$, and $Y*Y$ has the homotopy type of ΣY^2. Thus there is a direct sum decomposition

$$\pi_i \Sigma Y \cong \pi_i \Sigma Y^2 \oplus \pi_{i-1} Y\,.$$

We thus have, in our case,

$$\pi_i \Sigma Y_1 \cong \pi_i \Sigma Y_1^2 \oplus \pi_{i-1} Y_1\,, \quad \text{all } i\,,$$
$$\pi_i \Sigma Y_2 \cong \pi_i \Sigma Y_2^2 \oplus \pi_{i-1} Y_2\,, \quad \text{all } i\,.$$

But, by the main theorem and corollary, $\pi_i \Sigma Y_1 \cong \pi_i \Sigma Y_2$, $\pi_i \Sigma Y_1^2 \cong \pi_i \Sigma Y_2^2$ and all groups concerned lie in $\mathcal{A}b_0$. Thus Theorem 2.35 follows from Proposition 2.10.

Remarks (i) As we have already noted, for an additive functor $F\colon \Sigma\mathfrak{T}^1 \to \mathcal{A}b_0$, it is trivial that F factors through $\bar{G}(\Sigma\mathfrak{T}^1)$. Thus it follows that if h is any homology or cohomology theory with coefficients of finite type and if $X_1, X_2, P \in \Sigma\mathfrak{T}^1$ with $tX_1+P \simeq tX_2+P$, then $h(X_1) \cong h(X_2)$. Thus the hypotheses of the main theorem, in the form leading to the conclusion (2.12), also allow us to infer $h(X_1) \cong h(X_2)$. In other words, the polyhedra X_1 and X_2, while in general of different homotopy types, have isomorphic homotopy groups, homology groups and cohomology groups.

(ii) The argument holds provided only that the collection of equivalence classes of spaces forms a semiring, that $\pi_i \Sigma$ maps connected spaces to $\mathcal{A}b_0$, and that Lemma 2.25 holds. It is thus possible to replace $\mathfrak{T}$ by the collection of homotopy types of based polyhedra with countably many cells and finitely generated homology. This generalization may render Theorem 2.35 more interesting for we could then take Y_1, Y_2 to be loopspaces on 1-connected polyhedra.

3. The unstable Freyd construction. In this section we describe a simple way of constructing examples of polyhedra Y_1, Y_2 verifying the hypotheses of Theorem 2.20. Equivalently, we construct their suspensions X_1, X_2.

Let $\alpha \in \pi_{m-1}(S^n)$ be an element of finite order k, let l be prime to k and let $\beta = l\alpha$. Let

$$X_1 = S^n \cup_\alpha e^m\,, \qquad X_2 = S^n \cup_\beta e^m\,.$$

We show (compare [3])

THEOREM 3.1.

(i) $X_1 + S^m \simeq X_2 + S^m$;

(ii) $X_1 \simeq X_2$ *if and only if* (⁵) $\pm\beta = \alpha \circ (\pm 1)$.

Proof. (i) Consider $K = S^n \cup_\alpha e^m \cup_\beta e^m$. Since β is a multiple of α, $K \simeq (S^n \cup_\alpha e^m) + S^m = X_1 + S^m$. But since l is prime to k, α is a multiple of β so that $K \simeq X_2 + S^m$.

(ii) If $k = 1$, then $\alpha = \beta = 0$, $X_1 = X_2 = S^n + S^m$, so suppose $k > 1$. Then $m \geqslant n+2$, $n \geqslant 2$. Suppose $f\colon X_1 \simeq X_2$. Then we may assume that f maps S^n to S^n and, as such, has degree ± 1. Passing to quotients, f induces a map S^m to S^m, also of degree ± 1. Moreover, by a form of the Blakers-

(⁵) Recall that the composition $\alpha(\pm 1)$ means the homotopy class α *followed* by the class ± 1.

Massey-Theorem (see, e.g. p. 49 of [4]) it follows that there is a map $S^{m-1}\to S^{m-1}$ of degree ± 1 such that the diagram

$$\begin{tikzcd} S^{m-1} \arrow[r, "\alpha"] \arrow[d, "\pm 1"] & S^n \arrow[d, "\pm 1"] \\ S^{m-1} \arrow[r, "\beta"'] & S^n \end{tikzcd} \tag{3.2}$$

(homotopy) commutes; that is, $\pm\beta = \alpha(\pm 1)$.

Conversely, if $\pm\beta=\alpha(\pm 1)$, there are maps $S^{m-1}\to S^{m-1}$, $S^n\to S^n$ of degree ± 1 making (3.2) commutative and the induced map $X_1\to X_2$ is a homotopy equivalence.

Corollary 3.3. *Suppose, in addition, that α is a suspension. Then $X_1\simeq X_2$ if and only if $l\equiv\pm 1\ (\mathrm{mod}\, k)$.*

For then $X_1\simeq X_2$ if and only if $\beta=\pm\alpha$.

Theorem 3.4. *If α is a suspension then $X_1+S^n\simeq X_2+S^n$.*

Proof. Let

$$M=\begin{pmatrix} l & k \\ u & v\end{pmatrix}$$

be a unimodular matrix. Then M may be regarded as a map $S^n+S^n\to S^n+S^n$ and, as such, it is a homotopy equivalence. Let $(\alpha, 0)$ be the map attaching an m-cell to the first sphere in S^n+S^n by means of α. Then

$$(S^n+S^n)\cup_{(\alpha,0)} e^m \simeq (S^n+S^n)\cup_{(\alpha,0)M} e^m .$$

But since α is a suspension, $(\alpha,0)M=(\beta,0)$. Thus $X_1+S^n\simeq X_2+S^n$.

We move now towards a generalization of Corollary 3.3. Let $\alpha_i\in\pi_{m-1}(S^n)$ be of finite order k_i, $i=1,\dots,t$; let l_i be prime to k_i and let $\beta_i=l_i\alpha_i$. Let

$$X_{1i}=S^n\cup_{\alpha_i} e^m, \qquad X_{2i}=S^n\cup_{\beta_i} e^m . \tag{3.5}$$

Let $\boldsymbol{\alpha}$ be the $(t\times t)$-diagonal matrix with entries α_i down the diagonal and let $\boldsymbol{\beta}$ be similarly defined. Then Theorem 1 has the evident generalization.

Theorem 3.6.

(i) $X_{11}+\dots+X_{1t}+tS^m\simeq X_{21}+\dots+X_{2t}+tS^m$;

(ii) $X_{11}+\dots+X_{1t}\simeq X_{21}+\dots+X_{2t}$ *if and only if there are unimodular $(t\times t)$-matrices M and N such that $M\circ\boldsymbol{\beta}=\boldsymbol{\alpha}\circ N$.*

We now specialize the situation somewhat. Let $\alpha\in\pi_{m-1}(S^n)$ be a suspension element of order k and let $\alpha_i=a_i\alpha$, $\beta_i=b_i\alpha$ where a_i, b_i are

prime to k, $i=1,\dots,t$. Further, let $\gamma_j=c_j\alpha$, where c_j is prime to k, $j=1,\dots,u$ and set, in addition to (3.5),

$$Q_j=S^n\cup_{\gamma_j} e^m, \qquad j=1,\dots,u .$$

Let l_i satisfy $b_i\equiv l_i a_i\ (\mathrm{mod}\, k)$, $i=1,\dots,t$. We prove

Theorem 3.7. *The following statements are equivalent:*

(i) $l_1 l_2\dots l_t\equiv\pm 1\ (\mathrm{mod}\, k)$;

(ii) $X_{11}+\dots+X_{1t}\simeq X_{21}+\dots+X_{2t}$;

(iii) $X_{11}+\dots+X_{1t}+Q_1+\dots+Q_u\simeq X_{21}+\dots+X_{2t}+Q_1+\dots+Q_u$.

Proof. (i)$\Rightarrow$(ii): Let L_k be the matrix

$$L_k=\begin{pmatrix} l_1 & & & \\ & l_2 & & \\ & & \dots & \\ & & & l_t\end{pmatrix}$$

over Z_k. Since Z_k is a semilocal ring [6] and L_k is unimodular over Z_k, we may find a unimodular matrix L over Z which reduces to L_k over Z_k. The matrix L may be interpreted as a homotopy equivalence $L\colon tS^n\to tS^n$. Then, if C denotes the cone functor,

$$tS^n\cup_{\boldsymbol{\alpha}} CtS^{m-1}\simeq tS^n\cup_{\boldsymbol{\alpha}L} CtS^{m-1}.$$

But since α is a suspension, $\boldsymbol{\alpha}L=\boldsymbol{\beta}$, so that $X_{11}+\dots+X_{1t}\simeq X_{21}+\dots+X_{2t}$.

(ii)$\Rightarrow$(iii): Trivial.

(iii)$\Rightarrow$(i): By Theorem 3.6(ii) there are unimodular $(t+u)\times(t+u)$-matrices M and N such that, in an obvious notation,

$$M\begin{pmatrix}\boldsymbol{b}\alpha & 0\\ 0 & \boldsymbol{c}\alpha\end{pmatrix}=\begin{pmatrix}\boldsymbol{a}\alpha & 0\\ 0 & \boldsymbol{c}\alpha\end{pmatrix}N . \tag{3.8}$$

Since α is a suspension element. we may regard (3.8) as a matrix equality over Z_k. Taking determinants we have

$$\pm b_1 b_2\dots b_t c_1 c_2\dots c_u\equiv\pm a_1 a_2\dots a_t c_1 c_2\dots c_u\ (\mathrm{mod}\, k) .$$

Thus

$$l_1 l_2\dots l_t\equiv\pm 1\ (\mathrm{mod}\, k) .$$

We return now to the definition of X_1 and X_2 at the beginning of this section but assume now that α is a suspension element. We then have, as an immediate consequence of Theorem 3.7,

Corollary 3.9. *$tX_1\simeq tX_2$ if and only if $l^t\equiv\pm 1\ (\mathrm{mod}\, k)$. In particular,*

$$\varphi(k)X_1\simeq\varphi(k)X_2 ,$$

where φ is the Euler function.

---

[6] The author is indebted to S. U. Chase for this remark, simplifying an earlier argument.

The example given by Freyd consists essentially of taking $m = 9$, $n = 5$, α of order 8, $\beta = 3\alpha$. Then (see [4])

$$\begin{aligned}
&X_1 + S^9 \simeq X_2 + S^9 && \text{by Theorem 3.1(i)},\\
&X_1 + S^5 \simeq X_2 + S^5 && \text{by Theorem 3.4, since } \alpha \text{ is a suspension}.\\
&2X_1 \simeq 2X_2 && \text{by Corollary 3.9, since } 3^2 \equiv 1 \,(\mathrm{mod}\, 8).\\
&X_1 \not\simeq X_2 && \text{by Corollary 3.3},\\
&2X_1 \not\simeq X_1 + X_2 && \text{by Theorem 3.7}.
\end{aligned}$$

From any of the first three relations we infer, by the main theorem,

$$\pi_i X_1 = \pi_i X_2, \quad \text{all } i. \tag{3.10}$$

We may also construct examples by taking a prime p; then $m = n + 2p - 2$ in the stable range, α of order p, $\beta = l\alpha$ with $1 < l < p-1$. Then

$$\begin{aligned}
X_1 + S^{n+2p-2} &\simeq X_2 + S^{n+2p-2},\\
X_1 + S^{n} &\simeq X_2 + S^{n},
\end{aligned}$$

and if q is the smallest positive integer such that $l^q \equiv \pm 1 \,(\mathrm{mod}\, p)$, then $q|(p-1)$, $q > 1$, and

$$qX_1 \simeq qX_2,$$
$$rX_1 \not\simeq rX_2, \quad 1 \leqslant r < q.$$

Indeed qX_1, $(q-1)X_1 + X_2$, $(q-2)X_1 + 2X_2, \ldots, X_1 + (q-1)X_2$ all have different homotopy types. On the other hand they have isomorphic homotopy groups, since any two become homotopically equivalent on adding S^n. Again, (3.10) holds.

Let us now look for the example of lowest dimension which we may construct by the methods of this section. We require (i) that $\alpha \in \pi_{m-1}(S^n)$ be a suspension element of finite order k, (ii) that there exist l, prime to k, such that $l \not\equiv \pm 1 \,(\mathrm{mod}\, k)$. Then we wish m to be the smallest positive integer such that these requirements are fulfilled. Certainly $k > 2$ so that $m \geqslant n + 4$. Also $n \geqslant 3$ since $\pi_{m-1}(S^2)$ has no non-zero suspension elements. Thus we cannot do better than $n = 3$, $m = 7$. This, however, also fails, as does $n = 3$, $m = 8$ because there are then no suspension elements of order $k > 2$. However, we do find an example with $n = 4$, $m = 8$, and this example is unique up to homotopy type; namely we take $\alpha \in \pi_7(S^4)$ to be the suspension of a generator of $\pi_6(S^3)$. Then α is of order 12 and we take $\beta = 5\alpha$. We thus have the minimal example with respect to this construction, namely the suspension polyhedra X_1, X_2 where

$$X_1 = S^4 \cup_\alpha e^8, \quad X_2 = S^4 \cup_{5\alpha} e^8,$$

such that

$$\left.\begin{aligned}
X_1 + S^8 &\simeq X_2 + S^8\\
X_1 + S^4 &\simeq X_2 + S^4\\
2X_1 &\simeq 2X_2,
\end{aligned}\right\} \quad \text{so that } \pi_i X_1 \cong \pi_i X_2,$$
$$X_1 \not\simeq X_2.$$

Although this example is unstable in the sense that $\pi_7(S^4)$ is not in the stable range, it is evident nevertheless that X_1 and X_2 are of different S-types, for α suspends monomorphically, so that $\Sigma^j X_1 \not\simeq \Sigma^j X_2$ for all $j \geqslant 0$.

Remarks. (i) Except for Theorems 3.1 and 3.6, all the results of this section have involved the assumption that α is a suspension element; and our examples have also had α as a suspension element. This is necessary for the application of the main theorem since the polyhedra X_1 and X_2 must be suspensions (of Y_1 and Y_2 in the notation of Theorem 2.20). However, it is certainly too strong an assumption for the results of this section. For these results what is required is that the distributive law $\alpha(\gamma_1 + \gamma_2) = \alpha\gamma_1 + \alpha\gamma_2$ hold. In fact, we only need this when γ_1 and γ_2 are elements of $\pi_n(S_1^n)$, $\pi_n(S_2^n)$; but this is, actually, a characterization of those maps $\alpha \in \pi_{m-1}(S^n)$ which are homomorphic with respect to the comultiplications on S^{m-1}, S^n, or *H'-maps* (see [1]). For example, for any prime p, we may take α to generate Z_p in $\pi_{2p}(S^3)$. Then α is an H'-map but not a suspension. Corollary 3.3, Theorem 3.4, Theorem 3.7 and Corollary 3.9 remain valid if we replace the assumption that α be a suspension element by the weaker assumption that it be an H'-map. Even this weaker assumption is by no means always necessary. For example, if n is odd, then, for any integer s, and any $\alpha \in \pi_{m-1}(S^n)$,

$$\alpha s - s\alpha = H_0(\alpha)\frac{s(s-1)}{2}[\iota, \iota], \tag{3.11}$$

where H_0 is the (generalized) Hopf invariant and $[\iota, \iota] \in \pi_{2n-1}(S^n)$ is the Whitehead product element. Thus $\alpha s = s\alpha$ if and only if $H_0(\alpha)\dfrac{s(s-1)}{2}[\iota, \iota] = 0$. If n is odd then $2[\iota, \iota] = 0$ so that certainly $\alpha k = k\alpha$, $\alpha l = l\alpha$ if $k, l \equiv 0$ or $1 \bmod 4$. It may then be shown that the conclusion of Theorem 3.4 holds if n is odd, $m \leqslant 4n-2$, and $k, l \equiv 0$ or $1 \bmod 4$, with no further restriction on α. If $n = 3$ or 7, $m \leqslant 4n-2$, the conclusion holds with no further restriction on k, l because $[\iota, \iota] = 0$. It would thus be of interest to try to extend the main theorem to apply not just to $\Sigma\mathfrak{T}^1$ but perhaps even to the whole of $\mathfrak{T}^2$.

(ii) Our examples establish the fact that we can have polyhedra X_1, X_2 such that $X_1 \not\simeq X_2$ but $tX_1 \simeq tX_2$ for some $t > 1$. This, however, by no means implies that there is torsion in the Grothendieck group $G(\mathfrak{I})$. For, in all our examples, $X_1+P\simeq X_2+P$ for a suitable polyhedron P (indeed, a sphere). Thus the question of the existence of torsion in the Grothendieck group remains open.

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A minimal hyperdegree

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Two sets of natural numbers have the same hyperdegree if each is hyperarithmetic in the other. A non-hyperarithmetic set is said to have *minimal hyperdegree* if all sets of lower hyperdegree are hyperarithmetic. In this paper we construct a set that has minimal hyperdegree, and we study a certain class of models of the hyperarithmetic comprehension axiom. We draw upon ideas occurring in Spector's construction of a minimal degree of unsolvability [9] and in Feferman's application of forcing to analysis [2]. Our argument mixes Cohen's forcing method [1] with classical truth considerations; however, the use of forcing is not essential to the construction of a set of minimal hyperdegree. Instead of forcing with finite conditions in the style of Feferman [2], we force with infinite, hyperarithmetic conditions. As one might expect, forcing with infinite conditions is much closer to truth than forcing with finite conditions. A set generic with respect to our notion of forcing must necessarily have minimal hyperdegree.

All of our forcing is with respect to a second order language L(S), which is virtually isomorphic to Feferman's language L*(S) ([2], p. 335). L(S) is the language of first order number theory augmented by the constant symbol **S**, some second order variables, and the membership symbol ϵ. Let O_1 be a π_1^1 subset of O [4], the set of all notations for recursive ordinals, such that each recursive ordinal has precisely one notation in O_1 [3]; if b is the unique notation in O_1 for the recursive ordinal β, we write $|b|=\beta$. In addition, the relation $|b|<|c|$ is the restriction of some recursively enumerable relation to O_1. For each $b\,\epsilon\,O_1$, L(S) has set variables $X^b, Y^b, Z^b, \ldots$; L(S) also has set variables $X, Y, Z, \ldots$, number variables $x, y, z, \ldots$, a numeral $\bar{n}$ for each natural number n, and symbols for equality ($=$), successor ($'$), addition ($+$) and multiplication ($\cdot$).

For each $b\,\epsilon\,O_1$, the variable X^b is said to be *ranked*; the variable X is said to be *unranked*. A formula $\mathfrak{F}$ of L(S) is called *ranked* if every set

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