Or,

$$\mu\cdot P_{0,t}-\mu\cdot P_{t-1,t}=\sum_{k=1}^{t-1}[\mu\cdot P_{k-1,t}-\mu\cdot P_{k,t}],$$

donc

$$\lim_{t\to+\infty}\sum_{k=1}^{t-1}[\mu\cdot P_{k-1,t}-\mu\cdot P_{k,t}]=0,$$

c'est-à-dire (iii).

On peut également remarquer que $\mu\cdot P_{k-1,k}-\mu=M_{k-1,k}$ où $M_{k-1,k}$ est un vecteur dont la somme des composantes vaut 0, c'est-à-dire tel que

$$M_{k-1,k}(r)=-\sum_{i=1}^{r-1}M_{k-1,k}(i).$$

D'où

$$\mu\cdot P_{k-1,t}-\mu\cdot P_{k,t}=(\mu\cdot P_{k-1,k}-\mu)P_{k,t}=M_{k-1,k}\cdot P_{k,t},$$

c'est-à-dire

$$\sum_{i\in\mathscr{X}}M_{k-1,k}(i)p_{k,t}(i,j)=\sum_{i=1}^{r-1}M_{k-1,k}(i)[p_{k,t}(i,j)-p_{k,t}(r,j)].$$

La condition (iii) s'écrit alors

$$\lim_{t\to+\infty}\sum_{k=1}^{t-1}\sum_{i=1}^{r-1}M_{k-1,k}(i)[p_{k,t}(i,j)-p_{k,t}(r,j)]=0.$$

Nous retrouvons alors l'énoncé donné par Koźniewska dans [3].

Pour les exemples, le lecteur peut se reporter aux articles [2] et [3] et à ceux qu'ils citent en références. D'autre part, nous pensons que les résultats énoncés ci-dessus peuvent être étendus à des processus de Markov d'ordre $n>1$. Signalons enfin que dans [5], son auteur énonce une condition nécessaire et suffisante pour l'ergodicité forte uniforme et une autre pour l'ergodicité faible uniforme des processus de Markov d'ordre n.

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STUDIA MATHEMATICA, T. XXVIII. (1967)

Matrix transformations of Schauder bases

by

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Let X be a real or complex Banach space which has a Schauder basis, $\mathscr{X}=\{x_i: i=1,2,\ldots\}$, and let $\mathscr{F}=\{f_i: i=1,2,\ldots\}$ be the sequence of continuous linear functionals biorthogonal to $\mathscr{X}$. If $\mathscr{Y}=\{y_i\}$ is any sequence in X, there is an infinite matrix $A=(a_{ij})$ such that $\mathscr{Y}=\mathscr{X}A$ in the sense that

$$y_j=\sum_{i=1}^{\infty}a_{ij}x_i \quad \text{for} \quad j=1,2,\ldots$$

In this paper we intend to further the investigation begun in [3] concerning conditions on A which imply that $\mathscr{Y}$ is a basic sequence in $\mathscr{X}$ or a Schauder basis of X.

The notation used here will be the same as that in [3]. By $s=(s_i)$ we mean a scalar sequence which we handle as an infinite column vector. If S is a linear space of such sequences, S' is the β-dual of S, i.e. $\{t=(t_i): \sum_{i=1}^{\infty}s_it_i$ converges for each $s\in S\}$. Given an infinite matrix $C=(c_{ij})$ each row of which is in S' we write $C(S)$ for $\{t=Cs: s\in S\}$.

Let $S_x=\{s: \sum_{i=1}^{\infty}s_ix_i$ converges in $X\}=\{(f_i(x)): x\in X\}$; then S_x with norm $\|(f_i(x))\|=\|x\|$ is a Banach space isometric to X under the correspondence

$$\eta_x(f_i(x))=x. \tag{1}$$

Since x_i corresponds to $e^i=(\delta_{ij})_{j=1}^{\infty}$, $\{e^1,e^2,\ldots\}$ is a basis for S_x. Define S_y^0 to be $\{s: \sum_{i=1}^{\infty}s_iy_i$ converges in $X\}$; then S_y^0 is a Banach norm

$$\|s\|=\sup_n\Big\|\sum_{i=1}^{n}s_iy_i\Big\|$$

and $\{e^1,e^2,\ldots\}$ is a basis for S_y^0. Of course, we must assume that $y_i\neq 0$ for all i. We shall assume this condition satisfied throughout the remainder of this paper.

Let

$$Y = \Big\{y \in X\colon\, y = \sum_{i=1}^{\infty} t_i y_i\Big\};$$

then there is a function from S_y^0 onto Y given by

$$\eta_y(t) = \sum_{i=1}^{\infty} t_i y_i.$$

By Lemma 1.1 of [3] each row of A is in $(S_y^0)'$ and $A(S_y^0) \subseteq S_x$. Let the closed linear span of $\{y_i\colon i = 1, 2, \ldots\}$ in X be denoted by L.

The following diagram summarizes the relation among X, L, Y, S_x, S_y^0 and A:

$$\begin{tikzcd} Y \subseteq L \subseteq & X \\ A: S_y^0 \arrow[u, "\eta_y"] \arrow[r] & S_x \arrow[u, "\eta_x"'] \end{tikzcd}$$

In section 3 of [3] it was observed that with the correspondences

$$f \leftrightarrow \big(f(e^i)\big) f \in S_x^*, \qquad \big(f(e^i)\big) \in S_x', \tag{3}$$

$$g \leftrightarrow \big(g(e^i)\big) g \in (S_y^0)^*, \qquad \big(g(e^i)\big) \in (S_y^0)'. \tag{4}$$

S_x' and $(S_y^0)'$ can be given BK-topologies which make them equivalent to the conjugate spaces S_x^* and $(S_y^0)^*$ respectively. We here note that S_x' is also equivalent to X^* under the correspondence

$$f \leftrightarrow \big(f(x_i)\big) f \in X^*, \qquad \big(f(x_i)\big) \in S_x'. \tag{5}$$

LEMMA 1. *For each $f \in X^*$, $A^T\big(f(x_i)\big)$ exists and is equal to $\big(f(y_i)\big)$, where A^T is the transpose matrix of A.*

Proof. Given $f \in X^*$, we have

$$f(y_i) = f\Big(\sum_{j=1}^{\infty} a_{ji} x_j\Big) = \sum_{j=1}^{\infty} a_{ji} f(x_j).$$

But $\sum_{j=1}^{\infty} a_{ji} f(x_j)$ is the i-th coordinate of $A^T\big(f(x_i)\big)$. Also $\big(f(y_i)\big) \in S_y'$ since if $\sum_{i=1}^{\infty} t_i y_i$ converges so does

$$f\Big(\sum_{i=1}^{\infty} t_i y_i\Big) = \sum_{i=1}^{\infty} t_i f(y_i).$$

From Lemma 1 we observe that A^T represents the adjoint of the operator defined by A from S_y^0 into S_x in the sense that

$$\big(A^* f(e_i)\big) = A^T\big(f(e_i)\big) \quad \text{for} \quad f \in S_x^*. \tag{6}$$

THEOREM 1. *The following statements are equivalent:*

(a) *the sequence $\mathscr{Y}$ is basic in X;*

(b) *A maps S_y^0 one to one onto the image of L under the mapping η_x^{-1};*

(c) (i) *$A(S_y^0)$ is closed in S_x and* (ii) *$\sum_{i=1}^{\infty} t_i y_i = 0$ implies $t_i = 0$ for each i;*

(d) *$(S_y^0)' = \big\{\big(f(y_i)\big)\colon f \in X^*\big\}$;*

(e) *A^T maps S_x' onto $(S_y^0)'$;*

(f) (i) *$\sum_{i=1}^{\infty} t_i y_i = 0$ implies $t_i = 0$ for each i and* (ii) *$A^T(S_x')$ is closed in $(S_y^0)'$.*

Proof. (a) $\Leftrightarrow$ (b). By definition the sequence $\mathscr{Y}$ is basic if and only if each $y \in L$ has a unique expansion of the form $y = \sum_{i=1}^{\infty} t_i y_i$. By Lemma 1.1 of [3], if $t \in S_y^0$, $\eta_x^{-1}\eta_y(t) = A(t)$. Thus, A maps S_y^0 onto $\eta_x^{-1}L$ if and only if $Y = L$ and A is one to one if and only if $\sum_{i=1}^{\infty} t_i y_i = 0$ implies $t_i = 0$ for each i.

(b) $\Leftrightarrow$ (c). Since $\mathscr{Y}$ is dense in L, $A(S_y^0)$ is dense in $\eta_x^{-1}(L)$ so that (i) of (c) holds if and only if $A(S_y^0) = \eta_x^{-1}(L)$. That (ii) of (b) holds if and only if A is one to one was noted above.

(a) $\Leftrightarrow$ (d). By a theorem of Grinblyum [2], $\mathscr{Y}$ is a basis for L, the closed linear span of $\mathscr{Y}$ in X if and only if $(S_y^0)' = \{F(y_i)\colon F \in L^*\}$. But each $F \in L^*$ can be extended to X and each $f \in X^*$ can be restricted to L so that

$$\{F(y_i) : F \in L^*\} = \big\{\big(f(y_i)\big) : f \in X^*\big\}.$$

(d) $\Leftrightarrow$ (e). This follows immediately from the fact that the image of S_x' under A^T is precisely $\big\{\big(f(y_i)\big) : f \in X^*\big\}$.

(e) $\Rightarrow$ (f). Obvious.

(f) $\Rightarrow$ (c). From (ii) of (f) and equation (6) we conclude that the image of S_x^* under A^* is closed in $(S_y^0)^*$. By [1], Theorem 4, p. 488, $A(S_y^0)$ is closed in S_x

LEMMA 2. *The sequence $\mathscr{Y}$ is fundamental in X if and only if A^T is one to one on S_x'.*

Proof. Recall that $\mathscr{Y}$ is fundamental in X if and only if $f(y_i) = 0$ for each i implies $f = 0$ for $f \in X^*$. If A^T is one to one and $f(y_i) = 0$ for each i, then by Lemma 1, $f(x_i) = 0$ for each i so $f = 0$. If A^T is not one to one on S_x' there must be $f \neq 0$ in X^* such that $A^T\big(f(x_i)\big) = \big(f(y_i)\big) = 0$ so that $\mathscr{Y}$ is not fundamental.

THEOREM 2. *The following statements are equivalent:*

(a) *the sequence $\mathscr{Y}$ is a basis for X;*

(b) A^T *maps* S'_x *one to one onto* $(S^0_y)'$;

(c) $A(S^0_y) = S_x$ *and* $A^T(S'_x) = (S^0_y)'$;

(d) (i) $\sum_{i=1}^{\infty} t_i y_i = 0$ *implies* $t_i = 0$ *for each* i, (ii) A^T *is one to one on* S'_x, (iii) $A^T(S'_x)$ *is closed in* $(S^0_y)'$.

Proof. (a) $\Leftrightarrow$ (b). This follows from (a) $\Leftrightarrow$ (e) of Theorem 1 and Lemma 2. This statement is Theorem 3.2 of [3].

(a) $\Leftrightarrow$ (c). By (e) of Theorem 1, $A^T(S'_x) = (S^0_y)'$ if and only if $\mathcal{Y}$ is basic. In the proof of Theorem 2.1 of [3] it is shown that $A(S^0_y) = S_x$ if and only if $\mathcal{Y}$ is fundamental in X.

(a) $\Leftrightarrow$ (d). This follows from (f) $\Leftrightarrow$ (a) in Theorem 1 and from Lemma 2.

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On the characterization of sequence spaces associated with Schauder bases

by

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1. Introduction. An F-space which has a Schauder basis is essentially a space of sequences ([9], p. 207). This paper discusses the question: What kinds of sequence spaces are associated with a Schauder basis of a locally convex F-space? The chief results are contained in Theorems 3.1, 3.2 and 3.3. They are correspondences between (a) Schauder bases and γ-perfect FK-spaces (b) unconditional bases and α-perfect FK-spaces (c) symmetric bases and σ-perfect FK-spaces. (See 2.1 and 2.2 for definitions of γ, α and σ-perfect.)

1.1. Definition. An *F-space* is a complete linear metric space.

A sequence $\chi = \{x_1, x_2, \ldots\}$ is a *basis* for the F-space X if each point x of X has a unique representation

$$x = \sum_{i=1}^{\infty} t_i x_i \tag{1.1}$$

where (t_i) is a sequence of scalars.

The sequence χ is an *unconditional basis* if the convergence in (1.1) is unconditional.

In the sequel we shall limit our consideration to χ a basis for a locally convex F-space.

It is known ([9], p. 207) that the linear functionals defined by

$$f_n\Big(\sum_{i=1}^{\infty} t_i x_i\Big) = t_n$$

are continuous. Thus if the linear space of sequences

$$S = \Big\{(t_i)\colon \sum_{i=1}^{\infty} t_i x_i \text{ converges in } X\Big\}$$

is given the identity topology with respect to the isomorphism

$$\sum_{i=1}^{\infty} t_i x_i \leftrightarrow (t_i). \tag{1.2}$$