Toroidal decompositions of E^3 *

by

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1. Introduction. Some attention has been given to the problem of determining under what conditions certain upper semicontinuous decompositions of E^3 have decomposition spaces which are topologically equivalent to E^3. Some of the examples of decompositions given in the literature have been toroidal decompositions of E^3. See for example [3], [4], [8], and [10]. This paper considers such decompositions.

Much of the work included here is motivated by two papers of R. H. Bing, [4] and [8]. In [4] Bing constructs a homeomorphism between the sum of two solid Alexander horned spheres and a certain decomposition of S^3 which arises from a toroidal decomposition of E^3. He then shows that the sum of the solid horned spheres is homeomorphic with S^3 by showing that the toroidal decomposition of E^3 has a decomposition space which is homeomorphic with E^3. One might glance at his construction of the toroidal decomposition and describe it intuitively as being obtained by iterating an imbedding of two linked solid tori which circles a solid torus once. In [8] he iterates an imbedding of two linked solid tori which circles a solid torus twice to describe a decomposition of E^3 with a countable number of nondegenerate elements whose decomposition space differs topologically from E^3. In Sections 3-5 we shall study decompositions of E^3 which are suggested by these two examples.

Armentrout and Bing have recently described in [3] an example of a toroidal decomposition of E^3 into tame arcs and points whose decomposition space is not homeomorphic with E^3. The collection of tame arcs in this example is not a continuous collection. Keldyš has announced a proof in [12] that with toroidal decompositions of E^3 into tame arcs and points, the collection of arcs must fail to be continuous

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if the decomposition space is not homeomorphic with E^3. It is not known if there is a decomposition of E^3 into straight line intervals and points whose decomposition space is not homeomorphic with E^3. An example of a decomposition of E^3 into straight line intervals and points appears in [8]. Neither this example nor the one presented by McAuley in [15] is known to yield E^3. Each of these suggested examples is described by cubes with at least three handles rather than by solid tori. In Section 6 we show that toroidal decompositions of E^3 into straight line intervals and points have decomposition spaces which are homeomorphic with E^3.

Bing proves in [8] that the toroidal decomposition he describes does not yield E^3. He proves more than this by showing that some points in the decomposition space do not have small neighborhoods whose boundaries are 2-spheres. Recently, Lambert has shown in [13] that any toroidal decomposition of E^3 that does not yield E^3 has this property.

2. Definitions and notation. A collection G of disjoint closed subsets of a metric space S is said to be *upper semicontinuous* if for each $g \in G$ and open set $U \supset g$, there exists a positive number r such that $g' \subset U$ if $g' \in G$ and $\varrho(g', g) < r$. Such a collection G is said to be an *upper semicontinuous decomposition* of S if each point in S is in some element of G. If the elements of G are continua (i.e., compact and connected), G is said to be *monotone*. We shall be concerned here only with monotone upper semicontinuous decompositions of E^3, Euclidean 3-space. If G is an upper semicontinuous decomposition of E^3, the decomposition space of G, denoted by E^3/G, is the space whose points are elements of G, and whose open sets are those sets U for which $U' = \bigcup_{g \in U} g$ is open in E^3. The projection map $P\colon E^3 \to E^3/G$ is defined by $P(x) = g$ if and only if $x \in g \in G$. We denote the union of the nondegenerate elements of G by H_G.

If Z is a collection of point sets in E^3, we denote $\bigcup_{X \in Z} X$ by Z^*. An upper semicontinuous decomposition G of E^3 is said to be *toroidal* if and only if there exists a sequence $Z_1, Z_2, \ldots$ such that (1) for each i, Z_i is a finite collection of disjoint solid tori in E^3, (2) for each i, $Z_{i+1}^* \subset \operatorname{Int} Z_i^*$, and (3) $g \in G$ if and only if g is either a component of $A = Z_1^* \cap Z_2^* \cap \ldots$ or a point of $E^3 - A$. By virtue of Bing's approximation theorem for 2-manifolds in E^3, we may suppose that each solid torus in Z_i, $i = 1, 2, \ldots$, is polyhedral. We refer to the collection Z_i as the collection of solid tori at the ith stage in the construction of G.

If M is an n-manifold with boundary, we use $\operatorname{Int} M$ to denote the collection of points of M which have neighborhoods whose intersection with M are homeomorphic with E^n, Euclidean n-space. We denote $M - \operatorname{Int} M$ by $\operatorname{Bd} M$. If K is a compact connected 2-manifold in E^3, we also use $\operatorname{Int} K$ to denote the bounded complementary domain of K in E^3, but this should cause no confusion. The statement that ab is an arc means that ab is an arc from the point a to the point b. We use the notation $M_1 \simeq M_2$ to indicate that M_1 and M_2 are homeomorphic spaces.

3. Preliminary results. In this section we discuss the concept of the winding number of a simple closed curve (or a solid torus) imbedded in the interior of a solid torus. This is not a new concept and is discussed in a modified form, for example, in Chapter VII of [16]. We include a discussion here since our formulation leads to a convenient way of obtaining the results of Sections 4 and 5. The proofs of the theorems stated in this section are straightforward but tedious, and will not be included here.

Suppose that J is a polyhedral simple closed curve imbedded in the interior of a solid torus T, and D_1 and D_2 are disjoint polyhedral meridional disks of T in general position with respect to J. We may denote the points of $J \cap (D_1 \cup D_2)$ by $a_0, a_1, \ldots, a_n$ in such a way that $J = a_0 a_1 \cup a_1 a_2 \cup \ldots \cup a_{n-1} a_n \cup a_n a_0$ and $\operatorname{Int} a_i a_{(i+1) \bmod (n+1)}$ fails to intersect $\bigcup_{k=0}^{n} \{a_k\}$ $(i = 1, \ldots, n)$. Consider the sequence obtained from the sequence $a_0 a_1 \ldots a_n$ by replacing each a_i by the letter p if $a_i \in D_1$ and by q if $a_i \in D_2$. This sequence, which consists of a finite number of p's and q's, will be referred to as a characteristic sequence of J with respect to D_1 and D_2 and will be denoted by $C(J, D_1, D_2)$. For a given J, there is, of course, more than one way to choose such a sequence. We note that, since D_1 and D_2 are in general position with respect to J, $C(J, D_1, D_2)$ has an even number of terms.

Let X be the collection of all sequences which have a finite number of terms, each of which is the letter p or the letter q. An equivalence operation of Type I on $A \in X$ will be the interchange of the letters p and q in A. An equivalence operation of Type II on A will be the addition or deletion of the adjacent pair of letters pp or qq in A. Sequences $A, B \in X$ are said to be *equivalent*, written $A \approx B$, if A can be obtained from B by a finite sequence of Type I and Type II equivalence operations. We denote the empty sequence by $\{0\}$ and we have, for example, $ppqq \approx \{0\}$. It is easily seen that $\approx$ is an equivalence relation. Let X' be the subset of X consisting of those members of X with an even number of terms. If we denote by $[A]$ the equivalence class, under $\approx$, that contains A, we note that the equivalence classes of X' are $E_0 = [\{0\}]$, $E_1 = [pq]$, $E_2 = [pqpq], \ldots, E_n = [pqpq \ldots pq(n \text{ pairs})], \ldots$ We define the *winding number* of $A \in X'$, written $W(A)$, to be i if $A \in E_i$.

THEOREM 1. *Suppose that J is a polyhedral simple closed curve imbedded in the interior of a solid torus T, D_1 and D_2 are disjoint polyhedral meridional disks of T in general position with respect to J, $C(J, D_1, D_2)$ and $C'(J, D_1, D_2)$ are characteristic sequences of J with respect to D_1 and D_2. Then $C(J, D_1, D_2) \approx C'(J, D_1, D_2)$.*

DEFINITION. Now let J be a polyhedral simple closed curve imbedded in the interior of a solid torus T and let D_1 and D_2 be disjoint polyhedral meridional disks of T in general position with respect to J. Let $C(J, D_1, D_2)$ be a characteristic sequence of J with respect to D_1 and D_2. We define the *winding number* of J with respect to D_1 and D_2, written $W(J, D_1, D_2)$, to be $W(C(J, D_1, D_2))$. It follows from Theorem 1 that $W(J, D_1, D_2)$ is independent of the choice of $C(J, D_1, D_2)$. The following theorem shows that $W(J, D_1, D_2)$ is independent of the choice of D_1 and D_2.

THEOREM 2. *Suppose that J is a polyhedral simple closed curve imbedded in the interior of a solid torus T, D_1 and D_2 are disjoint polyhedral meridional disks of T, D_1' and D_2' are disjoint polyhedral meridional disks of T, and D_1, D_2, D_1', D_2' are in general position with respect to J. Then $W(J, D_1, D_2) = W(J, D_1', D_2')$.*

Henceforth we shall simply refer to $W(J, D_1, D_2)$ as the *winding number* of J in T and write $W(J, T)$ or just $W(J)$ where there is no chance for confusion.

COROLLARY 1. *If J is a polyhedral simple closed curve imbedded in the interior of a solid torus T and h is a piecewise linear homeomorphism of T onto itself fixed on $\mathrm{Bd}\,T$, then $W(J, T) = W(h(J), T)$.*

We may think of a polyhedral solid torus as a closed neighborhood of a polyhedral simple closed curve, which we shall refer to as the *center* of the solid torus. We denote the center of the solid torus T by $\mathrm{Cen}(T)$. If T is a solid torus, we shall refer to any simple closed curve which is equivalently imbedded in T under a homeomorphism fixed on $\mathrm{Bd}\,T$ to $\mathrm{Cen}(T)$ as a *central* simple closed curve of T. If T' is a polyhedral solid torus imbedded in the interior of a polyhedral solid torus T, we define the *winding number* of T' in T, written $W(T', T)$, to be $W(\mathrm{Cen}(T'), T)$. We note that if J is a polyhedral central simple closed curve of T', $W(T', T) = W(J, T)$.

THEOREM 3. *Suppose that T, T' and T'' are solid tori such that $T'' \subset \mathrm{Int}\,T' \subset T' \subset \mathrm{Int}\,T$. Then $W(T'', T) = W(T'', T')\cdot W(T', T)$.*

Remark. The above definitions and theorems consider only the polyhedral case. The more general case can be handled by using polyhedral approximations. We do not do this here since the polyhedral case suffices for the material considered in this paper.

4. The (m, n)-spaces. When we speak of linking simple closed curves, we refer to homological linking. A complete development of such linking is given in Chapter XV of [1]. Using this concept of linking, which is somewhat more general than that used by Bing [6], we note that Theorem 3 of [7] still is valid.

Suppose that J_1 and J_2 are linked polyhedral simple closed curves imbedded in the interior of a solid torus T, D_1 and D_2 are disjoint polyhedral meridional disks of T in general position with respect to J_1 and J_2 such that (1) $W(J_1) = 0$, (2) $J_2 \cap (D_1 \cup D_2) = \emptyset$, and (3) J_1 contains arcs A and B such that (a) A is an arc from p_1 to q_1, B is an arc from p_2 to q_2 where $\{p_1, q_1\} \subset D_1$, $\{p_2, q_2\} \subset D_2$, (b) $(\mathrm{Int}\,A \cup \mathrm{Int}\,B \cup J_2)$ lies in a component C of $T-(D_1 \cup D_2)$, (c) $J_1-(A \cup B)$ contains no arc with both endpoints on D_i ($i = 1$ or 2) whose interior lies in C, and (d) if p_1q_1 is an arc of D_1, p_2q_2 is an arc of D_2, then J_2 links each of the simple closed curves $A \cup p_1q_1$, $B \cup p_2q_2$.

If (1)-(3) are satisfied, we say that $Z\colon J_1, J_2$ is a *linked chain* of simple closed curves in T. If K is an arc in C from a point of A (or B) to a point of J_2 such that $(\mathrm{Int}\,K) \cap (J_1 \cup J_2) = \emptyset$, K is said to be a *connecting arc* from A (or B) to J_2. Suppose $a_1 \in A$, $a_2 \in B$, $b_1, b_2 \in J_2$. Let a_1a_2 be an arc of J_1, b_1b_2 be an arc of J_2 and a_1b_1, a_2b_2 be disjoint connecting arcs. The simple closed curve $J = a_1a_2 \cup a_2b_2 \cup b_1b_2 \cup a_1b_1$ is said to be a *counting curve* for Z. We define the *winding number* of Z in T, written $W(Z, T)$, to be $W(J, T)$.

The disjoint polyhedral simple closed curves J_1 and J_2 are said to be *simply linked* in T if (1) J_1 and J_2 are linked in T, and (2) there exists a 2-sphere S such that $(J_1 \cup J_2) \subset \mathrm{Int}\,S$, $S \subset \mathrm{Int}\,T$. $Z\colon J_1, \dots, J_m$ $(m > 2)$ is said to form a *linked chain* of simple closed curves in T provided J_i and J_j are simply linked if and only if either $|i-j| = 1$ $(i, j = 1, \dots, m)$, or $i = 1$ and $j = m$, or $i = m$ and $j = 1$. Suppose that J_1 and J_2 are simply linked simple closed curves in the interior of the solid torus T, and S is a 2-sphere as above. If K is an arc from a point of J_1 to a point of J_2 such that $K \subset \mathrm{Int}\,S$ and $(\mathrm{Int}\,K) \cap (J_1 \cup J_2) = \emptyset$, K is said to be a *connecting arc* from J_1 to J_2. Let $Z\colon J_1, \dots, J_m$ $(m > 2)$ be a linked chain, let $a_i, b_i \in J_i$ $(i = 1, \dots, m)$, let a_ib_i be an arc of J_i, and let $a_ib_{(i+1)\bmod m}$ be a connecting arc from J_i to $J_{(i+1)\bmod m}$ such that $(\mathrm{Int}\,a_ib_{(i+1)\bmod m}) \cap Z^* = \emptyset$ $(i = 1, \dots, m)$. The simple closed curve $J = a_1b_1 \cup a_1b_2 \cup \dots \cup a_mb_m \cup a_mb_1$ is said to be a *counting curve* for Z. We define the *winding number* of Z in T, $W(Z, T)$, to be $W(J, T)$.

It is not difficult to show that if $Z\colon J_1, J_2, \dots, J_m$ $(m \geqslant 2)$ is a linked chain of simple closed curves in the interior of the solid torus T, $W(Z, T)$ is independent of the choice of counting curve for Z.

The collection $Z\colon T_1, \dots, T_m$ of disjoint solid tori in the interior of a solid torus T is said to be a *linked chain* of solid tori in T with *winding number* $W(Z, T) = n$ if $Z'\colon \mathrm{Cen}(T_1), \dots, \mathrm{Cen}(T_m)$ is a linked chain of simple closed curves in T with $W(Z', T) = n$.

Let T_0 be a solid torus in E^3. Let $Z\colon T_1, \dots, T_m$ $(m \geqslant 2)$ be a linked chain of solid tori in T_0 with $W(Z, T_0) = n$ $(n \geqslant 0)$. We refer to the m solid tori in Z as the solid tori at the first stage. Let $Z_i\colon T_{i1}, \dots, T_{im}$ be a linked chain of solid tori in T_i with $W(Z_i, T_i) = n$, $Z_{ij}\colon T_{ij1}, \dots, T_{ijm}$ be a linked chain of solid tori in T_{ij} with $W(Z_{ij}, T_{ij}) = n$, $Z_{ijk}\colon \dots$ and so on. We have m^k solid tori at the kth stage and we define an upper semicontinuous decomposition G of E^3 by taking as elements of G the components of $A = (T_0) \cap (\bigcup_{i=1}^{m} T_i) \cap (\bigcup_{i,j=1}^{m} T_{ij}) \cap \ldots$ and the points of $E^3 - A$. The collection G is a toroidal decomposition of E^3, and the space E^3/G will be referred to as an *(m, n)-space*. We note that under this definition the space described by Bing in [4] is a $(2, 1)$-space and the one described in [8] is a $(2, 2)$-space.

A criterion for certain decompositions of E^3 to yield E^3 appears in [2]. We state it here for completeness.

Theorem 3 of [2]. *Suppose that G is a point-like decomposition of E^3 such that $P[H_G]$ is a compact 0-dimensional set. Then E^3/G is homeomorphic to E^3 if and only if for each open set U in E^3 containing H_G and each positive number ε, there is a homeomorphism h from E^3 onto E^3 such that* (1) *if $x \notin U$, $h(x) = x$ and* (2) *if g is any non-degenerate element of G,* $(\operatorname{diam} h[g]) < \varepsilon$.

Remark. The above theorem is valid if $P[H_G]$ in the hypothesis is replaced by $\operatorname{Cl} P[H_G]$. By Theorem 10 of [2], if H_G is definable by 3-cells-with-handles, the hypothesis that G is point-like may be replaced by the hypothesis that G is monotone. It is in this form that we use the theorem.

Theorem 4. *Suppose that E^3/G is an (m, n)-space and $m < 2n$. Then $E^3/G \not\approx E^3$.*

Proof. Using the notation above, let D^1 and D^2 be disjoint polyhedral meridional disks of T_0 and let h be a piecewise linear homeomorphism of T_0 onto itself fixed on $\operatorname{Bd} T_0$ such that for each integer j $(j = 1, \ldots, m)$, $h(\operatorname{Bd} T_j)$, and $(D^1 \cup D^2)$ are in general position. We shall show that some T_i $(i = 1, \ldots, m)$ has the property that the image, under h, of each of its central simple closed curves intersects both D^1 and D^2. We may suppose then that $h(T_i)$ has meridional disks D_1^1 and D_1^2 such that D_1^j is a subdisk of D^j $(j = 1, 2)$. By repeated use of this fact it follows that any homeomorphism of T_0 onto itself, fixed on $\operatorname{Bd} T_0$, carries some element of G onto a set that intersects both D^1 and D^2. Hence, by Theorem 3 of [2], $E^3/G \not\approx E^3$.

Suppose that there exist central simple closed curves $J_1, \ldots, J_m$ of $T_1, \ldots, T_m$ and a piecewise linear homeomorphism h of T_0 onto itself fixed on $\operatorname{Bd} T_0$ such that for each integer i $(i = 1, \ldots, m)$, either $h(J_i) \cap D^1 = \emptyset$ or $h(J_i) \cap D^2 = \emptyset$. We also suppose that D^1 and D^2 are in general position with respect to each $h(J_i)$. Then $E_1 = h^{-1}(D^1)$ and $E_2 = h^{-1}(D^2)$ are disjoint meridional disks of T_0 in general position with respect to $J_1, \ldots, J_m$ and J_i intersects at most one of the pair E_1, E_2 $(i = 1, \ldots, m)$.

Case 1. $m > 2$. Theorem 3 of [7] guarantees the existence of arcs $A_1, \ldots, A_m$ such that (1) A_i is a connecting arc from J_i to $J_{(i+1) \bmod m}$, (2) $(\operatorname{Int} A_i) \cap (\bigcup_{1}^{m} J_k) = \emptyset$, and (3) $A_i \cap (E_1 \cup E_2) = \emptyset$ $(i = 1, \ldots, m)$. We use the arcs $A_1, \ldots, A_m$ to construct a counting curve J for Z': $J_1, \ldots, J_m$ which is the sum of m arcs, disjoint except for endpoints, each of which intersects at most one of the pair E_1, E_2. Each of these arcs can contribute at most one letter to the reduced characteristic sequence $C(J, E_1, E_2)$, but by hypothesis, $C(J, E_1, E_2) \approx pq \ldots pq$ (n pairs) where $m < 2n$. This is a contradiction.

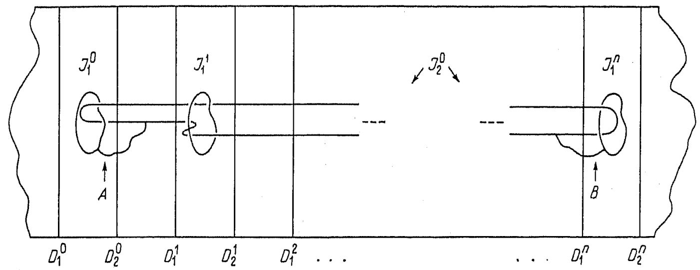

Fig. 1

Case 2. $m = 2$. The proof in this case is similar to the alternative proof Bing suggested in [8] to show that his $(2, 2)$-space is not E^3. Let D_1 and D_2 be disjoint meridional disks of T_0 as in the definition of the linked chain J_1, J_2. Let S be the universal covering space of T_0 and denote the copies of J_1, J_2, D_1 and D_2 in S by J_1^i, J_2^i, D_1^i, and D_2^i $(i = 0, \pm 1, \pm 2, \ldots)$. A portion of S is shown in Figure 1. We can now use Theorem 3 of [7] to construct arcs A and B such that (1) A is an arc from a point p of J_1^0 to a point q of J_2^0, B is an arc from a point r of J_2^0 to a point s of J_1^n, (2) A and B miss the copies of E_1 and E_2 in S, and (3) A and B project onto disjoint arcs of T_0. Let π denote the projection map from S onto T_0. Let C be an arc of J_1 from $\pi(p)$ to $\pi(s)$ and D be an arc of J_2 from $\pi(q)$ to $\pi(r)$. Let J be $\pi(A) \cup \pi(B) \cup C \cup D$. J is a simple closed curve in $\operatorname{Int} T_0$ which is the sum of two arcs $\pi(A) \cup D$ and $\pi(B) \cup C$, one of which misses E_1 while the other misses E_2. Since $W(J, T_0) = W(Z, T_0) = n$, this implies $n \leqslant 1$, contrary to the requirement that $2 = m < 2n$.

Corollary 2. *Suppose that G is a toroidal decomposition of E^3 generated by the sequence of collections of solid tori $Z_1, Z_2, \ldots$ Suppose further that for each $T \in Z_i$ the collection of elements of Z_{i+1} lying in T is a linked chain of $m(T)$ solid tori with winding number $n(T)$ in T and $m(T) < 2n(T)$. Then $E^3/G \not\approx E^3$.*

We wish now to state and prove a theorem which will provide a sufficient condition for an (m, n)-space to be homeomorphic with E^3. The theorem will be stated in more general fashion, for an arbitrary toroidal decomposition of E^3, and the result for (m, n)-spaces will be stated as a corollary. We will make use of the following lemma.

LEMMA 1. *Let G be a toroidal decomposition of E^3. Suppose that if T is any member of the collection of solid tori defining G and n is a positive integer, there exist disjoint polyhedral meridional disks $D_1, D_2, \ldots, D_n$ of T and a homeomorphism h of T onto itself such that* (1) *$h|\operatorname{Bd} T = I$, and* (2) *if $g \in G$ and $g \subset T$, $h(g)$ intersects at most one of the disks $D_1, \ldots, D_n$. Then $E^3/G \simeq E^3$.*

Indication of proof. Let T be a member of the collection of solid tori defining G and $\varepsilon > 0$. Using the hypothesis of Lemma 1 we can construct a homeomorphism h_ε of T onto itself such that (1) $h_\varepsilon|\operatorname{Bd} T = I$, and (2) if $g \in G$ and $g \subset T$, then $\operatorname{diam} h_\varepsilon(g) < \varepsilon$. Such a homeomorphism may be constructed by choosing an appropriate number of meridional disks of T, applying the homeomorphism h of the hypothesis, and then spreading the disks around T while pushing the elements of G toward the center of T. It follows from Theorem 3 of [2] that $E^3/G \simeq E^3$.

We now introduce some notation which will be used in the proof of Theorem 5. Let A denote the canonical solid torus in E^3 obtained by revolving the disk $D = \{(x, y, z) |\ x^2 + (y-2)^2 \leqslant 1,\ z = 0\}$ about the x-axis. We may think of A as being the union of the disjoint disks $D \times t$, $t \in [0, 2\pi)$ where $D \times t$ is the disk obtained by revolving D about the x-axis through the angle t. We refer to the disks $D \times t$ as *cross-sections* of A and we denote $D \times t$ by D_t. If $0 \leqslant r < s < 2\pi$, the *section* of A determined by D_r and D_s, which we denote by $S(r, s)$, is $\bigcup_{t \in [r,s]} D_t$. Every solid torus is a homeomorphic image of A. If T is a solid torus and f is a homeomorphism from A onto T, we refer to the disks $f(D_t)$, $t \in [0, 2\pi)$, as *cross-sections* of T and we denote $f(D_t)$ by $f(D)_t$.

DEFINITION. A collection Z of solid tori imbedded in the interior of a solid torus T_0 is said to be *simple* in T_0 if there exist disjoint polyhedral meridional disks E_1 and E_2 of T_0 such that if $T \in Z$, (1) either $T \cap E_1 = \emptyset$ or $T \cap E_2 = \emptyset$, and (2) if $T \cap E_i \neq \emptyset$ ($i = 1$ or 2), $T \cap E_i$ is the sum of two disjoint meridional disks of T.

THEOREM 5. *If* (1) *G is a toroidal decomposition of E^3 defined by the collections of solid tori $Z_1, Z_2, \ldots$, and* (2) *for each $T \in Z_i$, the collection of elements of Z_{i+1} which lie in T is simple in T, then $E^3/G \simeq E^3$.*

Proof. Let T be a member of some Z_i and let n be a positive integer. We wish to show the existence of disjoint polyhedral meridional disks $D_1, \ldots, D_n$ of T and a homeomorphism h of T onto itself such that (1) $h|\operatorname{Bd} T = I$, and (2) if $g \in G$ and $g \subset T$, then $h(g)$ intersects at most one of the disks $D_1, \ldots, D_n$. It will follow from Lemma 1 that $E^3/G \simeq E^3$.

Let Z'_{i+1} be the subcollection of Z_{i+1} consisting of those elements of Z_{i+1} which lie in T. By hypothesis, Z'_{i+1} is simple in T. Thus there exist disjoint polyhedral meridional disks E_1 and E_2 of T such that if $T' \in Z'_{i+1}$, (1) either $T' \cap E_1 = \emptyset$ or $T' \cap E_2 = \emptyset$, and (2) if $T' \cap E_j \neq \emptyset$ ($j = 1$ or 2), $T' \cap E_j$ is the sum of two disjoint meridional disks of T'. We may suppose that E_1 and E_2 are cross-sections of T under a homeomorphism $f\colon A \to T$, that $E_1 = f(D) = f(D)_0$, and that $E_2 = f(D)_1$.

We may suppose that there is a positive number $\delta_1 < 1$ such that if $0 \leqslant t \leqslant \delta_1$, and T' is an element of Z'_{i+1} which intersects $f(D)_0$, $T' \cap f(D)_t$ is a pair of disjoint meridional disks of T' which are cross-sections of T'. If Z'_{i+2} denotes the subcollection of Z_{i+2} consisting of those elements of Z_{i+2} which lie in T', Z'_{i+2} is simple in T'. Thus there exist disjoint polyhedral meridional disks F_1 and F_2 of T' such that if $T'' \in Z'_{i+2}$, (1) either $T'' \cap F_1 = \emptyset$ or $T'' \cap F_2 = \emptyset$, and (2) if $T'' \cap F_j \neq \emptyset$ ($j = 1$ or 2), $T'' \cap F_j$ is the sum of two disjoint polyhedral meridional disks of T''. There is a homeomorphism of T' onto itself fixed on $\operatorname{Bd} T'$ that pushes the intersection of Z'^{*}_{i+2} with F_1 onto one component of $T' \cap f(D)_0$ and the intersection of Z'^{*}_{i+2} with F_2 onto the other component of $T' \cap f(D)_0$. We may suppose that there is a positive number $\delta_2 \leqslant \delta_1$ such that if $0 \leqslant t \leqslant \delta_2$ and T'' is an element of Z'_{i+2} which intersects $f(D)_0$, $T'' \cap f(D)_t$ is a pair of disjoint meridional disks of T'' which are cross-sections of T''.

By following a procedure indicated by the above paragraph we can adjust the solid tori in T through the $(i+n-1)$st stage so that they are nicely imbedded in T near $f(D)_0$. In other words, we may make the following assumption: If Z is the collection of solid tori consisting of those members of $\bigcup_{j=i+1}^{i+n-1} Z_j$ which lie in T, there is a positive number δ such that (1) if $T_0 \in Z$ either $T_0 \cap f\big(S(0, \delta)\big) = \emptyset$ or $T_0 \cap f(D)_1 = \emptyset$, and (2) if $T_0 \in Z$ and $T_0 \cap f\big(S(0, \delta)\big) \neq \emptyset$, then for each $t \in [0, \delta]$, $T_0 \cap f(D)_t$ is a pair of disjoint meridional disks of T_0.

Now let $t_1, \ldots, t_{n-1}$ be positive numbers such that $0 = t_1 < t_2 < \ldots < t_{n-1} = \delta$ and let $t_n = 1$. Let $D_j = f(D)_{t_j}$ ($j = 1, \ldots, n$). Let T_0 be a member of Z'_{i+1} which intersects $f\big(S(0, \delta)\big)$. One can use the techniques of the proof of Lemma 1.2 of [10] to construct a homeomorphism of T_0 onto itself fixed on $\operatorname{Bd} T_0$ which takes each solid torus in T_0 at the $(i+n-1)$st stage onto a set that intersects at most one of the disks $D_1, \ldots, D_{n-1}$. If we apply such a homeomorphism to each element of Z'_{i+1} that intersects $f\big(S(0, \delta)\big)$, and extend the composition of these homeomorphisms as the identity on the rest of T, the resulting homeomorphism of T onto itself satisfies the requirements set forth in the first paragraph of the proof.

Corollary 3. *In the notation above, if E^3/G is an (m, n)-space with $Z_{ij...k}$ simple in $T_{ij...k}$, then $E^3/G \simeq E^3$.*

5. Examples. In the previous section it was shown that if E^3/G is an (m, n)-space with $m < 2n$, then $E^3/G \not\simeq E^3$. In this section we show that if $m \geqslant 2n$, there exist (m, n)-spaces M_1 and M_2 such that $M_1 \simeq E^3$ and $M_2 \not\simeq E^3$.

Example 1. *If m and n are nonnegative integers with $m \geqslant 2$ and $m \geqslant 2n$, there exists an (m, n)-space M_1 such that $M_1 \simeq E^3$.*

Construction. The case $m = 2$, $n = 1$ is covered in Bing's paper [4] on the sum of two solid horned spheres, so we may suppose $m > 2$. Let T_0 be a solid torus in E^3 and let D_1, D_2 be disjoint polyhedral meridional disks of T_0. We may construct polyhedral simple closed curves $J_1, \ldots, J_{2n}$ in $\operatorname{Int} T_0$ such that (1) J_i and J_k are simply linked in T_0 if and only if $|i-k| = 1$ $(i, k = 1, \ldots, 2n)$, (2) $J_i \cap (D_1 \cup D_2)$ consists of a pair of points of D_1 if i is odd, and (3) $J_i \cap (D_1 \cup D_2)$ consists of a pair of points of D_2 if i is even. See Figure 2, where C_1 and C_2 are the components of $T_0 - (D_1 \cup D_2)$. If $m = 2n$, we link J_1 and J_{2n} by adjusting the part of J_{2n} in C_1. If not we construct polyhedral simple closed curves $J_{2n+1}, \ldots, J_m$

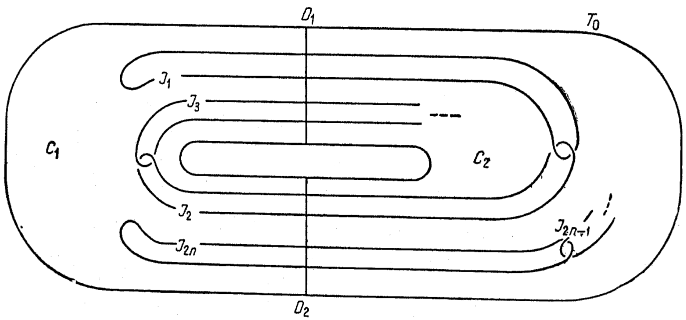

Fig. 2

in C_1 such that (1) J_i and J_k $(i, k = 1, \ldots, m)$ are simply linked in T_0 if and only if either $|i-k| = 1$, or $i = 1$ and $k = m$, or $i = m$ and $k = 1$. Thus Z': $J_1, \ldots, J_m$ is a linked chain of simple closed curves in T_0. We may construct connecting arcs for Z' which miss $(D_1 \cup D_2)$. This shows that $W(Z', T_0) = n$. If we expand each J_i slightly to get a solid torus T_i we obtain Z: $T_1, \ldots, T_m$, a linked chain of solid tori in T_0, such that $W(Z, T_0) = n$ and Z is simple in T_0. Let Z_i: $T_{i1}, \ldots, T_{im}$ $(i = 1, \ldots, m)$ be a linked chain of solid tori in T_i imbedded in T_i as Z is imbedded in T_0, let Z_{ij}: $T_{ij1}, \ldots, T_{ijm}$ be a linked chain of solid tori in T_{ij} imbedded in T_{ij}

as Z is imbedded in T_0, etc. This construction yields an (m, n)-space which we denote by M_1. If follows from Theorem 5 that $M_1 \simeq E^3$.

Example 2. *If m and n are nonnegative integers with $m \geqslant 2$ and $m \geqslant 2n$, there exists an (m, n)-space M_2 such that $M_2 \not\simeq E^3$.*

Construction. Let T_0 be a solid torus in E^3. A polyhedral simple closed curve J imbedded in $\operatorname{Int} T_0$ is said to have property $P(k)$ with respect to T_0 if k is a positive integer such that for any pair of disjoint polyhedral meridional disks D_1 and D_2 of T_0 in general position with respect to J, $C(J, D_1, D_2)$ has either $pq \ldots pq$ (k pairs) or $qp \ldots qp$ (k pairs) as a subsequence. In the following paragraph we construct, for any nonnegative integer j and any positive integer k, a polyhedral simple closed curve in $\operatorname{Int} T_0$ with winding number j and property $P(k)$ with respect to T_0.

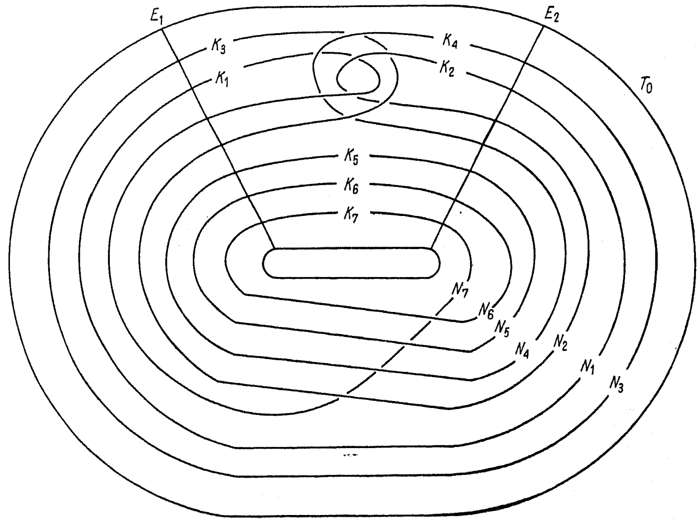

Fig. 3

Let E_1 and E_2 be disjoint polyhedral meridional disks of T_0, and denote the components of $T_0 - (E_1 \cup E_2)$ by C_1 and C_2. Let $K_1, K_2, \ldots, K_{2k+j}$ be disjoint polyhedral arcs such that (1) K_i has each of its endpoints on E_1, and $\operatorname{Int} K_i \subset C_1$ $(i = 1, 3, \ldots, 2k-1)$, (2) K_i has each of its endpoints on E_2, and $\operatorname{Int} K_i \subset C_1$ $(i = 2, 4, \ldots, 2k)$, (3) K_i has one endpoint on E_1, one endpoint on E_2, and $\operatorname{Int} K_i \subset C_1$ $(i = 2k+1, \ldots, 2k+j)$, and (4) if $K_r \in \{K_1, K_3, \ldots, K_{2k-1}\}$ and $K_s \in \{K_2, K_4, \ldots, K_{2k}\}$, K_r and K_s are linked in C_1 in the sense that if K is an arc in E_1 joining the endpoints of K_r, L is an arc in E_2 joining the endpoints of K_s, $K \cup K_r$ and $L \cup K_s$ are linked in T_0. Now let $N_1, \ldots, N_{2k+j}$ be disjoint polyhedral arcs in $\mathrm{Cl}(C_2)$ such that (1) one endpoint of N_i is on E_1 while the other lies on E_2 $(i = 1, \ldots, 2k+j)$, (2) $\mathrm{Int} N_i \subset C_2$ $(i = 1, \ldots, 2k+j)$, and (3) N_i joins one endpoint of K_i to an endpoint of $K_{(i+1) \bmod (2k+j)}$. We let $J = (\bigcup K_i) \cup \cup (\bigcup N_i)$. Figure 3 shows J in the case $j = 3$, $k = 2$.

Let m_0 be the least odd integer which is not less than m, and let J be a simple closed curve in $\mathrm{Int}\, T_0$ such that $W(J, T_0) = n$ and J has property $P((m_0+1)/2)$ with respect to T_0. We expand J slightly to obtain a polyhedral solid torus T' in $\mathrm{Int}\, T_0$ with J as its center. Inside T' we imbed a linked chain Z: $T_1, \ldots, T_m$ of solid tori smoothly, in the sense that there exists a counting curve for Z which is a central curve of T'. It follows from Theorem 3 that $W(Z, T_0) = n$. See Figure 4, where $m = 3$,

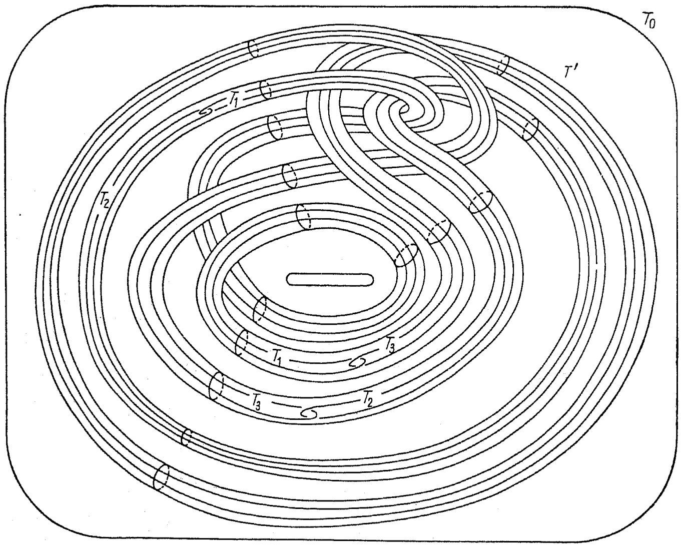

Fig. 4

$n = 1$, and T_1, T_2, T_3 are shown as simple closed curves. We iterate this process, imbedding $T_{i1}, \ldots, T_{im}$ $(i = 1, \ldots, m)$ in T_i as $T_1, \ldots, T_m$ are imbedded in T_0, etc. We denote the (m, n)-space obtained in this way by M_2. If D_1 and D_2 are disjoint polyhedral meridional disks of T and h is a homeomorphism of T_0 onto itself fixed on $\mathrm{Bd}\, T_0$, then some T_i has

the property that the image, under h, of each of its central simple closed curves intersects both D_1 and D_2. It follows, as in the proof of Theorem 4, that $M_2 \not\simeq E^3$.

6. Toroidal decompositions of E^3 into intervals and points. As was pointed out in Section 1, there has been some interest in the study of decompositions of E^3 whose only nondegenerate elements are straight line intervals. In this section we show that if G is a toroidal decomposition of E^3 such that each element of G is either a point or a straight line interval, then $E^3/G \simeq E^3$.

LEMMA 2. *Suppose that G is a toroidal decomposition of E^3 into straight line intervals and points, and that T is a member of the collection of solid tori defining G with the property that there is a plane π such that some component of $\pi \cap \mathrm{Bd}\, T$ is a meridional simple closed curve of $\mathrm{Bd}\, T$. Then if n is a positive integer there exist disjoint meridional disks $D_1, \ldots, D_n$ of T and a homeomorphism h of T onto itself fixed on $\mathrm{Bd}\, T$ such that if $g \in G$ and $g \subset T$, $h(g)$ intersects at most one of the disks $D_1, \ldots, D_n$.*

Proof. We may suppose that $\mathrm{Bd}\, T$ is in general position with respect to π and that π is the xy-coordinate plane. By hypothesis, if $g \in G$, then $g \cap \pi$ is connected. In fact, $g \cap \pi$ is either g or a point of g. This property is what enables us to prove Lemma 2. There is a homeomorphism f of E^3 onto itself such that (1) some component of $f(T) \cap \pi$ is a meridional disk of $f(T)$, say D', and (2) if $g \in G$, then $f(g) \cap \pi$ is connected. The homeomorphism f is obtained by pushing components of $\pi \cap \mathrm{Bd}\, T$ which bound disks on $\mathrm{Bd}\, T$ off π, as described in the following paragraph.

Let J be a component of $\pi \cap \mathrm{Bd}\, T$ which is a meridional simple closed curve of $\mathrm{Bd}\, T$, and denote by D' the disk in π bounded by J. We suppose that the components of $(\mathrm{Bd}\, T) \cap (\mathrm{Int}\, D')$ are simple closed curves which bound disks on $\mathrm{Bd}\, T$. Let J_1 be such a simple closed curve which has the property that if E_1 is the disk in π bounded by J_1, $E_1 \cap (\mathrm{Bd}\, T) = J_1$. Let E_2 be the disk on $\mathrm{Bd}\, T$ bounded by J_1. There is a homeomorphism of E^3 onto itself that moves E_2 onto E_1 and that moves no element of G which does not intersect π onto π. By then pushing E_1 slightly to one side of π, we reduce the number of components of $(\mathrm{Bd}\, T) \cap (\mathrm{Int}\, D')$ by one. Following this procedure, we remove the components of $(\mathrm{Bd}\, T) \cap \cap (\mathrm{Int}\, D')$ one at a time to obtain the homeomorphism f.

Let f_0 be a homeomorphism of A onto $f(T)$. We may suppose that there is a $\delta > 0$ such that (1) $D' = f_0(D)_0$, and (2) if $0 \leqslant t \leqslant \delta$ and $g \in G$ then $f_0(D)_t \cap f(g)$ is a connected set. Let $D_i' = f_0(D)_{t_i}$ where $0 = t_1 < t_2 < \ldots < t_n = \delta$. We shall now construct a homeomorphism h' of $f(T)$ onto itself fixed on $\mathrm{Bd} f(T)$ such that if $g \in G$ and $g \subset T$, $h'f(g)$ intersects at most one of the disks $D_1', \ldots, D_n'$. The disks $D_1, \ldots, D_n$ required in the conclusion of Lemma 2 will be $D_i = f^{-1}(D_i')$. We also denote by h' the homeomorphism of E^3 onto itself obtained by extending h' to the identity outside of $f(T)$, and we let $h = f^{-1}h'f$.

Suppose that i is an integer so that T is a solid torus at the ith stage of the construction of G. We may suppose that the boundaries of the solid tori in $f(T)$ at the $(i+1)$st stage are in general position with respect to $\bigcup_1^n D_i'$. We may further suppose that if T' is one of these solid tori, then T' intersects no more than one component of $f(T) \cap \pi \times t_i$, $i = 1, \ldots, n$. This follows from the fact that if $g \in G$ and $g \subset T$, then $f(g) \cap \pi \times t_i$ is connected, $i = 1, \ldots, n$. We may remove simple closed curves in $(\mathrm{Bd}\, T') \cap \cap (\bigcup_1^n D_i')$ which bound disks on $\mathrm{Bd}\, T'$ from $\bigcup_1^n D_i'$ in a manner similar to that used in the construction of the homeomorphism f. If $\mathrm{Bd}\, T'$ has a longitudinal simple closed curve J on some D_j' we can find a homeomorphism of T' onto itself fixed on $\mathrm{Bd}\, T'$ which pulls all of the nondegenerate elements of G in T' so near J that their images could not intersect $\bigcup_{i \neq j} D_i'$. We suppose then that $T' \cap (\bigcup_1^n D_i')$ is the sum of a finite number of meridional disks of T'. We adjust the solid tori in $f(T)$ at the $(i+2)$nd stage inside the solid tori at the $(i+1)$st stage in a manner similar to the way in which the solid tori at the $(i+1)$st stage were adjusted in T, and we perform similar adjustments through the $(i+n)$th stage. We suppose then that each solid torus in T through the $(i+n)$th stage intersects $\bigcup_1^n D_i'$ in a finite number of meridional disks. See Figure 5.

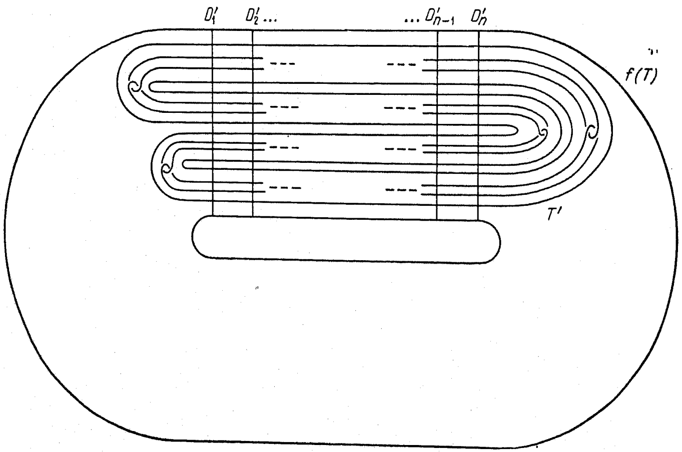

Fig. 5

Let T' be as in Figure 5. Let $A_1, \ldots, A_m$ be the components of $T' \cap \big(f(T) - f_0(S(0, \delta))\big)$. Each A_i is a section of T', and if T'' is a solid torus at the $(i+2)$nd stage inside T', then T'' does not intersect both end cross-sections of A_i. We may identify an orientation of T' which we refer to as "clockwise". There exists a homeomorphism h_1' of T' onto itself fixed on $\mathrm{Bd}\, T'$ and fixed on the part of $f_0(S(0, \delta))$, between D_2' and D_{n-1}' that takes each solid torus in T' at the $(i+2)$nd stage onto a set which intersects at most one of the disks D_1' and D_n'. We think of h_1' as being obtained by slipping the solid tori at the $(i+2)$nd stage inside T' in the clockwise direction keeping the counterclockwise end of each A_i' fixed. The intersections of the tori in $f(T)$ with D_i' for $2 \leqslant i \leqslant n-1$ is not changed by h_1'. See Figure 6. For each T' in $f(T)$ there is a homeomorphism such as h_1'. We extend the composition of these homeomorphisms as the identity

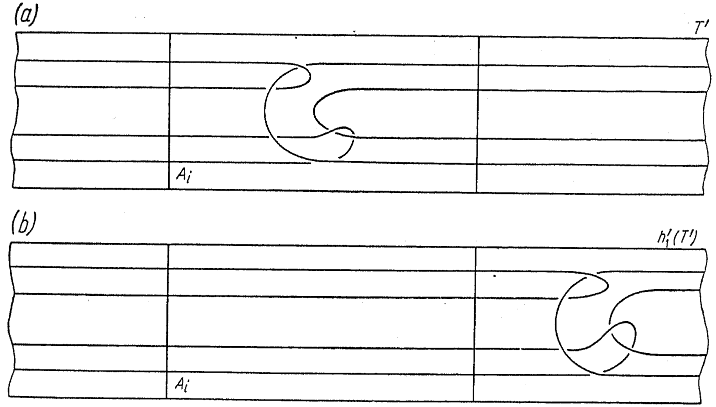

Fig. 6

outside the sum of the T''s to obtain a homeomorphism of $f(T)$ onto itself, which we also denote by h_1'. The homeomorphism h_1' is fixed on $\mathrm{Bd} f(T)$, fixed between D_2' and D_{n-1}', and has the property that each solid torus of $f(T)$ at the $(i+2)$nd stage is carried by h_1' onto a set which intersects at most $n-1$ of the disks $D_1', \ldots, D_n'$. We iterate this process to get h_2', so that each solid torus in $f(T)$ at the $(i+3)$rd stage is carried by $h_2'h_1'$ onto a set which intersects at most $n-2$ of the disks $D_1', \ldots, D_n'$, etc., and we let $h' = h_{n-1}' \ldots h_1'$. Then h' carries each solid torus in $f(T)$ at the $(i+n)$th stage onto a set which intersects at most one of the disks $D_1', \ldots, D_n'$. Thus h' is the homeomorphism required in the first paragraph of this proof.

LEMMA 3. *Under the hypothesis of Lemma 2, if $\varepsilon > 0$, there exists a homeomorphism h_0 of T onto itself fixed on* Bd T *such that if $g \in G$, $g \subset T$, then* $\operatorname{diam} h_0(g) < \varepsilon$.

Lemma 3 follows from Lemma 2 and Lemma 1.

THEOREM 6. *If G is a toroidal decomposition of E^3 into straight line intervals and points, then $E^3/G \simeq E^3$.*

Proof. Let T be a member of the collection of solid tori defining G and let n be a positive integer. Let $D_1, \ldots, D_{3n}$ be disjoint polyhedral meridional disks of T labeled consecutively around T. We shall construct a homeomorphism h of T onto itself fixed on Bd T such that if $g \in G$ and $g \subset T$, $h(g)$ intersects at most three of the disks $D_1, \ldots, D_{3n}$. It follows that $h(g)$ intersects at most one of the disks $D_3, D_6, \ldots, D_{3n}$ and hence from Lemma 1 that $E^3/G \simeq E^3$.

Let $N_1, \ldots, N_{3n}$ be disjoint neighborhoods of $D_1, \ldots, D_{3n}$ respectively. Let A be a nondegenerate element of G in T. There exist polyhedral meridional disks $D'_1, \ldots, D'_{3n}$ of T in $N_1, \ldots, N_{3n}$ respectively, such that A does not intersect S_1, the 1-skeleton of the triangulation of $\bigcup_1^{3n} D'_i$. Let T_A be a solid torus in the collection of solid tori defining G such that $T_A \supset A$ and $T_A \cap S_1 = \emptyset$. We may suppose that Bd T_A is in general position with respect to $\bigcup_1^{3n} D'_i$. We show that there is a homeomorphism h_A of T_A onto itself fixed on Bd T_A such that if $g \in G$ and $g \subset T_A$, $h_A(g)$ intersects at most one of the disks $D'_1, \ldots, D'_{3n}$. It will follow that $h_A(g)$ intersects at most three of the disks $D_1, \ldots, D_{3n}$.

Case 1. *There is a meridional simple closed curve of* Bd T_A *in* $\bigcup_{i=1}^{3n} D'_i$. Let ε be a positive number which is less than the distance between any pair of disks in the collection $D'_1, \ldots, D'_{3n}$. By Lemma 3 there is a homeomorphism h_A of T_A onto itself fixed on Bd T_A such that if $g \in G$ and $g \subset T_A$, then $\operatorname{diam} h_A(g) < \varepsilon$. Thus $h_A(g)$ intersects at most one of the disks $D'_1, \ldots, D'_{3n}$ for every $g \in G$ with $g \subset T_A$.

Case 2. *There is no meridional simple closed curve of* Bd T_A *in* $\bigcup_{i=1}^{3n} D'_i$. In this case we can find a longitudinal simple closed curve J of Bd T_A such that $J \cap (\bigcup_1^{3n} D'_i) = \emptyset$. Let ε be a positive number which is less than the distance from J to $\bigcup_1^{3n} D'_i$. There is a homeomorphism h_A of T_A onto itself fixed on Bd T_A such that if $g \in G$ and $g \subset T_A$, then every point of $h_A(g)$ is within ε of J.

There is a finite collection of disjoint solid tori such as T_A whose interiors cover the nondegenerate elements of G in T, say $T_1, \ldots, T_k$. We use the fact that $\mathrm{Cl}(H_G)$ is compact to conclude this. As shown in the above paragraph for T_A, there is, for each i, a homeomorphism h_i of T_i onto itself fixed on Bd T_i such that if $g \in G$ and $g \subset T_i$, then $h_i(g)$ intersects at most three of the disks $D_1, D_2, \ldots, D_{3n}$. Let $h = h_1 h_2 \ldots h_k$ be extended to the identity on $T - \bigcup_1^k T'_i$. Thus, for each element g of G in T, no more than one of the D_{3i}'s intersects $h(g)$.

References

[1] P. S. Aleksandrov, *Combinatorial Topology*, Vol. 3, Graylock Press, 1960.

[2] Steve Armentrout, *Decompositions of E^3 with a compact 0-dimensional set of nondegenerate elements*, Trans. Amer. Math. Soc. 123 (1966), pp. 165-177.

[3] — and R. H. Bing, *A toroidal decomposition of E^3*, Fund. Math. 60 (1967), pp. 81-87.

[4] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) 56 (1952), pp. 354-362.

[5] — *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) 65 (1957), pp. 363-374.

[6] — *Approximating surfaces with polyhedral ones*, Ann. of Math. (2) 65 (1957), pp. 456-483.

[7] — *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. (2) 65 (1957), pp. 484-500.

[8] — *Point-like decompositions of E^3*, Fund. Math. 50 (1962), pp. 437-453.

[9] B. G. Casler, *On the sum of two solid Alexander horned spheres*, Trans. Amer. Math. Soc. 116 (1965), pp. 135-150.

[10] L. O. Cannon, *Sums of solid horned spheres*, Trans. Amer. Math. Soc. 122 (1966), pp. 203-228.

[11] M. K. Fort, Jr., *A note concerning a decomposition space defined by Bing*, Ann. of Math. (2) 65 (1957), pp. 501-504.

[12] L. V. Keldyš, *Imbedding into E^4 of some monotone images of E^3*, Soviet Math. Dokl. 2 (1961), pp. 9-12.

[13] H. W. Lambert, *Toroidal decompositions of E^3 which yield E^3*, Fund. Math. 61 (1967), pp. 121-132.

[14] L. F. McAuley, *Some upper semicontinuous decompositions of E^3 into E^3*, Ann. of Math. (2) 73 (1961), pp. 437-457.

[15] — *Another decomposition of E^3 into points and straight line intervals*, Topology Seminar Wisconsin, 1965, Ann. of Math. Studies, 1966, pp. 33-51.

[16] M. H. A. Newman, *Topology of Plane Sets of Points*, Cambridge University Press, 1951.

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