ACTA ARITHMETICA
XIV (1968)

On lattice points with weight in high-dimensional ellipsoids

by

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§ 1. Introduction. Let r be a natural number, $r \geqslant 2$, and let

$$Q(u) = Q(u_j) = \sum_{l,j=1}^{r} a_{lj} u_l u_j \tag{1}$$

be a positive definite quadratic form, with the determinant D. Let further M_j, b_j and a_j be real numbers, $M_j > 0$ $(j = 1, 2, \ldots, r)$. For $x > 0$ let us consider the function

$$A(x) = \sum e^{2\pi i \sum_{j=1}^{r} a_j u_j} \tag{2}$$

where the summation runs over all systems $u = (u_1, u_2, \ldots, u_r)$ of real numbers, satisfying

$$u_j \equiv b_j \pmod{M_j}$$

$(j = 1, 2, \ldots, r)$ and

$$Q(u) \leqslant x.$$

If we put

$$M = \frac{\pi^{r/2}}{\sqrt{D} \prod_{j=1}^{r} M_j} \quad \text{and} \quad V(x) = \frac{M e^{2\pi i \sum_{j=1}^{r} a_j b_j} \delta}{\Gamma(\frac{1}{2}r+1)} x^{r/2}$$

($\delta = 1$ if all numbers $a_1 M_1, a_2 M_2, \ldots, a_r M_r$ are integers, $\delta = 0$ otherwise) there hold as known (e.g. [3], pp. 11-84) for the "lattice rest"

$$P(x) = A(x) - V(x)$$

the formulas

$$P(x) = O\left(x^{\frac{r}{2} - \frac{r}{r+1}}\right) \tag{3}$$

and (if $A(x)$ is not identically zero)

$$P(x) = \Omega\left(x^{\frac{r-1}{4}}\right). \tag{4}$$

The basic problem in the study of the function $P(x)$ is the finding of "exact" exponent in the O- and Ω-estimations, i.e. the finding of such a number f, that the formulas

$$P(x) = O(x^{f+\varepsilon}), \quad P(x) = \Omega(x^{f-\varepsilon})$$

hold for every $\varepsilon > 0$. If we exclude from our considerations the case where $A(x) \equiv 0$ identically, it is clear that

$$f = \limsup_{x\to+\infty} \frac{\lg|P(x)|}{\lg x}. \tag{5}$$

Landau's estimations (3) and (4) imply the generally valid inequality

$$\frac{r}{4} - \frac{1}{4} \leqslant f \leqslant \frac{r}{2} - \frac{r}{r+1}.$$

The exact value of f has been found in some special cases. The first definitive result follows from the work of Jarník (in Landau [3], p. 162), Landau ([3], p. 148) and Walfisz [6]. If $r > 4$,

$$M_j = 1, \quad b_j = a_j = 0, \quad j = 1, 2, \ldots, r \tag{6}$$

and if the form (1) has integer coefficients, then $f = \frac{1}{2}r - 1$ and, more precisely,

$$P(x) = O(x^{\frac{r}{2}-1}), \quad P(x) = \Omega(x^{\frac{r}{2}-1}). \tag{7}$$

Jarník obtained a number of definitive results for irrational diagonal ellipsoids (assuming (6)). Let us present the most interesting of the mentioned results: In the paper [1] Jarník investigates forms $Q(u)$ of the type

$$Q(u_j) = a_1(u_1^2 + u_2^2 + \ldots + u_{r_1}^2) + a_2(u_{r_1+1}^2 + u_{r_1+2}^2 + \ldots + u_{r_1+r_2}^2),$$

where $r_1 \geqslant 4$, $r_2 \geqslant 4$ are integers and $r = r_1 + r_2$, $a_1 > 0$, $a_2 > 0$.

Let us denote by $\gamma = \gamma(a_1, a_2)$ the supremum of all such numbers $\beta > 0$, that the inequality

$$\left| q\frac{a_1}{a_2} - p \right| \leqslant q^{-\beta}$$

is valid for infinitely many pairs $p = p_n$, $q = q_n$ $(n = 1, 2, \ldots)$ of natural numbers, such that $p_n \to +\infty$, $q_n \to +\infty$ (thus $1 \leqslant \gamma \leqslant +\infty$). Then, assuming (6), we have for the value (5)

$$f = \frac{r}{2} - 1 - \frac{1}{\gamma} \tag{8}$$

(for $\gamma = +\infty$ we put $1/\gamma = 0$).

Let us now confine ourselves to the case where the coefficients of the form (1) and the numbers b_j are integers, M_j natural numbers $(j = 1, 2, \ldots, r)$, $r > 4$. The numbers $a_1, a_2, \ldots, a_r$ can be arbitrary real numbers. From Landau's and Walfisz's papers ([3], p. 148 and [6]) it follows, that if the numbers $a_1, a_2, \ldots, a_r$ are rational, then always

$$P(x) = O(x^{\frac{r}{2}-1})$$

and if at least one of the corresponding generalized Gauss sums is non-vanishing, then also

$$P(x) = \Omega(x^{\frac{r}{2}-1})$$

([7], pp. 52-53). Thus, under these assumptions

$$f = \frac{r}{2} - 1.$$

As Walfisz shows ([7], p. 62, Theorem 3) making use of the theory of modular forms, in the opposite case (i.e. if all corresponding Gauss sums are zero)

$$f \leqslant \frac{r}{4} - \frac{1}{10}.$$

On the base of a generalization of the first Petersson theorem I have found in [5] some results for $r > 4$:

I. For arbitrary $a_1, a_2, \ldots, a_r$ is

$$P(x) = O(x^{\frac{r}{2}-1});$$

thus

$$f \leqslant \frac{r}{2} - 1.$$

II. If at least one of the numbers $a_1, a_2, \ldots, a_r$ is irrational, then

$$P(x) = o(x^{\frac{r}{2}-1}) \tag{9}$$

and this estimation cannot be generally improved, i.e. if $\varphi(x)$ is a positive increasing function defined for $x > 0$, $\varphi(x) = o(x^{r/2-1})$, there exists such a system $a_1, a_2, \ldots, a_r$ that (9) holds and

$$P(x) = \Omega(\varphi(x)).$$

Thus there exist "irrational" systems $a_1, a_2, \ldots, a_r$, for which

$$f = \frac{r}{2} - 1.$$

III. For almost all systems $\alpha_1, \alpha_2, \ldots, \alpha_r$ (in the sense of the Lebesgue measure in the r-dimensional Euclidean space E_r) is

$$P(x) = O(x^{r/4}\lg^{3r} x),$$

consequently

$$f \leqslant \frac{r}{4}.$$

The aim of this paper is to investigate the dependence of the value (5) on the properties of the system $\alpha_1, \alpha_2, \ldots, \alpha_r$. A simple inequality of this type is announced in [4]. Besides of some results which have a more general validity, it is possible to formulate the main theorem of this paper as follows:

THEOREM. *Let $r > 5$ and let the coefficients of the form* (1) *be integers, $M_1, M_2, \ldots, M_r$ natural numbers, $\alpha_1 = \alpha_2 = \ldots = \alpha_r = \alpha$, $b_1 = b_2 = \ldots \ldots = b_r = 0$. Let $\gamma = \gamma(\alpha)$ be the supremum of all numbers $\beta > 0$, for which the inequalities*

$$|q\alpha - p| \leqslant q^{-\beta}, \quad q > 0$$

take place for infinitely many pairs of integers p, q. Then for the value (5)

$$f = \left(\frac{r}{4} - \frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1} \tag{10}$$

holds, where we put $(2\gamma+1)/(\gamma+1) = 2$ for $\gamma = +\infty$ (1).

One part of this Theorem (the O-estimation) is a consequence of a general O-estimation from [5] (Theorem 2). For the Ω-estimation of the function $P(x)$ there were used three methods. Landau's ([3], p. 71), which leads to the result (4), Jarník's (applicable only for $\delta = 1$) which in our case leads (for $\delta = 1$) again to $P(x) = \Omega(x^{r/2-1})$ and finally a method, basing on an asymptotic investigation of the functions

$$\int_0^x |P(y)|^2\,dy, \qquad \sum_{1\leqslant n\leqslant x} |P(n)|^2 \quad \text{or} \quad \sum_{1\leqslant n\leqslant x} |A(n) - A(n-1)|^2$$

(Walfisz uses the latter function to find Ω-estimations for rational $\alpha_1, \alpha_2, \ldots, \alpha_r$). It is obvious, that an efficient application of these methods in our case is either impossible or unclear. Therefore, we shall use for Ω-estimations in § 4 a different method, based on properties of the corresponding theta-function in the neighbourhood of the imaginary axis.

§ 2. Notations and auxiliary assertions. Besides of the notations which we introduced in § 1 we shall keep (eventual changes will always be properly indicated) the following notations and agreements.

(1) Let us note, that it is interesting to compare (10) with Jarník's result (8).

The letters n and k (supplied with indices or prime if convenient) are natural numbers, j is a nonnegative integer, h and m (supplied with indices or prime if convenient) are integers, $u_1, u_2, \ldots, u_r$ are real numbers. If h and k are to appear simultaneously, they are always relatively prime, i.e. $(h, k) = 1$. For real t let $[t]$ be the integral part of t and

$$\langle t\rangle = \min(t-[t], 1-t+[t])$$

(distance between t and the set of integers). The letter c means (various) positive constants, which depend only on Q, M_j, b_j and α_j $(j = 1, 2, \ldots, r)$. $c(\varrho)$, $c(\psi, t)$ etc. are positive constants (various) depending only on ϱ, ψ and t, respectively, etc. The symbols O, o and Ω have their usual meaning and correspond to the limiting process $x \to +\infty$ and the constants involved are of the "type" c. $A \ll B$ means $|A| \leqslant cB$. If $A \ll B$ and $B \ll A$ we write $A \asymp B$. Let the number x be sufficiently large, i.e. $x > c$. $\bar{Q}$ means the quadratic form conjugated with Q. Let $0 \leqslant \lambda_1 < \lambda_2 < \ldots$ be the sequence of all values of the type $Q(m_j M_j + b_j)$ and

$$a_n = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j}$$

where the summation runs over all systems $u_1, u_2, \ldots, u_r$ which satisfy the relations $Q(u_j) = \lambda_n$, $u_j \equiv b_j \pmod{M_j}$ $(j = 1, 2, \ldots, r)$. Thus we can write

$$A(x) = \sum_{\lambda_n \leqslant x} a_n.$$

For complex s, $\operatorname{Re} s > 0$, let

$$\Theta(s) = \sum_{m_1, m_2, \ldots, m_r = -\infty}^{\infty} \exp\Big\{-sQ(m_j M_j + b_j) + 2\pi i \sum_{j=1}^{r} \alpha_j (m_j M_j + b_j)\Big\}, \tag{11}$$

i.e.

$$\Theta(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

The series in (11) is, as is known, almost uniformly and absolutely convergent in the half plane $\operatorname{Re} s > 0$, $\Theta(s)$ is consequently a holomorphic function in this domain.

LEMMA 1. *Let s be a complex number, $\operatorname{Re} s > 0$. Let the coefficients of the form Q and the numbers $b_1, b_2, \ldots, b_r$, $M_1, M_2, \ldots, M_r$ be integers, $M_1 > 0$, $M_2 > 0, \ldots, M_r > 0$. Then*

(12)

$$\Theta(s) = \frac{M}{k^r\left(s - \frac{2\pi i h}{k}\right)^{r/2}} \sum_{m_1, m_2, \ldots, m_r = -\infty}^{\infty} S_{h,k,(m)} \exp\left\{-\frac{\pi^2 \bar{Q}\left(\frac{m_j}{M_j} - \alpha_j k\right)}{k^2\left(s - \frac{2\pi i h}{k}\right)}\right\},$$

where

(13) $$S_{h,k,(m)} = S_{h,k,m_1,m_2,\dots,m_r} = \sum_{a_1,a_2,\dots,a_r=1}^{k} \exp\left\{-\frac{2\pi ih}{k} Q(a_jM_j+b_j) + \frac{2\pi i}{k}\sum_{j=1}^{r}\frac{m_j}{M_j}(a_jM_j+b_j)\right\}$$

and where $z^{r/2}$ means (here and also further) the branch of $z^{r/2}$ in the half plane $\operatorname{Re} z > 0$, *which is positive for positive values of z.*

Proof. Let $s = s' + \dfrac{2\pi ih}{k}$. Then

$$\Theta(s) = \sum_{m_1,m_2,\dots,m_r=-\infty}^{\infty}\ \sum_{a_1,a_2,\dots,a_r=1}^{k} \exp\Big\{-sQ\big((km_j+a_j)M_j+b_j\big) + 2\pi i\sum_{j=1}^{r} a_j\big((km_j+a_j)M_j+b_j\big)\Big\}$$

$$= \sum_{a_1,a_2,\dots,a_r=1}^{k} \exp\left\{-\frac{2\pi ih}{k}Q(a_jM_j+b_j)\right\}\times$$

$$\times \sum_{m_1,m_2,\dots,m_r=-\infty}^{\infty} \exp\Big\{-s'Q(km_jM_j+a_jM_j+b_j)+2\pi i\sum_{j=1}^{r}a_j\big((km_j+a_j)M_j+b_j\big)\Big\}.$$

Using the well known transformation formula for theta-function (e.g. [3], p. 239, Theorem 3, where we write s/π instead of s and put $\gamma_j = \dfrac{a_jM_j+b_j}{kM_j}$, $\delta_j = a_jkM_j$, $\alpha_{jl} = k^2M_jM\alpha_{jl}$) we get further

$$\Theta(s) = \sum_{a_1,a_2,\dots,a_r=1}^{k} \exp\left\{-\frac{2\pi ih}{k}Q(a_jM_j+b_j)\right\}\times$$

$$\times \frac{M}{s'^{r/2}k^r}\sum_{m_1,m_2,\dots,m_r=-\infty}^{\infty} \exp\left\{-\frac{\pi^2}{s'}\bar{Q}\left(\frac{m_j}{kM_j}-a_j\right)+\frac{2\pi i}{k}\sum_{j=1}^{r}\frac{m_j}{M_j}(a_jM_j+b_j)\right\}$$

and consequently (12).

LEMMA 2. *Under the assumption of Lemma 1 there is always*

(14) $$S_{h,k,(m)} \ll k^{r/2}.$$

Further

(15) $$|S_{h,k,(m)}|^2 = k^r\sum \exp\{2\pi i\varphi(h,k,d,m_j,f_j)\}$$

holds, where we put

$$d = \Big(k, 2D\prod_{j=1}^{r}M_i^2\Big),$$

$$\varphi(h,k,d,m_j,f_j) = \frac{hk}{d^2}Q(f_jM_j) - \frac{1}{d}\sum_{j=1}^{r}f_jm_j + \frac{2h}{d}\sum_{l,j=1}^{r}a_{lj}f_jM_jb_l$$

and sum over all systems $f_1, f_2, \dots, f_r$ of integers, $1 \leqslant f_j \leqslant d$ $(j = 1, 2, \dots, r)$ satisfying the congruences

$$2\sum_{j=1}^{r}a_{lj}M_lM_jf_j \equiv 0 \pmod{d}$$

$(l = 1, 2, \dots, r)$.

Proof. It is

$$|S_{h,k,(m)}|^2 = \sum_{c_1,c_2,\dots,c_r=1}^{k}\ \sum_{a_1,a_2,\dots,a_r=1}^{k}\exp\Big\{-2\pi i\frac{h}{k}Q(a_jM_j+b_j) - Q\big((a_j+c_j)M_j+b_j\big) - \frac{2\pi i}{k}\sum_{j=1}^{r}m_jc_j\Big\}$$

$$= \sum_{c_1,c_2,\dots,c_r=1}^{k}\exp\left\{2\pi i\frac{h}{k}Q(c_jM_j) - \frac{2\pi i}{k}\sum_{j=1}^{r}m_jc_j + 2\pi i\frac{h}{k}2\sum_{l,j=1}^{r}a_{lj}c_jM_jb_l\right\}\times$$

$$\times S'_{c_1,c_2,\dots,c_r},$$

where

$$S'_{c_1,c_2,\dots,c_r} = \sum_{a_1,a_2,\dots,a_r=1}^{k}\exp\left\{2\pi i\frac{h}{k}2\sum_{l,j=1}^{r}a_{lj}c_jM_ja_lM_l\right\}$$

$$= \prod_{l=1}^{r}\sum_{a_l=1}^{k}\left(\exp\left\{2\pi i\frac{h}{k}2\sum_{j=1}^{r}a_{lj}M_lM_jc_j\right\}\right)^{a_l}.$$

The latter expression is non-zero (and then equal to k^r) if and only if

$$2\sum_{j=1}^{r}a_{lj}M_lM_jc_j \equiv 0 \pmod{k}$$

$(l = 1, 2, \dots, r)$. This implies

$$2D\prod_{j=1}^{r}M_j^2c_l \equiv 0 \pmod{k}$$

$(l = 1, 2, \dots, r)$, i.e.

$$c_l \equiv 0 \left(\operatorname{mod}\frac{k}{d}\right)$$

...ber of solutions of this system mod k is at

$$d^r \leqslant \Big(2D\prod_{j=1}^{r} M_j^2\Big)^r \ll 1.$$

Herefrom we obtain immediately (14). If we write $c_l = f_l\dfrac{k}{d}$ $(l=1,2,\dots,r)$, where $1\leqslant f_l\leqslant d$ $(l=1,2,\dots,r)$ we obtain the second assertion of the Lemma.

We mention that analogous assertions has been proved in a similar way (for some special cases e.g. in [3], p. 150).

Moreover, we present a simple

LEMMA 3. *Let $\psi(u_1, u_2, \dots, u_r) = \psi(u_j)$ be a positive definite quadratic form, $V_1, V_2, \dots, V_r, z_1, z_2, \dots, z_r$ real numbers, $V_1>0, V_2>0, \dots, V_r>0$. For $t>0$ let*

$$\tau(t) = \sum 1,$$

where the summation runs over all systems $u_1, u_2, \dots, u_r$ satisfying

$$\psi(u_1, u_2, \dots, u_r)\leqslant t,\qquad u_j\equiv z_j(\mathrm{mod}\; V_j),\qquad j=1,2,\dots,r.$$

Then there exist such positive constants $c_1 = c(\psi)$, $c_2 = c(\psi)$, $c_3 = c(\psi, V_1, V_2, \dots, V_r)$, $c_4 = c(\psi, V_1, V_2, \dots, V_r)$, $c_5 = c(\psi, V_1, V_2, \dots, V_r)$ (and thus independent on $z_1, z_2, \dots, z_r$) that

(a) *$c_1\max\limits_{j=1,2,\dots,r}|u_j|^2\leqslant\psi(u_1, u_2, \dots, u_r)\leqslant c_2\max\limits_{j=1,2,\dots,r}|u_j|^2$ for all real systems $u_1, u_2, \dots, u_r$;*

(b) *$\tau(t)\leqslant c_3(t+1)^{r/2}$ for all $t>0$;*

(c) *if $|u_j|\leqslant c_4$ $(j=1,2,\dots,r)$ then*

$$\psi\left(\frac{u_j+m_j}{V_j}\right)-\psi\left(\frac{u_j}{V_j}\right)\geqslant c_5$$

for all integers $m_1, m_2, \dots, m_r$ which are not all simultaneously zero.

Proof. (a) Let c_1, c_2 be infimum, supremum respectively, of the function $\psi(u_1, u_2, \dots, u_r)$ on the set of all $u_1, u_2, \dots, u_r$ satisfying $\max\limits_{j=1,2,\dots,r}|u_j| = 1$. Obviously $c_1 = c(\psi)$, $c_2 = c(\psi)$ and it is sufficient to consider such $u_1, u_2, \dots, u_r$ for which $U = \max\limits_{j=1,2,\dots,r}|u_j|>0$. In this case for $v_j = u_j/U$ $(j=1,2,\dots,r)$

$$c_1\leqslant\psi(v_1, v_2, \dots, v_r)\leqslant c_2$$

holds, and the assertion is proved.

(b) Following (a) we have

$$\tau(t)\leqslant\prod_{j=1}^{r}\left(\frac{2\sqrt{t}}{V_j\sqrt{c_1}}+1\right)$$

and thus

$$\tau(t)\leqslant c_3(t+1)^{r/2}$$

for $t>0$.

(c) Let $m_1, m_2, \dots, m_r$ be integers, $\max\limits_{j=1,2,\dots,r}|m_j|>0$. If $\max\limits_{j=1,2,\dots,r}|u_j|<\frac{1}{2}$ then according to (a)

$$\psi\left(\frac{u_j+m_j}{V_j}\right)\geqslant\frac{1}{4}c_1\min_{j=1,2,\dots,r}\frac{1}{V_j^2} = 2c_5. \tag{16}$$

If now

$$|u_j|<\min\left(\sqrt{\frac{c_5}{c_1}}\min_{j=1,2,\dots,r}V_j,\ \frac{1}{2}\right) = c_4$$

$(j=1,2,\dots,r)$ then (16) holds and according to (a) we have

$$\psi(u_j/V_j)\leqslant c_5$$

and the assertion (c) is proved.

Let now

$$R_k = \min_{m_1, m_2, \dots, m_r}\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)$$

and

$$P_k = \max_{j=1,2,\dots,r}\langle a_jM_jk\rangle.$$

From the assertion (a) of Lemma 3 we obtain immediately

$$P_k^2\asymp R_k. \tag{17}$$

Let us now present a basic relation which will be used to find the O-estimations.

LEMMA 4. *Let $r>4$ and let the coefficients of the form (1) and the numbers $b_1, b_2, \dots, b_r$ be integers, $M_1, M_2, \dots, M_r$ natural numbers. Let $\min\left(A, \dfrac{1}{0}\right) = A$ for $A>0$. Then*

$$P(x) = O\left(x^{\frac{r}{4}-\frac{1}{2}}\sum_{k\leqslant\sqrt{x}}\min^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right)\lg 2k\right). \tag{18}$$

Proof. Cf. [5], Theorem 2.

Remark 1. (a) Similarly as in [5] let us define the numbers $S_{h,k}$ as follows: if the relation

$$R_k = \bar{Q}\left(\frac{m_j}{M_j} - a_j k\right) \tag{19}$$

is satisfied by only one system $m_1, m_2, \ldots, m_r$, let

$$S_{h,k} = S_{h,k,(m)}. \tag{20}$$

If (19) is for some k satisfied by more systems $m_1, m_2, \ldots, m_r$, then, for this k, we choose one of them and define the number $S_{h,k}$ as in (20). Let us note that the assertion (c) of the Lemma 3 implies that there exists such a constant $c_6 = c(Q, M_1, M_2, \ldots, M_r)$ that if $R_k < c_6$ or $P_k < c_6$ then (19) is satisfied by only one system $m_1, m_2, \ldots, m_r$ and in this case there also

$$\langle a_j M_j k\rangle = |a_j M_j k - m_j|$$

($j = 1, 2, \ldots, r$) holds. Thus we have a certain indetermination in the choice of $S_{h,k}$ only for those values of k, for which $R_k \geqslant c_6$.

(b) On the base of (17) we can write instead of (19)

$$P(x) = O\left(x^{\frac{r}{4}-\frac{1}{2}} \sum_{k\leqslant\sqrt{x}} \min^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{P_k^2}\right)\lg 2k\right). \tag{21}$$

From the proof of Theorem 2 in [5] it follows that in (18), respectively in (21), it is sufficient to sum only over those values of k, for which there exists such an h that $S_{h,k} \neq 0$, i.e. more precisely, e.g.

$$P(x) = O\left(x^{\frac{r}{4}-\frac{1}{2}} \sum \min^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right)\lg 2k + x^{\frac{r}{4}}\lg x\right) \tag{22}$$

holds, where the summation runs over all $k \leqslant \sqrt{x}$ for which $S_{h,k} \neq 0$ (i.e. $S_{h,k} \neq 0$ for suitable h).

§ 3. *O*-estimation. Let (only in this paragraph) $a_1 = a_2 = \ldots = a_r = a$. The aim of this paragraph is to investigate the dependence of the exponent in the O-estimation on the properties of a.

Remark 2. Let us present some known properties of continued fractions [2]. Let a be an irrational number and let

$$\{a_0; a_1, a_2, \ldots\}$$

[2] Cf. e.g. [2], pp. 240-242.

be the corresponding continued fraction. If we put $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$ then the recurrent relations

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned} \tag{23}$$

give the numerators and denominators of the convergents of a.

By means of induction we easily find that the numbers p_n and q_n satisfy the relation

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \tag{24}$$

(which implies $(p_n, q_n) = 1$). Let n be given. Then (for $n \geqslant m \geqslant 0$)

$$q_n \geqslant q_m 2^{(n-m)/2}. \tag{25}$$

Further we can write

$$a = \frac{r_{n+1}p_n + p_{n-1}}{r_{n+1}q_n + q_{n-1}}, \tag{26}$$

where

$$r_n = \{a_n; a_{n+1}, a_{n+2}, \ldots\} = a_n + \frac{1}{r_{n+1}}$$

and thus

$$a_n < r_n < a_n + 1. \tag{27}$$

(26) and (27) imply

$$a - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(r_{n+1}q_n + q_{n-1})}. \tag{28}$$

From these relations we can draw three important conclusions.

Let u and v be nonnegative integers and let

$$1 \leqslant k = uq_n + v < q_{n+1}, \qquad v < q_n.$$

(a) If $v = 0$, then according to (28)

$$ak - up_n = \frac{(-1)^n u}{q_n r_{n+1} + q_{n-1}},$$

i.e. according to (23) and (27)

$$\langle ak\rangle \asymp \frac{u}{q_{n+1}} \qquad (k = uq_n < q_{n+1}). \tag{29}$$

(b) If $v = q_{n-1}$ (and thus $u < a_{n+1}$) we have according to (24) and (28)

$$ak - up_n - p_{n-1} = \frac{(-1)^{n+1}(r_{n+1} - u)}{q_n r_{n+1} + q_{n-1}}$$

and thus according to (23) and (27)

$$(30)\qquad \langle ak\rangle \asymp \frac{a_{n+1}-u}{q_{n+1}} \qquad (k = uq_n+q_{n-1} < q_{n+1}).$$

(c) Let finally $v \neq 0$, q_{n-1}. According to (28) there is

$$ak-up_n-\frac{vp_n}{q_n} = (-1)^n\frac{uq_n+v}{q_n(q_nr_{n+1}+q_{n-1})}.$$

Since

$$\frac{uq_n+v}{q_nr_{n+1}+q_{n-1}} < 1$$

and

$$vp_n \not\equiv (-1)^{n+1} \pmod{q_n},$$

there exists such a natural $j \leqslant q_n/2$ that

$$(31)\qquad j/q_n < \langle ak\rangle < (j+1)/q_n \qquad (k = uq_n+v < q_{n+1},\ v\neq 0,\ q_{n-1}).$$

The number j depends only on v, n and a (and thus is independent on u). On the other hand: for each j $(0 < j \leqslant q_n/2)$ there exist for every $u \leqslant a_{n+1}$ at most two such values of v that for $k = uq_n+v < q_{n+1}$ (31) holds. In the following we shall use the notations and results of this remark.

THEOREM 1. *Let $a_1 = a_2 = \ldots = a_r = a$ be irrational numbers. Let*

$$(32)\qquad \langle ak\rangle \gg k^{-\beta}$$

hold for all k (and thus $\beta \geqslant 1$). Let the coefficients of the form (1) *and the numbers $b_1, b_2, \ldots, b_r$ be integers and $M_1, M_2, \ldots, M_r$ natural numbers. Then*

$$(33)\qquad P(x) = O\left(x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}\lg x\right)$$

for $r > 6$,

$$(34)\qquad P(x) = O\left(x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}\lg^2 x\right)$$

for $r = 6$ and

$$(35)\qquad P(x) = O\left(x^{\frac{r}{4}}\lg^2 x + x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}\lg x\right)$$

for $r = 5$.

Proof. Let the assumptions of the Theorem be satisfied and let $r > 4$. According to (21) it is sufficient to estimate the sum

$$(36)\qquad \sum_{k\leqslant\sqrt{x}} I_k \lg x,$$

where

$$(37)\qquad I_k = x^{\frac{r}{4}-\frac{1}{2}}\min{}^{\frac{r}{2}-1}\left(\frac{\sqrt{x}}{k}, \frac{1}{P_k}\right).$$

(a) First, let $M_1 = M_2 = \ldots = M_r = 1$. Thus $P_k = \langle ak\rangle$ and further

$$(38)\qquad I_k \leqslant \frac{x^{\frac{r}{2}-1}}{k^{\frac{r}{2}-1}}$$

and

$$(39)\qquad I_k \leqslant \frac{x^{\frac{r}{4}-\frac{1}{2}}}{\langle ak\rangle^{\frac{r}{2}-1}}$$

hold.

If $\langle ak\rangle \ll k/\sqrt{x}$ it is convenient to use (38); for $\langle ak\rangle \gg k/\sqrt{x}$ use (39). By (29) we obtain (for $u = 1$)

$$(40)\qquad q_{n+1} \ll q_n^{\beta}$$

for all natural n.

Let $x > c$ be given. Let N be such a natural number, that

$$q_N \leqslant \sqrt{x} < q_{N+1}$$

and for every n $(1 \leqslant n \leqslant N)$ let

$$(41)\qquad S_n = \sum I_k,$$

where the summation runs over all k, which satisfy the inequalities

$$(42)\qquad q_n \leqslant k < q_{n+1}, \qquad k \leqslant \sqrt{x}.$$

Let us now choose a certain n $(1 \leqslant n \leqslant N)$ and consider the sum S_n. For every k satisfying (42) it is possible unambiguously to find a natural u and such a nonnegative integer v that

$$(43)\qquad 1 \leqslant u \leqslant a_{n+1}, \quad 0 \leqslant v < q_n, \quad k = uq_n+v$$

holds.

Let us write

$$(44)\qquad S_n = S_n^{(1)}+S_n^{(2)},$$

where in $S_n^{(1)}$ the summation runs over all k satisfying (42) and (43), where $v = 0$ or $v = q_{n-1}$ and in $S_n^{(2)}$ over all other k satisfying (42).

In view of (29) and (30) we use (38) in $S_n^{(1)}$. For the corresponding k there is $k \geqslant uq_n$ and thus

$$S_n^{(1)} \ll x^{\frac{r}{2}-1} \sum_{n=1}^{\infty} \frac{1}{(uq_n)^{\frac{r}{2}-1}} \ll \left(\frac{x}{q_n}\right)^{\frac{r}{2}-1}. \tag{45}$$

According to (c) of the Remark 2 we can find for every k in $S_n^{(2)}$ (i.e. of the type (43), where $v \neq 0$, q_{n-1}), such a natural j, $0 < j \leqslant q_n/2$, that (31) holds and j depends only on v, a and n. If now

$$j \geqslant \frac{(u+1)q_n^2}{\sqrt{x}}$$

then

$$\frac{k}{\sqrt{x}} = \frac{uq_n+v}{\sqrt{x}} < \frac{(u+1)q_n}{\sqrt{x}} \leqslant \frac{j}{q_n} < \langle ak\rangle$$

and we use (39). For other j's, i.e. for $0 < j < \dfrac{(u+1)q_n^2}{\sqrt{x}}$ (so far they exist), we use (38). In view of (c) of the Remark 2 we can thus write

$$S_n^{(2)} \ll \sum_{1 \leqslant u \leqslant \frac{\min(\sqrt{x}, q_{n+1})}{q_n}} \left(x^{\frac{r}{4}-\frac{1}{2}} \sum_{j \geqslant \frac{(u+1)q_n^2}{\sqrt{x}}} \left(\frac{q_n}{j}\right)^{\frac{r}{2}-1} + x^{\frac{r}{2}-1} \sum_{1 \leqslant j < \frac{(u+1)q_n^2}{\sqrt{x}}} \frac{1}{(uq_n)^{\frac{r}{2}-1}} \right)$$

$$\ll \sum_{1 \leqslant u \leqslant \frac{\min(\sqrt{x}, q_{n+1})}{q_n}} \left(x^{\frac{r}{4}-\frac{1}{2}} q_n^{\frac{r}{2}-1} \left(\frac{\sqrt{x}}{q_n^2(u+1)}\right)^{\frac{r}{2}-2} + x^{\frac{r}{2}-1} \frac{(u+1)}{\sqrt{x}} q_n^2 \frac{1}{(uq_n)^{\frac{r}{2}-1}} \right),$$

i.e.

$$S_n^{(2)} \ll \frac{x^{\frac{r}{2}-\frac{3}{2}}}{q_n^{\frac{r}{2}-3}} \sum_{1 \leqslant u \leqslant \frac{\min(\sqrt{x}, q_{n+1})}{q_n}} u^{2-\frac{r}{2}}. \tag{46}$$

Now we determine such a natural R that

$$q_{R-1} \leqslant x^{\frac{1}{2(1+\beta)}} < q_R. \tag{47}$$

By (40) there is

$$q_R \ll x^{\frac{\beta}{2(1+\beta)}} < x^{\frac{1}{2}} \tag{48}$$

(and thus $R \leqslant N$). Using (25) we easily obtain from (45), (46) and (47)

$$\sum_{n=R}^{N} S_n^{(1)} \ll x^{\frac{r}{2}-1} \sum_{n=R}^{\infty} \frac{1}{q_n^{\frac{r}{2}-1}} \ll \left(\frac{x}{q_R}\right)^{\frac{r}{2}-1} \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}, \tag{49}$$

$$\sum_{n=R}^{N} S_n^{(2)} \tag{50}$$

$$\ll \begin{cases} x^{\frac{r}{2}-\frac{3}{2}} \sum\limits_{n=R}^{N} \dfrac{1}{q_n^{\frac{r}{2}-3}} \ll \dfrac{x^{\frac{r}{2}-\frac{3}{2}}}{q_R^{\frac{r}{2}-3}} \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}} & \text{for } r > 6, \\ x^{\frac{r}{2}-\frac{3}{2}} \sum\limits_{n=R}^{N} \lg\left(\dfrac{\min(\sqrt{x}, q_{n+1})}{q_n}+1\right) \ll x^{\frac{r}{2}-\frac{3}{2}} \lg x & \text{for } r = 6, \\ x^{\frac{r}{2}-\frac{3}{2}} \sum\limits_{n=R}^{N} \sqrt{q_n} \dfrac{\min(\sqrt[4]{x}, \sqrt{q_{n+1}})}{\sqrt{q_n}} \ll x^{r/4} \lg x & \text{for } r = 5. \end{cases}$$

It remains to estimate

$$\sum_{k<q_R} I_k.$$

According to (29), (39) and (48) there is

$$I_{q_{R-1}} \ll x^{\frac{r}{4}-\frac{1}{2}} q_R^{\frac{r}{2}-1} \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}.$$

We can associate to each $k < q_R$, $k \neq q_{R-1}$ (according to (c) of the Remark 2) such a natural j, $0 < j \leqslant q_R/2$ that

$$j/q_R < \langle ak\rangle < (j+1)/q_R$$

holds.

In view of this part of the Remark 2 we can write, according to (39) and (48),

$$\sum_{k<q_R} I_k = I_{q_{R-1}} + \sum_{\substack{k<q_R \\ k\neq q_{R-1}}} I_k \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}} + x^{\frac{r}{4}-\frac{1}{2}} \sum_{\substack{k<q_R \\ k\neq q_{R-1}}} \frac{1}{\langle ak\rangle^{\frac{r}{2}-1}}$$

$$\ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}} + x^{\frac{r}{4}-\frac{1}{2}} \sum_{1 \leqslant j \leqslant q_R/2} \left(\frac{q_R}{j}\right)^{\frac{r}{2}-1}$$

$$\ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}} + x^{\frac{r}{4}-\frac{1}{2}} q_R^{\frac{r}{2}-1} \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}},$$

i.e.

(51) $$\sum_{k<q_R} I_k \ll x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}.$$

Combining (36), (41), (44), (49), (50) and (51) we obtain the relations (33)-(35) for the case $M_1 = M_2 = \ldots = M_r = 1$.

(b) Let $S(x)$ be the sum (36) (for $M_1 = M_2 = \ldots = M_r = 1$) and let $M_1, M_2, \ldots, M_r$ be arbitrary natural numbers, $N = \max_{j=1,2,\ldots,r} M_j$. Successively we obtain

$$x^{\frac{r}{4}-\frac{1}{2}} \lg x \sum_{k\leqslant\sqrt{x}} \min{}^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{P_k^2}\right)$$
$$\ll x^{\frac{r}{4}-\frac{1}{2}} \lg x \sum_{k\leqslant\sqrt{x}} \sum_{j=1}^{r} \min{}^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{\langle aM_jk\rangle^2}\right)$$
$$\ll x^{\frac{r}{4}-\frac{1}{2}} \lg x \sum_{k\leqslant N\sqrt{x}} \min{}^{\frac{r}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{\langle ak\rangle^2}\right) \ll S(N^2x).$$

With respect to the part (a) the relations (33)-(35) holds, without the restriction $M_1 = M_2 = \ldots = M_r = 1$.

§ 4. The general Ω-estimation.

LEMMA 5. *For complex s, $\operatorname{Re} s > 0$,*

(52) $$\frac{\Theta(s)}{s} - \frac{Me^{2\pi i\sum_{j=1}^{n} a_jb_j}}{s^{r/2+1}}\delta = \int_0^\infty e^{-\xi s} P(\xi)\,d\xi$$

holds.

Proof. For $T > 0$ there is

$$s\int_0^T e^{-\xi s} A(\xi)\,d\xi = s\int_0^T e^{-\xi s} \sum_{\lambda_n\leqslant\xi} a_n\,d\xi$$
$$= \sum_{\lambda_n\leqslant T} a_n s\int_{\lambda_n}^T e^{-\xi s}\,d\xi = \sum_{\lambda_n\leqslant T} a_n e^{-\lambda_n s} - e^{-Ts}A(T).$$

Since according to (b) of Lemma 3 (we put $\psi = Q$, $M_j = V_j$, $z_j = b_j$ for $j = 1, 2, \ldots, r$) $A(T) \ll \tau(T) \ll (T+1)^{r/2}$ holds, we obtain, using the limiting process $T\to+\infty$,

(53) $$\Theta(s) = s\int_0^\infty e^{-\xi s} A(\xi)\,d\xi.$$

As

$$s\int_0^\infty e^{-\xi s} V(\xi)\,d\xi = \frac{Me^{2\pi i\sum_{j=1}^{r} a_jb_j}}{\Gamma\left(\frac{r}{2}+1\right)}\delta s\int_0^\infty e^{-\xi s}\xi^{r/2}\,d\xi = \frac{Me^{2\pi i\sum_{j=1}^{r} a_jb_j}}{s^{r/2}}\delta$$

we obtain immediately (52) from (53).

LEMMA 6. *Let the assumption of Lemma 3 take place and let d be a positive number. For $\sigma > 0$ let*

$$f(\sigma) = \sum e^{-\sigma\psi(u_1,u_2,\ldots,u_r)},$$

where the summation runs over all $u_1, u_2, \ldots, u_r$ which satisfy

(54) $$\psi(u_1, u_2, \ldots, u_r) \geqslant d$$

and

(55) $$u_j \equiv z_j \pmod{V_j}, \quad j = 1, 2, \ldots, r.$$

Then for $\sigma > 1$

$$f(\sigma) \leqslant c(\psi, V_1, V_2, \ldots, V_r)e^{-d\sigma/2}$$

holds.

Proof. For $\sigma > 1$ there is

$$f(\sigma) \leqslant e^{-d\sigma/2}\sum e^{-\frac{1}{2}\psi(u_1,u_2,\ldots,u_r)},$$

where the summation runs over all $u_1, u_2, \ldots, u_r$ which satisfy (55). In the notation of Lemma 3 there is (by (b) of this Lemma)

$$\tau(t) \leqslant c(\psi, V_1, V_2, \ldots, V_r)(t+1)^{r/2}$$

for $t > 0$ and thus

$$f(\sigma) \leqslant e^{-d\sigma/2}\sum_{m=0}^{\infty} e^{-m/2}\tau(m+1)$$
$$\leqslant c(\psi, V_1, V_2, \ldots, V_r)e^{-d\sigma/2}\sum_{m=0}^{\infty} e^{-m/2}(m+2)^{r/2} = c(\psi, V_1, V_2, \ldots, V_r)e^{-d\sigma/2},$$

q.e.d.

LEMMA 7. *Let, for a certain $\beta > 0$, be*

(56) $$P(x) = o(x^\beta).$$

Then for $s = \sigma + it$, $\sigma\to 0+$,

(57) $$\frac{\Theta(s)}{s} - \frac{Me^{2\pi i\sum_{j=1}^{r} a_jb_j}}{s^{r/2+1}}\delta = o\left(\frac{1}{\sigma^{\beta+1}}\right)$$

holds uniformly on $t\in(-\infty, \infty)$.

Proof. Let $\varepsilon > 0$. As we have assumed that there exists such a ξ_0 that for $\xi \geqslant \xi_0$

$$|P(\xi)| \leqslant \frac{\varepsilon \xi^\beta}{2\Gamma(\beta+1)}$$

holds, then also

$$\left|\int_0^\infty e^{-\xi s} P(\xi)\,d\xi\right| \leqslant \int_0^{\xi_0} |P(\xi)|\,d\xi + \frac{\varepsilon}{2\Gamma(\beta+1)} \int_{\xi_0}^\infty e^{-\xi\sigma}\xi^\beta\,d\xi$$

$$\leqslant \int_0^{\xi_0} |P(\xi)|\,d\xi + \tfrac{1}{2}\varepsilon\sigma^{-\beta-1}.$$

Thus, for sufficiently small $\sigma > 0$

$$\left|\int_0^\infty e^{-\xi s} P(\xi)\,d\xi\right| \leqslant \frac{\varepsilon}{\sigma^{\beta+1}}.$$

Herefrom and from (52) we obtain (57), q.e.d.

On the base of these lemmas we can easily prove in a different way the Walfisz Ω-estimation which is quoted in § 1:

THEOREM 2. *Let the form* (1) *have integer coefficients, let the numbers* $b_1, b_2, \ldots, b_r$ *be integers,* $M_1, M_2, \ldots, M_r$ *natural numbers and* $a_1, a_2, \ldots, a_r$ *rational numbers. Denote by* H *the least common denominator of the numbers* $a_1M_1, a_2M_2, \ldots, a_rM_r$. *Let there exist numbers* h *and* k *such that*

(58) $$k \equiv 0 \ (\mathrm{mod}\ H) \quad \textit{and} \quad S_{h,k} \neq 0.$$

Then

$$P(x) = \Omega(x^{\frac{r}{2}-1}).$$

Proof. Let $P(x) = o(x^{r/2-1})$. According to Lemma 7 there is for $s = \sigma + it$, $\sigma \to 0+$,

(59) $$\frac{\Theta(s)}{s} - \frac{Me^{2\pi i\sum_{j=1}^r a_jb_j}}{s^{r/2+1}}\,\delta = o(\sigma^{-r/2})$$

uniformly on $t \epsilon (-\infty, \infty)$.

Let h and k be numbers satisfying (58). If $H > 1$ then necessarily $h \neq 0$ (and $\delta = 0$). If $H = 1$ we can obviously choose $k = 1$ and h arbitrary. From the definition of $S_{h,k,(m)}$ (see (13)) it follows further, that $S_{h,k,(m)} = S_{h',k,(m)}$ for $h \equiv h' \ (\mathrm{mod}\ k)$. Thus we can choose the numbers h, k satisfying (58) so that $h > 0$. Put in (59) $s = \sigma + \dfrac{2\pi ih}{k}$ $(\sigma > 0)$. According to the relation (12) of Lemma 1 we obtain for $\sigma \to 0+$

(60) $$\frac{MS_{h,k}}{\left(\sigma+\frac{2\pi ih}{k}\right)k^r\sigma^{r/2}} + \frac{M}{k^r\sigma^{r/2}\left(\sigma+\frac{2\pi ih}{k}\right)}\sum{}' S_{h,k,(m)}\exp\left\{-\frac{\pi^2\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)}{k^2\sigma}\right\} - \frac{M\exp\{2\pi i\sum_{j=1}^r a_jb_j\}}{\left(\sigma+\frac{2\pi ih}{k}\right)^{r/2+1}}\,\delta = o(\sigma^{-r/2})$$

(the summation runs over all $m_1, m_2, \ldots, m_r$ with the exception of the system $a_1M_1k, a_2M_2k, \ldots, a_rM_rk$). In the sum there is obviously $\bar{Q}\left(\dfrac{m_j}{M_j} - a_jk\right) \geqslant c$ and thus according to (14) and Lemma 6 there is

$$\sum{}' S_{h,k,(m)}\exp\left\{-\frac{\pi^2\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)}{k^2\sigma}\right\} \leqslant k^{\frac{r}{2}}e^{-\frac{c}{k\sigma^2}}$$

for $\sigma < 1$. For $\sigma \to 0+$ there is clearly

$$\frac{M}{\left(\sigma+\frac{2\pi ih}{k}\right)k^r\sigma^{r/2}}\sum{}' S_{h,k,(m)}\exp\left\{-\frac{\pi^2\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)}{k^2\sigma}\right\} - \frac{M\exp\{2\pi i\sum_{j=1}^r a_jb_j\}}{\left(\sigma+\frac{2\pi ih}{k}\right)^{r/2+1}}\,\delta = o(\sigma^{-r/2}).$$

From (60) we thus obtain

$$\frac{MS_{h,k}}{2\pi ihk^{r-1}} = o(1)$$

for $\sigma \to 0+$; this is a contradiction.

LEMMA 8. *Let the coefficients of the form* (1) *and* $b_1, b_2, \ldots, b_r$ *be integers,* $M_1, M_2, \ldots, M_r$ *natural numbers. Suppose that, for a certain system* $h, k, m_1, m_2, \ldots, m_r$ *we have*

(61) $$S_{h,k,(m)} \neq 0.$$

Then we have

$$|S_{h,k,(m)}| \gg k^{r/2}. \tag{62}$$

Proof. Put

$$A = 2D\prod_{j=1}^{r} M_j^2.$$

By Lemma 2 is

$$|S_{h,k,(m)}|^2 = k^r \sum e^{2\pi i\varphi(h,k,d,m_j,f_j)} = k^r\mathfrak{Q}, \tag{63}$$

where $d = (k, A)$ and

$$\varphi(h, k, d, m_j, f_j) = \frac{hk}{d^2}Q(f_jM_j) - \frac{1}{d}\sum_{j=1}^{r} f_jm_j + \frac{2h}{d}\sum_{l,j=1}^{r} a_{lj}f_jM_jb_l \tag{64}$$

and the sum runs over all systems $f_1, f_2, \ldots, f_r$ of integers, $1 \leqslant f_j \leqslant d$ $(j = 1, 2, \ldots, r)$, satisfying the congruences

$$2\sum_{j=1}^{r} a_{lj}M_lM_jf_j \equiv 0 \ (\mathrm{mod}\ d)$$

$(l = 1, 2, \ldots, r)$.

In view of (64), $\mathfrak{Q}$ depends only on $h \bmod d^2$, on $k \bmod d^2$ and on m_j $(j = 1, 2, \ldots, r) \bmod d$. We have only c possibilities for d and, for every $d|A$, at most d^{r+4} different values of $\mathfrak{Q}$. Following (61) and (63), at least one value of $\mathfrak{Q}$ is positive; the minimum of the positive values of $\mathfrak{Q}$ is a "c", and so (61) implies (62).

THEOREM 3. *Let the coefficients of the form* (1) *and the numbers* $b_1, b_2, \ldots, b_r$ *be integers,* $M_1, M_2, \ldots, M_r$ *natural numbers. Let at least one of the numbers* $a_1, a_2, \ldots, a_r$ *be irrational. Let* β *be a positive real number and let there exist the sequences*

$$h_1, h_2, \ldots, \qquad k_1 < k_2 < \ldots \tag{65}$$

such that for $h = h_n$, $k = k_n$ *and all* n *is*

$$h \ll 1, \tag{66}$$

$$R_k \ll k^{-2\beta} \tag{67}$$

and

$$S_{h,k} \neq 0. \tag{68}$$

Then

$$P(x) = \Omega\left(x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}\right).$$

Proof. According to Lemma 8 we can assume that for $h = h_n$, $k = k_n$ even (62) hold and obviously $h \neq 0$ for $n > 1$. Put

$$w = \left(\frac{r}{4} - \frac{1}{2}\right)\frac{2\beta+1}{\beta+1}$$

and suppose that

$$P(x) = o(x^w).$$

For $s = \sigma + \dfrac{2\pi ih}{k}$, $\sigma \to 0+$ we have, according to Lemma 5,

$$\frac{\Theta(s)}{s} = o(\sigma^{-w-1}) \tag{69}$$

(in view of the assumptions there is $\delta = 0$) uniformly with respect to h and k. Now, let $k = k_n$, $h = h_n$, where n is sufficiently large (consequently, by (67), (65) R_k is sufficiently small). According to Lemma 1 we can write

$$\Theta(s) = \frac{MS_{h,k}}{k^r\sigma^{r/2}}e^{-\frac{\pi^2R_k}{k^2\sigma}} + \frac{M}{k^r\sigma^{r/2}}\sum{}' S_{h,k,(m)}e^{-\frac{\pi^2\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)}{k^2\sigma}}, \tag{70}$$

where the summation runs over all systems $m_1, m_2, \ldots, m_r$ satisfying $\bar{Q}\left(\dfrac{m_j}{M_j} - a_jk\right) \neq R_k$; according to (a) of the Remark 1 this inequality is satisfied for all systems with a unique exception, if R_k is small enough, and thus if n is sufficiently large. According to (c) of Lemma 3 we have then $\bar{Q}\left(\dfrac{m_j}{M_j} - a_jk\right) > c$ in the sum $\sum'$. According to (14) and Lemma 6 for $k^2\sigma < 1$ there is

$$\frac{M}{k^r\sigma^{r/2}}\sum{}' S_{h,k,(m)}e^{-\frac{\pi^2\bar{Q}\left(\frac{m_j}{M_j}-a_jk\right)}{k^2\sigma}} \ll \frac{e^{-\frac{c}{k^2\sigma}}}{\sigma^{r/2}k^{r/2}} \tag{71}$$

Now, put

$$\sigma = R_k/k^2.$$

Thus $\sigma \to 0+$ for $n \to +\infty$. From (70) and (71) we obtain

$$|\Theta(s)| \geqslant \frac{M}{k^r\sigma^{r/2}}\left(|S_{h,k}|e^{-\pi^2} - ck^{r/2}e^{-c/R_k}\right)$$

for sufficiently large n, $h = h_n$, $k = k_n$. As

$$|s| \leqslant \frac{R_k}{k^2} + 2\pi \frac{|h|}{k} \ll \frac{|h|}{k},$$

we obtain using (69) and (62),

$$\frac{\sigma^{w+1} k}{h(\sigma k)^{r/2}} = o(1)$$

(for $n \to +\infty$), i.e. according to (66)

$$\frac{\sigma^{w+1-r/2}}{k^{r/2-1}} = o(1)$$

and thus after substitution

$$\frac{1}{R_k k^{2\beta}} = o(1)$$

for $n \to \infty$. This is a contradiction with (67). Thus the theorem is proved.

LEMMA 9. *Let the coefficients of the form* (1) *be integers and* $M_1, M_2, \ldots, M_r$ *natural numbers,* $b_1 = b_2 = \ldots = b_r = 0$. *If* $a_1, a_2, \ldots, a_r$ *are rational numbers, then the assumptions of Theorem* 2 *are satisfied. If at least one of the numbers* $a_1, a_2, \ldots, a_r$ *is irrational and if for a certain* $\beta > 0$ *the inequality*

$$R_k \ll \frac{1}{k^{2\beta}} \tag{72}$$

is satisfied for infinitely many k, *then the assumptions of Theorem* 3 *are satisfied* (*with the same value* β).

Proof. If $a_1, a_2, \ldots, a_r$ are rational numbers, let H be the least common denominator of the numbers $a_1 M_1, a_2 M_2, \ldots, a_r M_r$, $k = H$ and so $R_k = \bar{Q}\left(\frac{m_j}{M_j} - a_j k\right) = 0$ (with $m_j = a_j M_j k$, $j = 1, 2, \ldots, r$). In the opposite case put $H = 1$ and let k be a sufficiently large number satisfying (72). Denote $A = 2D\prod_{j=1}^{r} M_j^2$. In both cases there is $H^2 A^2 k \equiv 0 \pmod{H}$ and according to (c) of the Remark 1

$$R_k = \bar{Q}\left(\frac{m_j}{M_j} - a_j k\right)$$

implies in the first case $R_{kA^2H^2} = R_k = 0$ and in the second case

$$R_{kA^2H^2} = \bar{Q}\left(\frac{m_j A^2 H^2}{M_j} - a_j k A^2 H^2\right) = A^4 H^4 R_k \ll \frac{1}{(kA^2H^2)^{2\beta}}.$$

Put $h = 1$. By Lemma 2 there is ($d = (kA^2H^2, A) = A$)

$$|S_{h,kA^2H^2}|^2 = k^r A^{2r} H^{2r} \sum 1,$$

where the summation runs over all systems $f_1, f_2, \ldots, f_r$ of integers which satisfy $0 < f_j \leqslant A$ ($j = 1, 2, \ldots, r$) and

$$2 \sum_{j=1}^{r} a_{lj} M_l M_j f_j \equiv 0 \pmod{A}$$

($l = 1, 2, \ldots, r$). Choosing $f_1 = f_2 = \ldots = f_r = A$ we obtain

$$S_{h,kA^2H^2} \neq 0.$$

The lemma is proved.

Now, question arises, whether it is possible to omit the assumptions (68) from Theorem 3. First, we prove the following lemma, making use of one of Walfisz's ideas (see [7], Lemma 4, p. 50):

LEMMA 10. *Let* $r > 4$,

$$Q(u_1, u_2, \ldots, u_r) = u_1^2 + Q_1(u_2, \ldots, u_r),$$

where Q_1 *is a positive definite quadratic form with integer coefficients. Let* $a_1 = p_1/q_1$, $a_2 = p_2/q_2$, *where* $0 \leqslant p_1 < q_1$, $0 \leqslant p_2 < q_2$ *are integers,* $(p_1, q_1) = 1$, $(p_2, q_2) = 1$. *Let* $a_3, a_4, \ldots, a_r$ *be arbitrary real numbers. Let further* $b_1 = 1$, $b_2 = b_3 = \ldots = b_r = 0$ and $M_1, M_2, \ldots, M_r$ *be natural numbers,* $M_1 > 2$, $M_2 = 1$, $q_2 \equiv 0 \pmod{4M_1^2 q_1}$. *Then*

$$A(x) \not\equiv 0$$

and

$$P(x) = O(x^{r/4} \lg x).$$

Proof. If $k \not\equiv 0 \pmod{q_2}$, then necessarily $R_k \geqslant c$. Let $k \equiv 0 \pmod{q_2}$. If

$$R_k = \bar{Q}\left(\frac{m_j}{M_j} - a_j k\right)$$

then either $R_k \geqslant c$ or $m_1 = \frac{p_1}{q_1} M_1 k$. Consider the last case. Then obviously

$$S_{h,k} = \sum_{a_1=1}^{k} \exp\left\{-\frac{2\pi i h}{k}(a_1 M_1 + 1)^2 + \frac{2\pi i}{k}\frac{m_1}{M_1}(a_1 M_1 + 1)\right\} \times$$
$$\times \sum_{a_2, a_3, \ldots, a_r = 1}^{k} \exp\left\{-\frac{2\pi i h}{k} Q_1(a_j M_j) + \frac{2\pi i}{k} \sum_{j=2}^{r} \frac{m_j}{M_j}(a_j M_j)\right\}.$$

Let us consider the sum

$$S = \sum_{a_1=1}^{k} \exp\left\{-\frac{2\pi i h}{k}(a_1M_1+1)^2 + \frac{2\pi i}{k}\frac{m_1}{M_1}(a_1M_1+1)\right\}.$$

Similarly as in Lemma 2 we obtain that

$$|S|^2 = k\sum \exp\left\{2\pi i\left(\frac{hk}{d^2}(f_1M_1)^2 - \frac{1}{d}f_1m_1 + \frac{2h}{d}f_1M_1\right)\right\},$$

where $d = (k, 2M_1^2) = 2M_1^2$ and the summation runs over all integers f_1, $1 \leqslant f_1 \leqslant d$, satisfying

$$2M_1^2 f_1 \equiv 0 \pmod{d}.$$

In view of the assumptions of the lemma we have

$$|S|^2 = k\sum_{f_1=1}^{2M_1^2} \exp\left\{2\pi i\frac{hf_1}{M_1}\right\} = 0.$$

Thus there exists such a constant c, that for $R_k \leqslant c$ there is $S_{h,k} = 0$ for all k. From (22) of the Remark 1 we obtain

$$P(x) = O\left(x^{\frac{r}{4}-\frac{1}{2}}\sum_{\substack{k\leqslant\sqrt{x}\\ R_k\geqslant c}}\frac{\lg 2k}{R_k^{r/4-1/2}} + x^{r/4}\lg x\right) = O(x^{r/4}\lg x).$$

As the relations

$$Q(u_1, u_2, \ldots, u_r) = u_1^2 + Q(u_2, u_3, \ldots, u_r) \leqslant 1,$$
$$u_j \equiv b_j \pmod{M_j}, \quad j = 1, 2, \ldots, r,$$

are satisfied only for

$$u_1 = 1, \quad u_2, u_3, \ldots, u_r = 0,$$

we have

$$A(1) = e^{2\pi i a_1} \neq 0.$$

The lemma is proved.

Let now a $\beta > 0$ be given, $r > 4$ and let us choose the numbers $\gamma_3, \gamma_4, \ldots, \gamma_r$ in such a way that at least one of them is irrational and the inequality

(73) $$\max_{j=3,4,\ldots,r}\langle\gamma_j k'\rangle \leqslant \frac{1}{k'^{\beta}}$$

is satisfied for infinitely many k'. Let the assumptions of Lemma 10 be satisfied and put $a_j = \gamma_j/M_jq_2$ $(j = 3, 4, \ldots, r)$. Put $k = k'q_1q_2$; from the Remark 1 (part (a)) it easily follows that the inequality

$$R_k \leqslant \frac{1}{k^{2\beta}}$$

is satisfied for infinitely many k, but according to Lemma 10 there is $A(x) \not\equiv 0$ and $P(x) = O(x^{r/4}\lg x)$. For sufficiently large β we thus obtain that

$$P(x) = \Omega\left(x^{\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\beta+1}{\beta+1}}\right)$$

cannot hold and thus the assumption (68) of Theorem 3 cannot be generally excluded.

§ 5. Conclusion. From the Theorems 1-3, Lemma 9 and I of § 1 we obtain our main Theorem:

THEOREM 4. *Let the coefficients of the form* (1) *and* $b_1, b_2, \ldots, b_r$ *be integers,* $M_1, M_2, \ldots, M_r$ *natural numbers. Let* $a_1 = a_2 = \ldots = a_r = a$. *Let* $\gamma = \gamma(a)$ *be the supremum of all* $\beta > 0$, *for which the relation*

$$\langle ak\rangle \leqslant \frac{1}{k^{\beta}}$$

is satisfied for infinitely many k. *For* $\gamma = +\infty$ *put* $(2\gamma+1)/(\gamma+1) = r$ *and let*

$$f = \left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1}.$$

Then

$$P(x) = O(x^{f+\varepsilon})$$

for $r > 5$,

$$P(x) = O(x^{\max(f, r/4)+\varepsilon})$$

for $r = 5$, *where* ε *is an arbitrary positive real number.*

If $b_1 = b_2 = \ldots = b_r = 0$, *then for every positive* ε

$$P(x) = \Omega(x^{f-\varepsilon}).$$

(The constants in O- and Ω-relations depend on ε.)

Remark 3. Let the coefficients of the form (1), the numbers b_j and M_j be integers, $M_j > 0$ $(j = 1, 2, \ldots, r)$. Let $\gamma = \gamma(a_1, a_2, \ldots, a_r)$ be the supremum of all numbers $\beta > 0$ which satisfy

$$\liminf_{k\to+\infty} R_k k^{-2\beta} < +\infty.$$

The Theorem 3 gives (under certain assumptions about sums $S_{h,k}$) the lower estimation $f \geqslant \left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1}$ (for $b_1 = b_2 = \ldots = b_r = 0$ the assumtions about $S_{h,k}$ are always satisfied by Lemma 9). In the special case $a_1 = a_2 = \ldots = a_r = a$ we obtain also by Theorem 1 the inequality (for $r > 5$)

$$f \leqslant \left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1}. \tag{74}$$

In general case I have presented in [4] the following upper estimation (for $r > 4$) $f \leqslant \left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1-\frac{2}{r}}$. On the base of Lemma 4 this estimation can be improved; we obtain, e.g.

$$f \leqslant \left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1}+\frac{1}{2(\gamma+1)}.$$

The exact value of f has not been found in the general case. First of all we observe that γ can assume (see (17)) all values from the interval $[1/r, +\infty]$. For $r > 3$, $\gamma < 1/(r-3)$ we have

$$\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1} < \frac{r-1}{4}$$

and thus for those γ (and $A(x) \neq 0$) in view of (4) the estimation (74) cannot hold. For $r > 4$, $\gamma < 2/(r-4)$ we have

$$\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1} < \frac{r}{4},$$

but using Lemma 4 we cannot obtain a better O-estimation than $O(x^{r/4}\lg x)$. Finally, let us note, that for $r = 2$ and $r = 3$ the Theorem 3 does not give better results than Landau's estimation (4).

Remark 4. In connection with Lemma 10, there arises a question whether the assumption (68) in Theorem 3 is needed in the case of irrational $a_1, a_2, \ldots, a_r$.

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Reçu par la Rédaction le 18. 9. 1967