for then, using Lemma 1,

$$\varphi(\bigvee_{i\in I} b_i) = 1 \wedge \varphi(\bigvee_{i\in I} b_i) = \big(\bigvee_{n} \varrho(n, s)\big) \wedge \varphi(\bigvee_{i\in I} b_i)$$

$$= \bigvee_{n} \big(\varrho(n, s) \wedge \varphi(\bigvee_{i\in I} b_i)\big) \leqslant \bigvee_{i\in I} \varphi(b_i)\,.$$

Let $f \in \varrho(n, s) \cap \varphi(\bigvee_{i\in I} b_i)$. Then $f(n) = s$. Let $\sigma = \langle f(1), \ldots, f(n)\rangle$, and let $\sigma' = \langle f(1), \ldots, f(n-1)\rangle$ (empty, if $n = 1$). Then either $\tau(\sigma) = \tau(\sigma') \wedge b_i$ for some $i \in I$ or $\tau(\sigma) = \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$. Suppose $\tau(\sigma) = \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$. Then

$$\mathfrak{D}(\sigma) \leqslant \varphi(-\bigvee_{i\in I} b_i) = -\varphi(\bigvee_{i\in I} b_i)\,.$$

Since $f \in \mathfrak{D}(\sigma), f \in -\varphi(\bigvee_{i\in I} b_i)$, which contradicts our hypothesis that $f \in \varphi(\bigvee_{i\in I} b_i)$. This contradiction shows that $\tau(\sigma) \neq \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$; so for some $i \in I, \tau(\sigma) = \tau(\sigma') \wedge b_i$. Hence $\tau(\sigma) \leqslant b_i$, so $\mathfrak{D}(\sigma) \leqslant \varphi(b_i)$, hence $f \in \varphi(b_i)$, and hence $f \in \bigvee_{i\in I} \varphi(b_i)$. This shows that

$$\varrho(n, s) \wedge \varphi(\bigvee_{i\in I} b_i) \leqslant \bigvee_{i\in I} \varphi(b_i)\,,$$

as desired.

Finally, we need to show that φ is a monomorphism. Let $b \in B$, $b \neq 0$; we show $\varphi(b) \neq 0$. Let σ be the one-termed sequence $\langle\{b\}\rangle$. Then since $b \neq 0$, $\tau(\sigma) = 1 \wedge b = b$. Hence $\mathfrak{D}(\sigma) \leqslant \varphi(b)$. Since $\mathfrak{D}(\sigma)$ is non-empty, $\varphi(b) \neq 0$. Q.E.D.

Theorem 1 includes the result of Gaifman and Hales, since there are Boolean algebras of arbitrarily high cardinality; it also includes the known result that *every Boolean algebra can be completely embedded in a complete Boolean algebra*, which is usually proved by other methods (see [4], section 21).

References

[1] R. Solovay, *New proof of a theorem of Gaifman and Hales*, Bull. Amer. Math Soc. 72 (1966), pp. 282-284.

[2] H. Gaifman, *Infinite Boolean polynomials. I*, Fund. Math. 54 (1964), pp. 230-250.

[3] A. W. Hales, *On the non-existence of free complete Boolean algebras*, Fund. Math. 54 (1964), pp. 45-66.

[4] P. R. Halmos, *Lectures on Boolean algebras*, Princeton, N. J. (U. S. A.), 1963.

[5] P. J. Cohen, *The independence of the continuum hypothesis I*, Proc. Nat. Acad. Sci., U. S. A., 50 (1963), pp. 1143-1148, II, *ibid.* 51 (1964), pp. 105-110.

PRINCETON UNIVERSITY
SOCIETY OF FELLOWS, HARVARD UNIVERSITY

Reçu par la Rédaction le 5. 7. 1966

Core decompositions of continua

by

R. W. FitzGerald (Madison, Wis.) and **P. M. Swingle** * (Miami, Fla.)

Let S be the space consisting of the $\sin(1/x)$ curve, $0 < x \leqslant 1$, and its limit continuum C on the y-axis. Shrinking C to a point gives rise to a decomposition G of S. The decomposition G is monotone with an arc as hyperspace, and if H is any other monotone decomposition of S whose hyperspace is an arc, then G refines H. Thus we say that G is the *core* decomposition of S with respect to the property — "Is monotone with an arc as hyperspace". (See Definition 1.1.).

Let P be a property of decompositions and $\mathcal{S}$ a class of topological spaces. By a *method of core decomposition for* $\mathcal{S}$ *with respect to* P, we mean a method of decomposition which, when applied to any $S \in \mathcal{S}$, yields the core decomposition of S with respect to P. Thus Kuratowski, by his decomposition into "tranches" ([7], p. 248), has described a method of core decomposition for *the class* $\mathcal{S}$ of compact continua irreducible between two points and P the property — "Is monotone with locally connected hyperspace". W. A. Wilson's decomposition into "oscillatory sets" ([15], p. 381) is a method of core decomposition for P as above and $\mathcal{S}$ the class of compact, one-dimensional, m-cyclic, and separable continua.

The principle result of this work is a method of core decomposition for the class $\mathcal{S}$ of compact Hausdorff continua and P the property — "Is monotone with semi-locally-connected hyperspace".

If M and N are set functions on a set S, then M is an *expansion* of N if and only if $M(A) \supset N(A)$ for all $A \subset S$. Our method consists of three successive expansions of the useful set function T of [1], [2] and of Definition 1.2 below. We first expand T to its minimal closure T^* ([4], p. 61) of Definition 3.1 below and use T^* to obtain a modification of H. Hahn's prime parts of a continuum ([5], p. 225). But $\{T^*(x)\colon x \in S\}$ is not a decomposition for every compact continuum S. We next, by a chaining process, expand T^* to its recursive chain closure $M\mathrm{Ch}T$ of Definition 4.4 below. For compact continuum S, $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is

* This work was done under National Science Foundation Grant G 19672, with partial support from the University of Wisconsin Center System.

a decomposition of S into continua, and it is identical with the core decompositions of Kuratowski and Wilson when S is restricted, respectively, to the continua they consider; but this decomposition is not upper semi-continuous for every compact Hausdorff continuum. The direct limit expansion of MChT overcomes this last obstacle, by use of MChθ, and gives our desired core decomposition. In general our expansion procedure will yield core decompositions provided property P can be appropriately associated with an expansive set function.

Much of our basic theory is given for any expansive set function N, with the aposyndetic function T and the Hausdorff function θ as concrete examples of the use of the theory; our general result in Theorem 7.2 illustrates that one can go a long way by the elementary operations of iteration and chaining in developing theory of core decompositions.

1. Definitions and the set functions T and θ. We denote the empty set by Ø. A *continuum* is a closed and connected point set, and *space* means topological space.

Let S be a set and G be any decomposition of S. For $x \in S$ we let $G(x)$ denote the unique element of G that contains x. If H is a decomposition of S, we say that G refines H, and write $G < H$, if and only if $G(x) \subset H(x)$ for all $x \in S$. If $\{G_\alpha\colon G_\alpha \in \mathcal{M}\}$, is a collection of decompositions of S, let the common intersection element $M(x) = \bigcap\{G_\alpha(x)\colon G_\alpha \in \mathcal{M}\}$; and note that the class $\{M(x)\colon x \in S\}$ is a decomposition of S. We denote this decomposition by $\bigwedge\{G_\alpha\colon G_\alpha \in \mathcal{M}\}$; it is precisely the greatest lower bound of the G_α under the partial order $<$ on the set of all decompositions of S.

Definition 1.1. Let S be a set and $\mathcal{H}$ be the family of all decompositions of S that have a certain property P. We say that G is the *core decomposition of S with respect to* P if and only if $G = \bigwedge\{H_\alpha\colon H_\alpha \in \mathcal{H}\}$ and $G \in \mathcal{H}$. We say that G is *atomic with respect to* P if and only if

(1) $G \in \mathcal{H}$ and

(2) $H \in \mathcal{H}$ and $H < G$ implies $H = G$.

Note that core and atomic are distinct concepts, and that the core decomposition of S with respect to P is characterized by having property P and refining all other decompositions of S that have property P.

We need the terminology of set functions and the properties of the set functions T and θ given below.

If S is a set, then *a set function on S* is a function N which assigns to each subset A of S a unique subset $N(A)$ of S. A set function N on S is said to be [4]: *enlarging*, if $A \subset N(A)$ for all $A \subset S$; *isotonic*, if $A \subset B \subset S$ implies $N(A) \subset N(B)$; *expansive*, if N is both enlarging and isotonic; *idempotent*, if $N(N(A)) = N(A)$ for all $A \subset S$; *a closure function*, if N is both expansive and idempotent. A subset A of S is *N-closed* means $N(A) = A$.

Definition 1.2. The set function T is defined on any space S as follows: for $A \subset S$, $T(A)$ is the set of all $y \in S$ for which there *does not exist* both an open set Q and continuum W such that $y \in Q \subset W \subset S - A$.

The set function T was defined in [2] and has its origin in Jones' concept of aposyndetic [6]. The first part of Lemma 1.3, below, follows directly from the definition of T (or see Lemmas 1 and 3 of [2], pp. 114, 116); the second part is Corollary 1.1 of [2], p. 115.

Lemma 1.3. (i) *If S is any space then T is expansive on S and $T(A)$ is closed for all $A \subset S$.*

(ii) *If A is a connected subset of a compact Hausdorff continuum then $T(A)$ is a continuum.*

We extend Whyburn's definition of *semi-locally-connected* (which we abbreviate s.l.c. as in [12], p. 19) and say S is s.l.c. at $A \subset S$ if and only if for every open subset U of S, such that $A \subset U$, there exists an open subset V of S such that $A \subset V \subset U$ and $S - V$ has only a finite number of components; a space S is s.l.c. if and only if it is s.l.c. at each of its points. The relation between T-closed sets and Whyburn's s.l.c. is given in Lemma 1.4, whose proof parallels those of Jones' Theorems 3 and 4 in [6], pp. 546-547.

Lemma 1.4. *If S is a compact Hausdorff space and $A \subset S$, then A is T-closed if and only if A is closed and S is* s.l.c. *at A. Consequently, S is* s.l.c. *if and only if $T(x) = x$ for all $x \in S$.*

Definition 1.5. The set function θ is defined on any space S as follows: for $A \subset S$, $\theta(A)$ is the set of all $y \in S$ for which there *does not exist* an open set Q such that $y \in Q \subset \bar{Q} \subset S - A$.

The set function θ comes straight from the definition of the Hausdorff property, and we have at once the following:

Lemma 1.6. *If S is any space, then θ is expansive on S, $\theta(A)$ is closed for all $A \subset S$, and S is Hausdorff if and only if $\theta(p) = p$ for all $p \in S$.*

Example 1.7. Let the embedding space be the plane and d its usual metric. Let $R(x, \varepsilon) = \{y\colon d(x, y) < \varepsilon\}$, where ε is always a positive real number. For $i = 1, 2, \ldots$, let $A_i = \{(x, y)\colon x = 1/i,\ 0 < y \leqslant 1/i\}$ and let $p_i = (x_i, 0)$ be chosen such that $1/(i+1) < x_i < 1/i$, and let $p = (0, 0)$. Let $S = p \cup (\bigcup p_i) \cup (\bigcup A_i)$. The basic open sets for p are the sets $R(p, \varepsilon) \cap S$, the basic open sets for $a \in A_i$ are the sets $R(a, \varepsilon) \cap A_i$, and the basic open sets for p_i are the sets $p_i \cup \{(x, y) \in A_i\colon y < \varepsilon\} \cup \{(x, y) \in A_{i+1}\colon y < \varepsilon'\}$. Let S have the topology generated by the above basis. Then S is compact and connected, and points are closed sets. Note $\theta(p_1) = p_1 \cup p_2$, $\theta^2(p_1) = \theta(\theta(p_1)) = p_1 \cup p_2 \cup p_3$, and in general $\theta^n(p_1) = p_1 \cup p_2 \ldots \cup p_{n+1}$. Thus θ is not idempotent on this S.

EXAMPLE 1.8. Let P be the closed triangular region of the plane determined by the points $(0,1)$, $(1,0)$ and $(2,0)$. For $i = 0, 1, 2, \ldots$, let $A_i = P \cap \{(x, y): 1-1/2^i \leqslant y \leqslant 1-1/2^{i+1}\}$, and let I_i be an indecomposable continuum such that $I_i \subset A_i$. Furthermore, let the I_i be such that for all n, $I_0, I_1, \ldots, I_n$ is a simple chain and such that the line segment joining $(1/2, 1/2)$ to $(1, 0)$ lies in I_0. Let $A = (0,1) \cup (\bigcup I_i)$ and let $A' = (0,1) \cup (\bigcup I_i')$ be the reflection of A about the y-axis. Let M be the line segment joining $(-1,0)$ to $(1,0)$, and for $i = 1, 2, \ldots$, let M_i be the horizontal line segment irreducible from I_0' to I_0 and passing through $(0, 1/2^i)$. Let $S = A \cup A' \cup M \cup (\bigcup M_i)$. Then S is a compact continuum, such that $T(x) = x$ for all $x \in S-(A \cup A')$, and for $x = (0,1)$. For $x \in (I_0 - I_1)$, $T(x) = I_0$, $T^2(x) = I_0 \cup I_1$, and $T^m(x) = I_0 \cup I_1 \cup \ldots \cup I_{n-1}$. The sets A and A' are both T-closed but $A \cup A'$ is not, because $(0,0) \in T(A \cup A')$.

Example 1.8 is a very critical type in the development of our theory below, because $A \cup A'$ is not T-closed.

2. Existence of core decompositions. We establish here the existence of our basic core decompositions.

The *hyperspace* ([9], pp. 42-43) of a space S with respect to a decomposition G of S is denoted by the ordered pair (S, G), and π denotes the decomposition map of S onto (S, G). We abbreviate *upper semi-continuous* by u.s.c. and say that G is *monotone* if and only if G is u.s.c. and each $g \in G$ is a continuum.

THEOREM 2.1. *Let S be any space, $\{G_a: G_a \in \mathcal{M}\}$ a collection of decompositions of S, and $G = \bigwedge \{G_a: G_a \in \mathcal{M}\}$. If, for each G_a, (S, G_a) is a Hausdorff space, then (S, G) is a Hausdorff space.*

Proof. Consider distinct points x' and y' in (S, G). For some x and y in S, $G(x) = \pi^{-1}(x')$, $G(y) = \pi^{-1}(y')$ and $G(x) \neq G(y)$. Then for some G_a, $G_a(x) \neq G_a(y)$. Because (S, G_a) is Hausdorff, there exist disjoint open sets Q and V of S such that $G_a(x) \subset Q$, $G_a(y) \subset V$, and such that Q and V are each unions of elements of G_a. Since $G < G_a$, both Q and V are exactly unions of elements of G, and so $\pi(Q)$ and $\pi(V)$ are disjoint open sets in (S, G) containing, respectively, x' and y'. Thus (S, G) is Hausdorff.

COROLLARY 2.2. *If S is any space, then there exists a core decomposition G of S with respect to the property: "The hyperspace is Hausdorff".*

THEOREM 2.3. *If G is an* u.s.c. *decomposition of a compact Hausdorff continuum S and each $g \in G$ is T-closed in S, then (S, G) is semi-locally-connected.*

Proof. By Lemma 1.4, it is sufficient to show $T(x') = x'$ for all x' in (S, G). Let x' and y' be distinct points in (S, G). Then, for some x and y in S, $G(x) = \pi^{-1}(x')$, $G(y) = \pi^{-1}(y')$ and $G(x) \neq G(y)$. Since $G(y)$ is compact and $G(y) \subset S - G(x) = S - T(G(x))$, there exists a finite collection of open sets Q_i and corresponding continua W_i $(i = 1, 2, \ldots, n)$ such that $G(y) \subset \bigcup Q_i \subset \bigcup W_i \subset S - G(x)$. Let $V = \bigcup \{g \in G: g \subset \bigcup Q_i\}$. Since G is u.s.c., $\pi(V)$ is open in (S, G). We have $y' \in \pi(V) \subset \bigcup \pi(W_i) \subset (S, G) - x'$, and so y' lies in the interior of one of the m components of $\bigcup \pi(W_i)$, where $m \leqslant n$. Thus $y' \notin T(x')$.

THEOREM 2.4. *If G is a monotone decomposition of a compact Hausdorff continuum S then (S, G) is semi-locally-connected if and only if each $g \in G$ is T-closed in S.*

Proof. Theorem 2.4 follows from Theorem 2.3 and the fact that π is a monotone map when G is a monotone decomposition.

THEOREM 2.5. *If N is any expansive set function on a compact Hausdorff continuum S, then there exists a core decomposition G of S with respect to the property: "G is* u.s.c. *with N-closed elements".*

Proof. Let $\mathcal{N}$ denote the collection of all u.s.c. decompositions of S into N-closed sets. Since $\{S\} \in \mathcal{N}$, we see that $\mathcal{N} \neq \emptyset$. Let $G = \bigwedge \{G_a: G_a \in \mathcal{N}\}$. It is well known that a decomposition H of a compact space S is u.s.c. if and only if (S, H) is Hausdorff. Since each $G_a \in \mathcal{N}$ is u.s.c., and hence each (S, G_a) is Hausdorff, it follows from Theorem 2.1 that (S, G) is Hausdorff, and hence that G is u.s.c. Let $G(x) = \bigcap \{G_a(x): G_a \in \mathcal{N}\}$ be an element of G. Then $G(x)$ is N-closed because each $G_a(x)$ is N-closed and, for N expansive, the intersection of any collection of N-closed sets is N-closed. Thus $G \in \mathcal{N}$, and so G is the desired core decomposition.

LEMMA 2.6. *If S is a compact Hausdorff continuum and C is a component of a T-closed subset A of S then C is T-closed.*

Proof. By hypothesis $T(A) = A$, and, by Lemma 1.3, T is expansive. Thus $C \subset T(C) \subset T(A) = A$. Since C is a component of A and, by Lemma 1.3, $T(C)$ is a continuum, it follows $T(C) = C$.

THEOREM 2.7. *If S is a compact Hausdorff continuum there exists a unique decomposition G of S such that:*

(1) *G is the core decomposition of S with respect to being* u.s.c. *with T-closed elements;*

(2) *G is monotone with* s.l.c. *hyperspace; and*

(3) *G is the core decomposition of S with respect to being monotone with* s.l.c. *hyperspace.*

Proof. Let G be the core decomposition of S with respect to being u.s.c. with T-closed elements, which exists by Theorem 2.5. Thus G satisfies (1). By definition of core, G is u.s.c. with T-closed elements, and hence, by Theorem 2.3, (S, G) is s.l.c. Let H be the decomposition of S into the components of elements of G. It follows that H is u.s.c. and, by Lemma 2.6, the elements of H are T-closed. By the definitions of H and G, both the inequalities $H < G$ and $G < H$ hold, and hence $G = H$ and G is monotone. Thus G also satisfies (2). If H is any monotone decomposition of S with s.l.c. hyperspace then, by Theorem 2.4, the elements of H are T-closed, and hence $G < H$. Thus G satisfies (3).

We have established, in Corollary 2.2 and Theorem 2.7, the existence of our basic core decompositions. We proceed now to describe our method of core decomposition, which is the systematic expansion of T and θ. Since T and θ need not be idempotent, as shown in Examples 1.8 and 1.7, our first step is to expand T and θ to closure functions.

3. Minimal closure expansion. If N is an expansive set function on a set S, then the intersection of any collection of N-closed sets is itself N-closed. Hence, for N expansive and $A \subset S$, $\bigcap \{A_a\colon A \subset A_a = N(A_a)\}$ is the unique minimal N-closed subset of S containing A.

DEFINITION 3.1. If N is an expansive set function on a set S, then the set function N^* is defined as follows: for $A \subset S$, $N^*(A)$ is the unique minimal N-closed subset of S containing A.

LEMMA 3.2. *If N is an expansive set function on a set S, then N^* is a closure function.*

Proof. It follows directly from Definition 3.1 that N^* is enlarging, isotonic, and idempotent.

For N expansive on S and $A \subset S$, the set $N^*(A)$ may be realized through iterated composition ([4], pp. 60-61). Let $N^0(A) = A$. For a non-limit ordinal a, let $N^a(A) = N(N^{a-1}(A))$, and, for a limit ordinal a, let $N^a(A) = \bigcup \{N^\lambda(A)\colon \lambda < a\}$. There is no difficulty in proving Lemma 3.3 below.

LEMMA 3.3. *If N is an expansive set function on a set S and $A \subset S$, then there exists a first ordinal number λ such that $N^\lambda(A) = N^{\lambda+1}(A)$ and, for this λ, $N^*(A) = N^\lambda(A)$.*

EXAMPLE 3.4. Let S be the space of Example 1.8, $p = (0,1)$, $x \in A - A'$, and $y \in A' - A$. Then $T^*(x) = T^{\omega+1}(x) = A$, $T^*(y) = T^{\omega+1}(y) = A'$, and $T^*(p) = p$. Thus $T^*(x) \cap T^*(y) = T^*(p)$, and so $\{T^*(x)\colon x \in S\}$ is not a decomposition of S. In the space of Example 1.7, $\theta^*(x) = x$ for $x \notin \bigcup p_i$, and $\theta^*(x) = \bigcup p_i$ for $x \in \bigcup p_i$.

LEMMA 3.5. *If A is a connected subset of a compact Hausdorff continuum then $T^*(A)$ is a continuum.*

Proof. Since $T^*(A)$ is a T-closed set, it is, by Lemma 1.4, a closed set. Since $T^*(A)$ is minimal for being T-closed and containing A, it follows from Lemma 2.6 that $T^*(A)$ is connected.

Only under strong hypothesis do the sets $T^*(x)$ constitute the core decomposition G of Theorem 2.7.

LEMMA 3.6. *If S is a compact Hausdorff continuum, n is an integer, $\{T^n(x)\colon x \in S\}$ is a decomposition of S, and $T^*(x) = T^n(x)$ for all $x \in S$, then $\{T^n(x)\colon x \in S\}$ is the core decomposition G of Theorem 2.7.*

Proof. For a compact metric S this follows from Theorems 3.3, 3.6 and Corollary 3.5 of [3], pp. 113-115. Here we outline proof for Hausdorff S and need first prove $\{T^n(x)\colon x \in S\}$ is u.s.c.; thus we must prove for $T^n(x) \subset U$ open ([12], p. 122), there exists open V such that $T^n(x) \subset V \subset \overline{V} \subset U$ and if $T^n(z) \cap V \neq \emptyset$, then $T^n(z) \subset U$. Since $T^n = T^*$, $T^n(x)$ is closed and T-closed; thus by compact Hausdorff properties and Lemma 1.4, there exists open V' such that $T^n(x) \subset V' \subset \overline{V}' \subset U$ and $S - V' = \bigcup C_f$ $(f = 1, 2, \ldots, h)$, C_f a continuum. Let $U = V_n$ and $V' = V_n$. This argument can be repeated, with V_n in place of U, to obtain V_{n-1} such that $T^n(x) \subset V_{n-1} \subset \overline{V}_{n-1} \subset V_n$ and finally V_i $(i = 1, 2, \ldots, n)$ such that $T^n(x) \subset V_i \subset \overline{V}_i \subset V_{i+1}$ and $S - V_i = \bigcup C_{if}$ $(f = 1, 2, \ldots, h_f)$ where C_{if} is a continuum.

Let $y \in T^n(z) \cap V_1$. Suppose $y' \in C_{1f} \cap (S - \overline{V}_1)$; then $y' \in Q$ (open) $\subset C_{1f} \subset S - V_1$ and so $y' \notin T(V_1) \subset \overline{V}_1$. Since $y \in V_1$, by Lemma 1.3,

$$T(y) \subset T(V_1) \subset \overline{V}_1 \subset V_2\,.$$

Similarly

$$T^2(y) \subset T^2(V_1) \subset T(V_2) \subset \overline{V}_2 \subset V_3\,,$$

and finally

$$T^n(y) = T^n(z) \subset T^n(V_1) \subset T^{n-1}(V_2) \subset \ldots \subset T(V_n) \subset \overline{V}_n \subset V_{n+1} = U\,.$$

Thus V_1 is the desired V above, and so $\{T^n(x)\colon x \in S\}$ is u.s.c., with T-closed elements. Thus $G(x) \subset T^n(x)$ for all $x \in S$. Each $G(x)$ is T-closed; hence, by Definition 3.1, $T^n(x) = T^*(x) \subset G(x) \subset T^n(x)$ and $G = \{T^n(x)\colon x \in S\}$.

THEOREM 3.7. *If S is a compact Hausdorff continuum irreducible between two of its points and S contains no n-indecomposable subcontinuum then $\{T^{n+1}(x)\colon x \in S\}$ is the core decomposition G of Theorem 2.7.*

Proof. By Theorems 5 and 3 of [2], S satisfies the hypothesis of Lemma 3.6. One can show, under conditions entirely analogous to those in Lemma 3.6, that $\{\theta^n(x)\colon x \in S\}$ is the core decomposition of Corollary 2.2. For giving core decompositions, θ^* behaves no better than does T^*.

The prime parts decomposition of Hahn ([5], p. 223) suggests the following general method of decomposition.

THEOREM 3.8. *Let S be any compact Hausdorff continuum, $B = \{x \in S\colon T(x) \neq x\}$, and H be the decomposition of S whose elements are the components of $T^*(B)$ and the points of $S - T^*(B)$. Then H is monotone, (S, H) is s.l.c., and if S is s.l.c. then $(S, H) = S$.*

Proof. The elements of H are continua, and, since the non-degenerate elements of H are components of the T-closed set $T^*(B)$, H is u.s.c.. The elements of H are either points x where $T(x) = x$ or components of the T-closed set $T^*(B)$. Hence, by Lemma 2.6, the elements of H are T-closed. By Theorem 2.3, (S, H) is s.l.c.. If S is s.l.c., then $B = \emptyset$; hence $T^*(B) = \emptyset$ and $(S, H) = S$.

The left figure on p. 132 of [9] shows that H in Theorem 3.8 need not be core (nor atomic) with respect to being monotone with s.l.c. hyperspace.

4. Decompositions and the set function $M\mathrm{Ch}N$. As our next step in building up from the set function T to the core decomposition of Theorem 2.7, we expand T to a set function $M\mathrm{Ch}T$ having the property that $M\mathrm{Ch}T(x)$ is a continuum and $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the finest decomposition of S into T-closed sets. While these sets fail to be u.s.c. in every compact Hausdorff continuum, they nevertheless are sufficient to give the classic decomposition of Wilson ([15], pp. 385, 386) and Kuratowski ([7], p. 248).

DEFINITION 4.1. We say that C *can be F-chained to* D if and only if there exists a simple chain from C to D whose links are of the type $F(x_1), F(x_2), \ldots, F(x_n)$, where C and D are subsets of a set S and F is a set function on S. For a set function N on S we define, for all $A \subset S$, $\mathrm{Ch}_0N(A) = N(A)$ and $\mathrm{Ch}_1N(A) = N(\{z\colon z$ can be N-chained to $A\})$. We next define, for all $A \subset S$, $\mathrm{Ch}_2N(A) = N\{z\colon z$ can be Ch_1N-chained to $A\}$, noting this definition is permissible, since Ch_1N is defined for all subsets of S. We continue in this way and define, for α a nonlimit ordinal and $A \subset S$, $\mathrm{Ch}_\alpha N(A) = N\{z\colon z$ can be $\mathrm{Ch}_{\alpha-1}N$-chained to $A\}$, and for α a limit ordinal and $A \subset S$, $\mathrm{Ch}_\alpha N(A) = \bigcup\{\mathrm{Ch}_\lambda N(A)\colon 0 \leqslant \lambda < \alpha\}$. It follows by transfinite induction that for each ordinal α, $\mathrm{Ch}_\alpha N$ is a well defined set function on S.

LEMMA 4.2. *If N is an expansive set function on a set S, $A \subset S$, and $0 \leqslant \alpha < \beta$, then $\mathrm{Ch}_\alpha N(A) \subset \mathrm{Ch}_\beta N(A)$.*

Proof. Lemma 4.2 is obviously true for a limit ordinal β and easily proved for $\beta = 1$. We assume $\mathrm{Ch}_\alpha N(X) \subset \mathrm{Ch}_{\beta-1}N(X)$ for $0 \leqslant \alpha < \beta-1$ and for all $X \subset S$, where β is a nonlimit ordinal and $\beta > 1$. Since $\mathrm{Ch}_\alpha N(x) \subset \mathrm{Ch}_{\beta-1}N(x)$ for all $x \in S$ and $\alpha < \beta-1$, each $z \in S$ that can be $\mathrm{Ch}_\alpha N$-chained to A can also be $\mathrm{Ch}_{\beta-1}N$-chained to A and, therefore, $\mathrm{Ch}_{\alpha+1}N(A) \subset \mathrm{Ch}_\beta N(A)$, for all $\alpha < \beta-1$. From the induction hypothesis, since $\beta > 1$ and $\mathrm{Ch}_0N(A) \subset \mathrm{Ch}_1N(A)$, it follows that $\mathrm{Ch}_{\beta-1}N(A) = \bigcup\{\mathrm{Ch}_{\alpha+1}N(A)\colon \alpha < \beta-1\}$. Hence $\mathrm{Ch}_{\beta-1}N(A) \subset \mathrm{Ch}_\beta N(A)$, and Lemma 4.2 is true by transfinite induction.

THEOREM 4.3. *If N is an expansive set function on a set S, then there exist a first ordinal number α such that $\mathrm{Ch}_\alpha N(A) = \mathrm{Ch}_{\alpha+1}N(A)$ for all $A \subset S$.*

Proof. Let β be an ordinal number whose cardinal number exceeds the cardinal number of the set of subsets of S. Suppose for some $A \subset S$, $\mathrm{Ch}_\beta N(A) \neq \mathrm{Ch}_{\beta+1}N(A)$. By Lemma 4.2, $\{\mathrm{Ch}_\lambda N(A)\colon \lambda < \beta\}$ is a well ordering of distinct subsets of S, which gives a contradiction. Thus β is an ordinal such that $\mathrm{Ch}_\beta N(A) = \mathrm{Ch}_{\beta+1}N(A)$ for all $A \subset S$. Let α be the first such ordinal number.

DEFINITION 4.4. Let N be an expansive set function on a set S. The set function $M\mathrm{Ch}N$ is the function $\mathrm{Ch}_\alpha N$, where α is the first ordinal such that $\mathrm{Ch}_\alpha N(A) = \mathrm{Ch}_{\alpha+1}N(A)$ for all $A \subset S$. To avoid confusion we will sometimes write $M\mathrm{Ch}(N)$ for $M\mathrm{Ch}N$.

THEOREM 4.5. *If N is an expansive set function on a set S, then $M\mathrm{Ch}N$ is expansive and $\{M\mathrm{Ch}N(x)\colon x \in S\}$ is the core decomposition of S with respect to having N-closed elements.*

Proof. Let α be the ordinal such that $M\mathrm{Ch}N = \mathrm{Ch}_\alpha N$. The expansive property of $\mathrm{Ch}_\alpha N$ follows easily by induction from Definition 4.1 and Lemma 4.2.

By Definition 4.1, $\mathrm{Ch}_\alpha N(x) \cap \mathrm{Ch}_\alpha N(y) \neq \emptyset$ implies $\mathrm{Ch}_\alpha N(x) \subset \mathrm{Ch}_{\alpha+1}N(y)$. By the definition of $M\mathrm{Ch}N$, $\mathrm{Ch}_\alpha N(y) = \mathrm{Ch}_{\alpha+1}N(y)$. Therefore, for all x and y in S, $\mathrm{Ch}_\alpha N(x) \cap \mathrm{Ch}_\alpha N(y) \neq \emptyset$ implies $\mathrm{Ch}_\alpha N(x) \subset \mathrm{Ch}_\alpha N(y)$. Hence $\{\mathrm{Ch}_\alpha N(x)\colon x \in S\}$ is a decomposition of S.

Since $\mathrm{Ch}_\alpha N = \mathrm{Ch}_{\alpha+1}N$, we have, for all $x \in S$, $\mathrm{Ch}_\alpha N(x) = N\{z\colon z$ can be $\mathrm{Ch}_\alpha N$-chaind to $x\}$. However, since $\{\mathrm{Ch}_\alpha N(x)\colon x \in S\}$ is a decomposition of S,

$$\{z\colon z \text{ can be } \mathrm{Ch}_\alpha N\text{-chained to } x\} = \mathrm{Ch}_\alpha N(x)\,, \quad \text{for all } x \in S\,.$$

Consequently,

$$\mathrm{Ch}_\alpha N(x) = N\big(\mathrm{Ch}_\alpha N(x)\big) \quad \text{for all } x \in S\,.$$

We have proved that $\{M\mathrm{Ch}N(x)\colon x \in S\}$ is a decomposition of S into N-closed sets. Let G be any other such decomposition of S. Since the elements of G are disjoint and N-closed, every simple chain $N(x_1)$, $N(x_2), \ldots, N(x_k)$ must have all of its links in but one element of G. Hence, for all $x \in S$, $\{z\colon z$ can be N-chained to $x\} \subset G(x)$. Since $N\big(G(x)\big) = G(x)$ and N is isotonic, it follows that $\mathrm{Ch}_1N(x) \subset G(x)$ for all $x \in S$. By induction, $\mathrm{Ch}_\alpha N(x) \subset G(x)$ for all $x \in S$, and so $\{M\mathrm{Ch}N(x)\colon x \in S\} < G$.

LEMMA 4.6. *If S is a compact Hausdorff continuum, then $M\mathrm{Ch}T(x)$ is a continuum for all $x \in S$.*

Proof. Lemma 4.6 is a direct consequence of Lemma 2.6 and the fact that $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the finest decomposition of S into T-closed sets.

For an expansive N we can replace N in its every occurrence in Definition 4.1 by N^* of Definition 3.1, and thus define $\mathrm{Ch}_\alpha(N^*)$ and $M\mathrm{Ch}(N^*)$. It follows that $M\mathrm{Ch}(N^*)$ is a closure function such that $N^*(A)$

$\subset M\mathrm{Ch}(N^*)(A)$ and, for $x \in S$, $M\mathrm{Ch}(N^*)(x) = M\mathrm{Ch}N(x)$ (see below); but these functions can disagree at $A \subset S$. Hence we may use either N or N^* in the recursive chaining process of Definition 4.1 to obtain the core decomposition of S into N-closed sets.

For the space S of Example 1.8, $M\mathrm{Ch}T(x) = \mathrm{Ch}_3(Tx)$ for all $x \in S$ and $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is u.s.c. Example 4.7, typical in theory trouble below, shows that $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is not always u.s.c.

EXAMPLE 4.7. We modify the continuum C described and illustrated in [9], p. 195. The vertical segments used in the construction of C divide each horizontal segment $M_n = \{(x, y)\colon 0 \leqslant x \leqslant 1,\ y = 1/2^n\}$ $(n = 0, 1, 2, \ldots)$ of C into 2^{n+1} subintervals $J_{n,k}$, $(k = 1, 2, \ldots, 2^{n+1})$ each of length $1/2^{n+1}$. Let C' be the continuum obtained by replacing each $J_{n,k}$ by an indecomposable continuum $I_{n,k}$ such that $I_{n,k}$ contains both end-points of $J_{n,k}$ and such that $I_{n,k}$ lies entirely in the closed circular disk whose center is the midpoint of $J_{n,k}$ and whose radius is $1/2^{n+2}$. In C', $M\mathrm{Ch}T = \mathrm{Ch}_1 T$. Let $C_n = \bigcup\{I_{n,k}\colon 1 \leqslant k \leqslant 2^{n+1}\}$. Then $\{M\mathrm{Ch}T(x)\colon x \in S\}$ has precisely the sets C_n as non degenerate elements and is clearly not u.s.c. at any point $(x, 0)$ in C'.

Since concepts of Definitions 3.1, 4.1 and 4.4 can be intuitively tricky, we give the following example and two lemmas with proofs.

LEMMA 4.8. *If N is an expansive set function on space S, then $M\mathrm{Ch}(N^*)(x) = M\mathrm{Ch}N(x)$ for $x \in S$.*

Proof. Let $H^* = \{M\mathrm{Ch}(N^*)(x)\colon x \in S\}$ and $H = \{M\mathrm{Ch}N(x)\colon x \in S\}$. By Theorem 4.5, H^* is the core decomposition of S into N^*-closed elements and by Lemma 3.3 $N^* = N^\lambda$; also H is decomposition of S into N-closed, and hence N^*-closed elements. Thus $H^* < H$. But by Theorem 4.5, H is core decomposition of S into N-closed elements, and H^* is a decomposition of S into N-closed elements. Therefore $H < H^* < H = H^*$.

LEMMA 4.9. *If N is an expansive set function on S and $A \subset S$, then*

(1) $M\mathrm{Ch}N(A) = N\{z\colon M\mathrm{Ch}N(z) \cap A \neq \emptyset\}$ and $M\mathrm{Ch}(N^*)(A) = N^*\{z\colon M\mathrm{Ch}(N^*)(z) \cap A \neq \emptyset\}$;

(2) if also $N(x) = x$ for all $x \in S$, then $M\mathrm{Ch}N(x) = x$ and $M\mathrm{Ch}N(A) = N(A)$ and $M\mathrm{Ch}(N^*)(A) = N^*(A)$.

Proof. Let λ of Lemma 3.3 be δ for N and β for N^*. Then by Definition 4.1, $M\mathrm{Ch}N(A) = N\{z\colon z$ can be $\mathrm{Ch}_\lambda N(x_i)$-chained to $A\}$ and $M\mathrm{Ch}(N^*)(A) = N^*\{z\colon z$ can be $\mathrm{Ch}_\delta N^*(x_i)$-chained to $A\}$. But $\mathrm{Ch}_\alpha N = M\mathrm{Ch}N$ and $\mathrm{Ch}_\delta N^* = M\mathrm{Ch}(N^*)$; and by Lemma 4.8, $M\mathrm{Ch}N(x_i) = M\mathrm{Ch}(N^*)(x_i)$. Thus $M\mathrm{Ch}N(A) = N\{z\colon z$ can be $M\mathrm{Ch}N(x_i)$-chained to $A\}$; and since H above is a decomposition, only one link chains are possible. Therefore $M\mathrm{Ch}N(A) = N\{z\colon M\mathrm{Ch}N(z) \cap A \neq \emptyset\}$, and similarly (1) of the lemma is true for N^*; then (2) follows at once from (1).

EXAMPLE 4.10. Let S_1 be a "ladder" type of set with countable infinite number of "rungs" as in Example 2 of [2], p. 122, and one such is also a subset of C in Example 4.7 above with the M_n's as rungs — let M be the arc limiting set of these rungs, with c and d as end points of M. Let S_2 be the union of two Cantor triangles ([1], p. 267) with common base, one with vertex at a, the other with vertex at a' as in Example 4 of [2], p. 125. Let $S = S_1 \cup S_2$, taken so that $S_1 \cap S_2 = a \cup a' \subset M - c - d$. Let in (2) of Lemma 4.9, $A = c \cup d$, $N = T$. Then $T(x) = x$ for all $x \in S$, $T^* = T^2$, $T^2(A) \not\subset \mathrm{Ch}_2 T(A) = T(A) = M\mathrm{Ch}T(A)$. But $M\mathrm{Ch}(T^*)(A) = M \cup S_2 = T^*(A)$. Hence this is an S and A such that $M\mathrm{Ch}T(A) \neq M\mathrm{Ch}(T^*)(A)$.

One may easily verify the following characterization of the sets $M\mathrm{Ch}N(x)$.

THEOREM 4.11. *If N is an expansive set function on a set S then $\{M\mathrm{Ch}N(x)\colon x \in S\}$ consists of exactly those subsets B of S such that*

(1) B is N-closed;

(2) B is not the union of two or more disjoint non empty N-closed sets;

(3) B is maximal with respect to (1) and (2).

5. Relation to classical core decompositions. We show here that our decomposition $\{M\mathrm{Ch}T(x)\colon x \in S\}$ gives the classical decompositions of Kuratowski [7] and Wilson [15].

DEFINITION 5.1. A continuum S is *connected im kleinen* ([11], p. 89) *at* $B \subset S$ if and only if for every open set U such that $B \subset U \subset S$ there exists an open set V and a continuum W such that $B \subset V \subset W \subset U$.

There is no difficulty in proving Theorem 5.2 below.

THEOREM 5.2. *If G is a monotone decomposition of a compact Hausdorff continuum S, then (S, G) is locally connected if and only if S is connected im kleinen at each $g \in G$.*

THEOREM 5.3. *If S is a compact Hausdorff continuum such that $T(B) = \bigcup\{T(b)\colon b \in B\}$ for all closed subsets B of S and if G is any decomposition (not necessarily u.s.c.) of S into connected T-closed sets, then S is connected im kleinen at each $g \in G$.*

Proof. Let $g \subset U \subset S$ where U is open and $g \in G$. Consider any $x \in g$. For each $y \in S - U$, $g \cap G(y) = \emptyset$ and $T(y) \subset T(G(y)) = G(y)$. Thus $x \notin \bigcup\{T(y)\colon y \in S - U\}$. From the hypothesis on closed subsets, it follows that $x \notin T(S - U)$. Thus for each $x \in g$ there exists open Q_x and continuum W_x such that $x \in Q_x \subset W_x \subset U$. Since g is compact and connected, there exists open Q and continuum W such that $g \subset Q \subset W \subset U$.

THEOREM 5.4. *If S is a compact Hausdorff continuum such that $T(B) = \bigcup\{T(b)\colon b \in B\}$ for all closed subsets B of S, then there exists a core decomposition G of S with respect to the property: "is monotone with* *locally connected hyperspace". Furthermore, G is identical with the core decomposition of S with respect to being monotone with* s.l.c. *hyperspace.*

Proof. Let G be the core decomposition of S with respect to being monotone with s.l.c. hyperspace given by Theorem 2.7. By Theorem 2.7, G is monotone with T-closed elements. Hence, by Theorem 5.3 and 5.2, (S, G) is locally connected. If H is any monotone decomposition of S such that (S, H) is locally connected, then H is monotone with s.l.c. hyperspace, and hence, by Theorem 2.7, $G < H$.

DEFINITION 5.5. A continuum is *hereditarily finitely coherent* (*k-coherent*) if and only if the intersection of any two of its subcontinua has at most a finite number of components (at most k-components).

LEMMA 5.6. *If S is a compact Hausdorff hereditarily finitely coherent continuum then $T(B) = \bigcup\{T(b)\colon b \in B\}$ for all closed subsets B of S.*

Proof. Since T is expansive we need only prove $T(B) \subset \bigcup\{T(b)\colon b \in B\}$. Consider $x \notin \bigcup\{T(b)\colon b \in B\}$. For each $b \in B$ there exists an open set Q_b and continuum W_b such that $x \in Q_b \subset W_b \subset S - b$. The sets $S - W_b$ cover the compact set B. Hence there exists a collection $\{W_i\} \subset \{W_b\}$ $(i = 1, 2, \ldots, n)$ and corresponding Q_i such that $x \in \bigcap Q_i \subset \bigcap W_i \subset S - B$. Since S is hereditarily finitely coherent, $\bigcap W_i$ has but a finite number of components $C_1, C_2, \ldots, C_m$ with, say, $x \in C_1$. There exists open $V \subset S$ such that $x \in V \subset S - \bigcup C_j$ $(j = i$ for $i \neq 1)$. Then $x \in V \cap (\bigcap Q_i) \subset C_1 \subset S - B$, and so $x \notin T(B)$.

LEMMA 5.7. *If S is a compact Hausdorff hereditarily k-coherent continuum and G is a decomposition of S into connected T-closed sets, then G is* u.s.c.

Proof. Let $g_1 \neq g_2$ be distinct elements of G. It follows from Lemma 5.6 and Theorem 5.3 that there exist disjoint continua W_1 and W_2 and corresponding open sets Q_1 and Q_2 with $g_1 \subset Q_1 \subset W_1$ and $g_2 \subset Q_2 \subset W_2$. Since S is hereditarily k-coherent, at most k elements $g_a \in G$ meet both Q_1 and Q_2, and so we may take Q_1 and Q_2 such that no $g_a \in G$ meets both Q_1 and Q_2. Thus G, as an equivalence relation, is a closed subset of $S \times S$ and consequently G is u.s.c.

THEOREM 5.8. *If S is a compact Hausdorff hereditarily k-coherent continuum, then $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the core decomposition of S with respect to the property: is monotone with locally connected hyperspace.*

Proof. By Theorem 4.5, Lemma 4.6, and Lemma 5.7, it follows that $\{M\mathrm{Ch}T(x)\}$ is u.s.c. Consequently, by Theorem 4.5, and Theorem 2.7, $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the core decomposition of S with respect to being monotone with s.l.c. hyperspace. Theorem 5.8 follows from this last fact, Lemma 5.6 and Theorem 5.3 ([14], p. 41).

An important property of T-closed sets is given in Lemma 5.9, below, whose proof is the same as that of Theorem 16 ([1], p. 274).

LEMMA 5.9. *If S is any space, P is a T-closed subset of S, and $A \cup B \subset S - P$, then either there exists a continuum $C \subset S - P$ joining A to B, or $S - P = H \cup K$ where H and K are mutually separate* ([11], p. 1) *with $A \subset H$ and $B \subset K$.*

THEOREM 5.10. *If S is a compact Hausdorff continuum irreducible between $a \in S$ and $b \in S$ and G is a decomposition of S into connected T-closed sets, then G is* u.s.c.

Proof. We will prove that for g and g' in G there exists a g'' in G such that g'' separates g from g'. This will show that G is saturated, and therefore, by Theorem 4.21 of [12], p. 128, that G is u.s.c.

Let g and g' be in G, $g \neq g'$. Consider the case where $(a \cup b) \subset S - g$. In this case, by Lemma 5.9, $S - g$ is not connected, and hence, by Theorem (II, 3) of [9], p. 133, $S - g = H \cup K$ where both H and K are connected open sets, $H \cap K = \emptyset$, $G(a) \subset H$ and $G(b) \subset K$. We may assume $g' \subset K$. Since H and K are connected, there can exist no separation $S - g = A \cup B$ with $g' \subset A$ and $G(b) \subset B$. Hence, by Lemma 5.9, there exists a continuum $M \subset K$ joining g' to $G(b)$. Since K is open, $K \neq M \cup g' \cup G(b)$. Let $z \in K - (M \cup g' \cup G(b))$ and take $g'' = G(z)$. There can be no continuum C joining g and g' in $S - g''$, for then $(H \cup g) \cup (C \cup M \cup g' \cup G(b))$ would be a proper subcontinuum of S containing a and b. Hence, by Lemma 5.9, there is a separation $S - g'' = A \cup B$, $g \subset A$, $g' \subset B$. By symmetry, the only remaining case is where $a \in g$ and $b \in g'$. The proof for this case follows directly from the above argument.

THEOREM 5.11. *If S is a compact Hausdorff continuum irreducible between $a \in S$ and $b \in S$, then $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the core decomposition of S with respect to the property: "is monotone with locally connected hyperspace".*

Proof. By Theorem 6 of [2], p. 119, $T(B) = \bigcup\{T(b)\colon b \in B\}$ for all closed $B \subset S$, and, by Theorem 5.10 above, $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is u.s.c. Hence the proof of Theorem 5.11 is parallel to that of Theorem 5.8.

THEOREM 5.12. *If S is a compact separable metric continuum irreducible between two of its points, then $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is the decomposition of S into its tranches* ([7], p. 250).

Proof. For $x \in S$, let $\mathrm{Tr}(x)$ denote the tranche of S containing x. It is shown in [7] that $\{\mathrm{Tr}(x)\colon x \in S\}$ is a monotone decomposition with an arc or single point as hyperspace. Hence, by Theorem 5.11 above, $M\mathrm{Ch}T(x) \subset \mathrm{Tr}(x)$ for all $x \in S$. Theorem 5.11 and its proof show, in this separable metric case, that $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is semi-continue et linéaire ([7], p. 227); hence, by Théoréme Fondamental ([7], p. 259), $\mathrm{Tr}(x) \subset M\mathrm{Ch}T(x)$ for all $x \in S$.

THEOREM 5.13. *If S is a compact, separable, one-dimensional, m-cyclic, metric continuum, then $\{MChT(x)\colon x \in S\}$ is the decomposition of S into its irreducible elements* ([15], p. 385).

Proof. This follows from Lemma 5.8, above, and Theorem 18 of [15], p. 385.

6. Direct limit and $MChN$ spectra. We show here that the core decomposition of Corollary 2.2 may be realized as a direct limit of $MCh\theta$ decompositions and that the core decomposition of Theorem 2.7 may be realized as a direct limit of $MChT$ and $MCh\theta$ decompositions. We denote direct limit by $\varinjlim$.

DEFINITION 6.1. A collection $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$ of decompositions of a space is *ascending* if and only if given G_α and G_β in $\mathcal{G}$ there exists G_λ in $\mathcal{G}$ such that both $G_\alpha < G_\lambda$ and $G_\beta < G_\lambda$. If $\{G_\alpha\colon \alpha \in A\}$ is ascending, then we define $\bigvee \{G_\alpha\colon \alpha \in A\}$ to be the decomposition G defined by

$$G(x) = \{y \in S\colon \text{for some } \alpha \in A\,,\ G_\alpha(x) = G_\alpha(y)\}\,.$$

The following remarks will be helpful in Definition 6.2 below. Let $\{G_\alpha\colon \alpha \in A\}$ be an ascending collection of decompositions of a space S. The index set A becomes a directed set if we define, for α and β in A, $\alpha < \beta$ if and only if $G_\alpha < G_\beta$. For $\alpha < \beta$ the decomposition maps $\pi_\alpha\colon S \to (S, G_\alpha)$ and $\pi_\beta\colon S \to (S, G_\beta)$ induce a map $f_{\alpha,\beta} = \pi_\beta \pi_\alpha^{-1}$ of (S, G_α) onto (S, G_β). It is easy to verify that $\{(S, G_\alpha), f_{\alpha,\beta}, A\}$ is a direct limit system such that $(S, \bigvee \{G_\alpha\colon \alpha \in A\} = \varinjlim\{(S, G_\alpha), f_{\alpha,\beta}, A\}$.

DEFINITION 6.2. Let N be a set function, such as T or θ, which is defined and expansive on any set S. If S is any space, then we let $(S, MChN)$ denote the hyperspace of S with respect to the decomposition $\{MChN(x)\colon x \in S\}$, and call the associated decomposition map an $MChN$ *decomposition map*. If $f\colon X \to Y$ is any mapping we let $[f^{-1}]$ denote $\{f^{-1}(y)\colon y \in Y\}$. For an ordinal α, $\Omega(\alpha) = \{\lambda\colon 0 \leqslant \lambda \leqslant \alpha\}$. By the $MChN$ *spectrum for* S we mean the direct limit system $\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$ obtained as follows: Let $S_0 = S$ and let $S_1 = (S, MChN)$ with $f_{0,1}$ the $MChN$ decomposition map. Let $S_2 = (S_1, MChN)$, $f_{1,2}$ be the $MChN$ decomposition map and $f_{0,2} = f_{1,2} f_{0,1}$. By finite induction we continue this way and define $S_n = (S_{n-1}, MChN)$, $f_{n-1,n}$ the $MChN$ decomposition map and $f_{0,n} = f_{n-1,n} \times f_{0,n-1}$. The decompositions $[f_{0,n}^{-1}]$ are ascending, so we let $S_\omega = (S, \bigvee [f_{0,n}^{-1}])$. By transfinite induction and the preceding remarks, we may continue this process out to any given ordinal β and obtain a direct limit system $\{S_\lambda, f_{\lambda,\mu}, \Omega(\beta)\}$. Since the collection $\{[f_{0,\lambda}^{-1}]\colon \lambda \leqslant \beta\}$ is ascending, there exists a first ordinal α such that $S_\alpha = S_{\alpha+1}$. Then $\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$ is a direct limit system such that for a nonlimit ordinal $\lambda \leqslant \alpha$, $f_{\lambda-1,\lambda}$ is the $MChN$ map and for a limit ordinal $\beta \leqslant \alpha$, $S_\beta = (S, \bigvee \{f_{0,\lambda}^{-1}\colon \lambda < \beta\})$.

THEOREM 6.3. *Let S be any space and G be the core decomposition of S with respect to the property: "the hyperspace is Hausdorff". Then $(S, G) = \varinjlim\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$, where $\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$ is the $MCh\theta$ spectrum for S.*

Proof. We note that $\varinjlim\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\} = S_\alpha$ and $S_\alpha = (S, [f_{0,\alpha}^{-1}])$. Since $S_\alpha = S_{\alpha+1}$, $MCH\theta$ is the identity on S_α. Hence S_α is a Hausdorff space and $[f_{0,\alpha}^{-1}]$ is a decomposition of S into θ closed sets. Thus $G < [f_{0,\alpha}^{-1}]$. We must show that $[f_{0,\alpha}^{-1}] < G$, which we do by induction. Since $[f_{0,1}^{-1}] = \{MCh\theta(x)\colon x \in S\}$, clearly by Lemma 1.6 and Theorem 4.5, $[f_{0,1}^{-1}] < G$. Suppose for a nonlimit ordinal $\lambda < \alpha$, $[f_{0,\lambda-1}^{-1}] < G$. Then $\pi f_{0,\lambda-1}^{-1}\colon S_{\lambda-1} \to (S, G)$ is a map, where $\pi\colon S \to (S, G)$ is the decomposition map. Since (S, G) is Hausdorff, $[(\pi f_{0,\lambda-1}^{-1})^{-1}]$ is a decomposition of $S_{\lambda-1}$ into θ closed sets. Since $f_{\lambda-1,\lambda}$ is the $MCh\theta$ map, $[f_{\lambda-1,\lambda}^{-1}] < [(\pi f_{0,\lambda-1}^{-1})^{-1}]$, from which it follows, since $f_{0,\lambda} = f_{\lambda-1,\lambda} f_{0,\lambda-1}$, that $[f_{0,\lambda}^{-1}] < [\pi^{-1}] = G$. If $\beta < \alpha$ is a nonlimit ordinal and $[f_{0,\lambda}^{-1}] < G$ for all $\lambda < \beta$, then clearly $[f_{0,\beta}^{-1}] = \bigvee \{[f_{0,\lambda}^{-1}]\colon \lambda < \beta\} < G$. Thus by transfinite induction $[f_{0,\alpha}^{-1}] = G$.

DEFINITION 6.4. By the *Hausdorff map* we mean the decomposition map $h\colon S \to (S, G)$ where G is the core decomposition of S with respect to giving a Hausdorff hyperspace. Let S be any space and N as in Definition 6.2. The h-$MChN$ *spectrum for* S is the direct limit system $\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$ obtained by modifying the construction of the $MChN$ spectrum as follows: for a nonlimit ordinal $\lambda \geqslant 1$ let S_λ be obtained from $S_{\lambda-1}$ by $f_{\lambda-1,\lambda} = h\varphi_{\lambda-1}$ where $\varphi_{\lambda-1}$ is the $MChN$ map and h is the Hausdorff map, and we define $S'_{\lambda-1}$ and S_λ by

$$S_{\lambda-1} \xrightarrow{\varphi_{\lambda-1}} S'_{\lambda-1} \xrightarrow{h} S_\lambda;$$

for a limit ordinal β let $S'_\beta = (S, \bigvee \{[f_0^{-1}]\colon \lambda < \beta\})$ and $S_\beta = h(S'_\beta)$.

THEOREM 6.5. *Let S be a compact Hausdorff continuum and G be the core decomposition of S with respect to being* u.s.c. *with* s.l.c. *hyperspace. Then $(S, G) = \varinjlim\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$, where $\{S_\lambda, f_{\lambda,\mu}, \Omega(\alpha)\}$ is the h-$MChT$ spectrum of S.*

Proof. We have $S \xrightarrow{\varphi_0} (S, MChT) \xrightarrow{h} S_1$ and $f_{0,1} = h\varphi_0$. It follows, because S is compact, $[f_{0,1}^{-1}] = \bigwedge \{G_\alpha\colon \{MChT(x)\} < G_\alpha$ and G_α is u.s.c.$\}$, Then, since each $MChT(x)$ is a continuum, $[f_{0,1}^{-1}]$ is a monotone decomposition and $f_{0,1}$ is a monotone map. By induction it follows that all maps $f_{\lambda,\mu}$ in the h-$MChT$ spectrum are monotone, and that each S_λ is a compact Hausdorff continuum. Thus, by following the proof of Theorem 6.3 and making use of Theorem 2.4, one obtains a proof of Theorem 6.5. (For S of Example 4.7, $S'_0 = (S, MChT)$ is not a Hausdorff space, but $S_1 = (S'_0, MCh\theta)$ is Hausdorff and $S_1 = (S, G)$ of Theorem 6.5.)

Generalizations of Theorem 6.3 and Theorem 6.5 are clearly suggested by the above proofs, but we will not labor this here. Instead we give a method for attaining our basic core decompositions which is closely related to the direct limit method, but which arises from a considerably different point of view.

7. Upper-semi-continuity by modifications of $MChN$. We show here that a simple modification of Definition 4.1 leads to a set function $MCHN$ such that $\{MCHN(x): x \in S\}$ is the core decomposition of S with respect to being u.s.c. with N-closed elements, where S is any compact space and N any expansive set function on S.

DEFINITION 7.1. For an expansive set function N on a space S we define, for all $A \subset S$, $\mathrm{CH}_0N(A) = N(A)$. For $x \in S$, define $F_0(x)$ by $y \notin F_0(x)$ if and only if there exist disjoint open sets U_y and V_x such that $\mathrm{CH}_0N(y) \subset U_y$, $\mathrm{CH}_0N(x) \subset V_x$ where U_x and V_x are both unions of sets $\mathrm{CH}_0N(z)$. Let $L_0 = \{F_0(x): x \in S\}$. Now define, for $A \subset S$, $\mathrm{CH}_1N(A) = N\{z: z$ can be simply chained to A by sets of type $\mathrm{CH}_0N(x)$ or elements of $L_0\}$. For a nonlimit ordinal α, define $\mathrm{CH}_\alpha N(A) = N\{z: z$ can be simply chained to A by links of either type $\mathrm{CH}_{\alpha-1}N(x)$ or elements of $L_{\alpha-1}\}$, where $L_{\alpha-1}$ is defined with respect to $\mathrm{CH}_{\alpha-1}N$ exactly as L_0 is defined with respect to CH_0N. For a limit ordinal α, define $\mathrm{CH}_\alpha N(A) = \bigcup\{\mathrm{CH}_\lambda N(A): \lambda < \alpha\}$. We define $MCHN(A) = \mathrm{CH}_\alpha N(A)$, where α is the first ordinal number such that $\mathrm{CH}_\alpha N(B) = \mathrm{CH}_{\alpha+1}N(B)$ for all $B \subset S$. (The existence of such an α follows by exactly the same arguments used to prove Lemmas 4.2 and 4.3.)

THEOREM 7.2. *If S is any space and N any expansive set function on S, then $\{MCHN(x): x \in S\}$ is the core decomposition of S with respect to having N-closed elements and giving a Hausdorff hyperspace.*

Proof. Let G be the core decomposition of S with respect to having N-closed elements and giving a Hausdorff hyperspace. That G exists follows from Theorem 2.1 and the last part of the proof of Theorem 2.5. By definition 7.1, $\mathrm{CH}_0N(x) \subset G(x)$ for each $x \in S$. Since G gives a Hausdorff hyperspace, it follows $F_0(x) \subset G(x)$ for each $x \in S$. Consequently no simple chain with links of type either $\mathrm{CH}_0N(x_i)$ or $F_0(x_i)$ can meet two different elements of G. Thus $\mathrm{CH}_1N(x) \subset G(x)$, for all $x \in S$. It follows by induction that $MCHN(x) \subset G(x)$ for all $x \in S$. It is obvious from Definition 7.1 that $\{MCHN(x): x \in S\}$ is a decomposition of S into N-closed elements and gives a Hausdorff hyperspace. Hence, by definition of core, $G = \{MCHN(x): x \in S\}$.

In the case of a compact S, $\{MCHN(x): x \in S\}$ is u.s.c. (because the hyperspace is Hausdorff) and obviously gives the decompositions of Corollary 2.2 and Theorem 2.7.

The $MCHN$ and direct limit decompositions of Section 6 do not agree in general. For example, there exists a locally compact Hausdorff continuum S where $\{MCHT(x): x \in S\}$ does not give a s.l.c. hyperspace while the direct limit decomposition does give a s.l.c. hyperspace. (These examples are tricky. The basic cause of this disagreement is that in $MCHN$ everything happens in S, while in the limit process the space keeps changing.)

8. Relation to McAuley's decomposition. We show here that the decomposition given by McAuley in his Theorem 3.1 ([10], p. 3) is our core (and hence atomic) decomposition of Theorem 2.7.

Lemma 8.1 below, whose proof is straight forward, is helpful in understanding McAuley's Theorem 3.1. Definition 8.2 below is due to McAuley; it is a simplification of Definition 1.1 in [10], p. 2.

LEMMA 8.1. *Let S be a compact Hausdorff space and K be any given collection of closed separators of S, and let, for each $x \in S$, $L(x) = \{y \in S$: no $k \in K$ separates x from $y\}$. Then the following are equivalent:*

(1) *If $k \in K$ separates the points a and b in S and if $c \in k$ then either some $k' \in K$ separates a from c or some $k' \in K$ separates b from c.*

(2) *$\{L(x): x \in S\}$ is a decomposition of S.*

(3) *$\{L(x): x \in S\}$ is an* u.s.c. *decomposition of S.*

DEFINITION 8.2. Let S be any space, and for each $p \in S$ let $M(p)$ denote the set of all points $y \in S$ for which there *does not exist* a collection K of closed separators of S such that (1) some $k \in K$ separates p from y and (2) for $k \in K$ and any separation $S - k = A \cup B$ and points $a \in A$, $b \in B$ and $c \in k$ there exists $k' \in K$, open set Q, and continuum W *such that $a \in Q \subset W$ and k' separates $c \cup W$ from b.*

THEOREM 8.3. *If S is a compact Hausdorff continuum, $M = \{M(p)$: $p \in S\}$, and G is the core decomposition of S with respect to being* u.s.c. *with T-closed elements, then $M = G$.*

Proof. For each collection of closed separators K_α which has property (2) of Definition 8.2, let $M_\alpha = \{L_\alpha(x): x \in S\}$ where L_α is defined with respect to K_α as in Lemma 8.1. Then by Lemma 8.1 and condition (2) of Definition 8.2 each M_α is an u.s.c. decomposition of S into T-closed sets. Furthermore, $M = \bigwedge\{M_\alpha\}$ and so by the proof of Theorem 2.5 M is u.s.c. with T-closed elements. Thus $G < M$.

Suppose $M \not< G$ and so for some x and y in S we have $y \in M(x)$ but $y \notin G(x)$. Since $y \notin G(x)$, there exists a decomposition H of S, which by Theorem 2.7 we may assume to be monotone with (S, H) s.l.c., such that $H(x) \neq H(y)$. Let π be the decomposition map associated with (S, H), and $K = \{\pi^{-1}(k)$: k is a closed separator of $S\}$. Because π is a monotone map K has both properties of Definition 8.2 with respect to x any y, and so $y \notin M(x)$ which is a contradiction. Thus $M < G$, and so $M = G$.

The decomposition given in Theorem 8.2 of [10], p. 9, is a core decomposition, but *not with respect to any of the properties we have considered*;

this decomposition is core with respect to satisfying McAuley's condition K_3 ([10], p. 9), but is not core (or atomic) with respect to giving an aposyndetic hyperspace, nor is it core with respect to having T-closed elements.

References

[1] H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Semigroups, continua and the set functions* T^n, Duke Math. Jour. 29 (1962), pp. 265-280.

[2] — *Properties of the set function* T^n, Portugal. Math. 21 (1962), pp. 113-133.

[3] H. S. Davis and P. M. Swingle, *The iteration and recursion of set functions*, Portugal. Math. 25 (1964), pp. 103-129.

[4] P. C. Hammer, *Extended topology: Set-valued set-functions*, Nieuw Archief voor Wiskunde (3), X, (1962), pp. 55-77.

[5] F. Hausdorff, *Mengenlehre*, Berlin 1927.

[6] F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. Jour. Math. 67 (1941), pp. 545-553.

[7] C. Kuratowski, *Théorie des continus irréductibles entre deux points* II, Fund. Math. 10 (1927), pp. 225-275.

[8] — *Les décompositions semi-continues d'espaces métriques compacts*, Fund. Math. 11 (1928), pp. 169-185.

[9] — *Topologie, II*, Warszawa 1961.

[10] L. F. McAuley, *An atomic decomposition of continua into aposyndetic continua*, Trans. Amer. Math. Soc. 88 (1958), pp. 1-11.

[11] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications 13 (1932).

[12] G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28 (1942).

[13] — *Semi-locally-connected sets*, Amer. J. Math. 61 (1939), p. 738.

[14] R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1949).

[15] W. A. Wilson, *On upper semi-continuous decompositions of compact continua*, Amer. Jour. Math. 54 (1932), pp. 377-386.

THE UNIVERSITY OF WISCONSIN, AND MIAMI

Reçu par la Rédaction le 14. 7. 1966

Spaces in which sequences suffice II

by

S. P. Franklin (Pittsburgh, Pennsylvania)

4. Introduction. In this paper we continue the work begun in [6] presenting some new facts on sequential and Fréchet spaces (Sections 5 and 6) and paying particular attention to those sequential spaces which are not Fréchet spaces (Section 7).

5. Sequential spaces II. The category of sequential spaces fails to have two important permanence properties; it is neither hereditary ([6], Example 1.8) nor productive ([6], Example 1.11). There is another example of a non-sequential subspace of a sequential space (due essentially to Arens [1]) which plays a critical role in what follows.

5.1. EXAMPLE. *There is a countable, normal sequential space M with a non-sequential subspace.*

Proof. Let $M = (\boldsymbol{N}\times\boldsymbol{N})\cup\boldsymbol{N}\cup\{0\}$ with each $(m, n)\in\boldsymbol{N}\times\boldsymbol{N}$ an isolated point, where $\boldsymbol{N}$ denotes the set of natural numbers. For a basis of neighborhoods at $n_0\in\boldsymbol{N}$, take all sets of the form $\{n_0\}\cup\{(m, n_0)|\ m\geqslant m_0\}$. U will be a neighborhood of 0 if and only if $0\in U$ and U is a neighborhood of all but finitely many $n\in\boldsymbol{N}$. One verifies routinely that M is normal and sequential. We shall show that $\{0\}$ is sequentially open but not open in $M\setminus\boldsymbol{N}$.

Since $0\in\mathrm{cl}_M(\boldsymbol{N}\times\boldsymbol{N})$, $\{0\}$ is not open in $M\setminus\boldsymbol{N}$. If $\{(m_i, n_i)\}$ is any sequence in $\boldsymbol{N}\times\boldsymbol{N}$, either there is some $n_0\in\boldsymbol{N}$ such that $n_i = n_0$ for infinitely many i, or there is no such n_0. In the first case, $\{(m_i, n_i)\}$ has a cluster point in the set $\{n_0\}\cup\{(m, n_0)|\ m\in\boldsymbol{N}\}$ and hence does not converge to 0. In the second case one easily constructs a neighborhood of 0 disjoint from $\{(m_i, n_i)\}$.

We are left by these examples with the problem of characterizing those subspaces of a sequential space X which are themselves sequential. Such a characterization can be effected in terms of X as a quotient (under the quotient map φ_X) of X^*, the topological sum of its convergent sequences (see 1.12, [6]) as follows.

5.2. PROPOSITION. *A subspace Y of a sequential space X is sequential iff $\varphi_X|\overset{-1}{\varphi_X}(Y)$ is a quotient map.*