On some numerical constants associated with abstract algebras II

by

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1. Introduction. In this paper we use the terminology and notation of [4]. E. Marczewski introduced in [2] the order of enlargeability and the arity or the order of reducibility of abstract algebras. We recall his definition of these concepts. Let $\mathfrak{A} = (A; \mathbf{F})$ be an abstract algebra and let $\mathbf{G}$ run over all families of operations in A. Put

$$\varepsilon(\mathfrak{A}) = \min\{n\colon \bigwedge_{\mathbf{G}} \big([\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})] \Rightarrow [\mathbf{A}(\mathbf{F}) \supset \mathbf{A}(\mathbf{G})]\big)\}$$

and

$$\varrho(\mathfrak{A}) = \min\{n\colon \bigwedge_{\mathbf{G}} \big([\mathbf{A}^{(n)}(\mathbf{F}) = \mathbf{A}^{(n)}(\mathbf{G})] \Rightarrow [\mathbf{A}(\mathbf{F}) \subset \mathbf{A}(\mathbf{G})]\big)\}$$

where the minimum of an empty set is assumed to be ∞. The constant $\varepsilon(\mathfrak{A})$ is called the *order of enlargeability* of the algebra $\mathfrak{A}$. The constant $\varrho(\mathfrak{A})$ (denoted in [2] by $\beta(\mathfrak{A})$) is called the *arity* or the *order of reducibility* of the algebra $\mathfrak{A}$. In the sequel we shall sometimes write ε and ϱ instead of $\varepsilon(\mathfrak{A})$ and $\varrho(\mathfrak{A})$ respectively when no confusion will arise.

In [4] a relationship between the order of enlargeability and a substitute of the minimal number of generators was discussed. The aim of the present paper is to give a description of all possible pairs (ε, ϱ) for abstract algebras.

The p-enlargement $\mathfrak{E}_p(\mathfrak{A})$ $(p \geqslant 1)$ of the algebra $\mathfrak{A}$ was defined in [4], Chapter 2. A relationship between the concepts of the order of enlargeability and the p-enlargement is given by the following simple theorem ([4], Theorem 2.2):

(i) *The inequality* $\varepsilon(\mathfrak{A}) \leqslant p$ *holds if and only if* $\mathfrak{A} = \mathfrak{E}_p(\mathfrak{A})$.

If $\mathfrak{A} = (A; \mathbf{F})$, then by $\mathfrak{R}_p(\mathfrak{A})$ $(p \geqslant 1)$ we shall denote the p-reduct $(A; \mathbf{A}^{(p)}(\mathbf{F}))$ of $\mathfrak{A}$. It is clear that

(ii) *The inequality* $\varrho(\mathfrak{A}) \leqslant p$ *holds if and only if* $\mathfrak{A} = \mathfrak{R}_p(\mathfrak{A})$.

Many algebras usually treated in mathematics have small arity. We say that an algebra $\mathfrak{A}$ is *rigid* if the inequality $\varepsilon(\mathfrak{A}) < \varrho(\mathfrak{A})$ holds. As an example of rigid algebras we quote complete algebras over an at least two-element set, i.e. algebras for which every operation is algebraic. In fact, for complete algebras we have the formulas $\varepsilon = 0$ and $\varrho = 2$ (see [3], [5]).

2. A class of rigid algebras with finite arity. In this section we assume that p and q are arbitrary positive integers satisfying the inequality

$$q \geqslant p+2 . \tag{2.1}$$

Consider a $2(q+1)$-element set $A_q = \{a_1, a_2, \dots, a_{q+1}, b_1, b_2, \dots, b_{q+1}\}$. Put $B_k = \{a_1, a_2, \dots, a_{k-1}, b_k, a_{k+1}, \dots, a_{q+1}\}$ $(k = 1, 2, \dots, q+1)$. Further, for every n-tuple $v_1, v_2, \dots, v_n$ $(n = 1, 2, \dots)$ of elements of the set A_q we define a set $D_p(v_1, v_2, \dots, v_n)$ as follows:

1. $D_p(v_1, v_2, \dots, v_n) = \{v_1, v_2, \dots, v_n\}$ if one of the following cases holds

a. $\operatorname{card}\{v_1, v_2, \dots, v_n\} < p$,

b. $\operatorname{card}\{v_1, v_2, \dots, v_n\} = p$ and $\{v_1, v_2, \dots, v_n\} \not\subset B_k$ for every index k $(k = 1, 2, \dots, q+1)$,

c. there exist two different indices k and s $(k, s = 1, 2, \dots, q+1)$ such that $\{v_1, v_2, \dots, v_n\} \subset B_k \cap B_s$.

2. $D_p(v_1, v_2, \dots, v_n) = B_k$ if $\operatorname{card}\{v_1, v_2, \dots, v_n\} \geqslant p$, $\{v_1, v_2, \dots, v_n\} \subset B_k$ and $\{v_1, v_2, \dots, v_n\} \not\subset B_s$ for every index $s \neq k$ $(s = 1, 2, \dots, q+1)$.

3. $D_p(v_1, v_2, \dots, v_n) = A_q$ in the remaining cases, i.e. if

$$\operatorname{card}\{v_1, v_2, \dots, v_n\} > p \quad \text{and} \quad \{v_1, v_2, \dots, v_n\} \not\subset B_k$$

for every index k $(k = 1, 2, \dots, q+1)$.

It is easy to verify the following inclusions

$$D_p(u_1, u_2, \dots, u_m) \subset D_p(v_1, v_2, \dots, v_n) \quad \text{if} \quad \{u_1, u_2, \dots, u_m\} \subset \{v_1, v_2, \dots, v_n\} , \tag{2.2}$$

$$D_p(v_1, v_2, \dots, v_n) \subset B_k \quad \text{if} \quad \{v_1, v_2, \dots, v_n\} \subset B_k . \tag{2.3}$$

Hence we get the inclusion

$$D_p(u_1, u_2, \dots, u_m) \subset D_p(v_1, v_2, \dots, v_n) \quad \text{if} \quad \{u_1, u_2, \dots, u_m\} \subset D_p(v_1, v_2, \dots, v_n) . \tag{2.4}$$

LEMMA 2.1. *If $n > q$ and $u \in D_p(v_1, v_2, \dots, v_n)$, then there exists an index i $(1 \leqslant i \leqslant n)$ such that $u \in D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.*

Proof. First consider the case $D_p(v_1, v_2, \dots, v_n) = \{v_1, v_2, \dots, v_n\}$. Then $u = v_s$ for some index s $(1 \leqslant s \leqslant n)$. Taking $i \neq s$ $(1 \leqslant i \leqslant n)$ we have the formula

$$u \in \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} = D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n) .$$

Now consider the case $D_p(v_1, v_2, \dots, v_n) = B_k$ $(k = 1, 2, \dots, q+1)$. Then either $b_k \in \{v_1, v_2, \dots, v_n\}$, $\{v_1, v_2, \dots, v_n\} \subset B_k$ and $\operatorname{card}\{v_1, v_2, \dots, v_n\} \geqslant p$ or $\{v_1, v_2, \dots, v_n\} = B_k \setminus \{b_k\}$. Taking into account (2.1) and the inequality $n > q$, we infer that there exists an index i $(1 \leqslant i \leqslant n)$ satisfying the condition $v_i \neq c_k$ and $\operatorname{card}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \geqslant p$ if $b_k \in \{v_1, v_2, \dots, v_n\}$ and the condition $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} = B_k \setminus \{b_k\}$ otherwise. In both cases we have the equation $D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n) = B_k$ which implies the relation $u \in D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

Finally consider the case

$$D_p(v_1, v_2, \dots, v_n) = A_q . \tag{2.5}$$

If $\operatorname{card}\{v_1, v_2, \dots, v_n\} < n$, then there exist two different indices i and j $(1 \leqslant i, j \leqslant n)$ such that $v_i = v_j$. Hence it follows that $D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n) = A_q$ and, consequently, $u \in D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. Suppose now that

$$\operatorname{card}\{v_1, v_2, \dots, v_n\} = n . \tag{2.6}$$

If there exists an index i $(1 \leqslant i \leqslant n)$ such that $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \not\subset B_k$ for every index k $(1 \leqslant k \leqslant q+1)$, then, by (2.1), (2.6) and the inequality $n > q$, we have the formula $D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n) = A_q$. Thus $u \in D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. In the opposite case, for every index i $(1 \leqslant i \leqslant n)$ there exists an index k_i $(1 \leqslant k_i \leqslant q+1)$ such that $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \subset B_{k_i}$. We shall prove that $k_i \neq k_j$ whenever $i \neq j$. Indeed, the equation $k_i = k_j$ would imply the relation

$$\{v_1, v_2, \dots, v_n\} = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \cup \{v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} \subset B_{k_i}$$

and, consequently, by (2.3), the inclusion $D_p(v_1, v_2, \dots, v_n) \subset B_{k_i}$ which would contradict (2.5). Thus $k_i \neq k_j$ if $i \neq j$. Hence and from the inequality $n > q$ it follows that

$$n = q+1 . \tag{2.7}$$

Moreover, by the inequality $q \geqslant p+2 \geqslant 3$, for every index j $(1 \leqslant j \leqslant n)$ we have the relation

$$v_j \in \bigcap_{i \neq j} \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \subset \bigcap_{i \neq j} B_{k_i} \subset \{a_1, a_2, \dots, a_{q+1}\} .$$

Thus, by (2.6) and (2.7), the n-tuple $v_1, v_2, \dots, v_n$ is a permutation of $a_1, a_2, \dots, a_{q+1}$. If $u = a_s$ or $u = b_s$, then taking an index i such that $v_i \neq a_s$ and $v_i = a_s$ respectively, we get the formula $u \in D_p(v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ which completes the proof.

Let $\mathbf{F}_p$ be the class of all operations f on A_q such that $f(v_1, v_2, \dots, v_n) \in D_p(v_1, v_2, \dots, v_n)$ for all n-tuples of elements of A_q. The class $\mathbf{F}_p$ contains all trivial operations and, by (2.4), is closed under the composition. Put $\mathfrak{A}_{p,q} = (A_q; \mathbf{F}_p)$ for any pair of positive integers p, q satisfying condition (2.1). Of course, $\mathbf{A}(\mathfrak{A}_{p,q}) = \mathbf{F}_p$.

LEMMA 2.2. *For every $(q+1)$-ary algebraic operation f in $R_{q-1}(\mathfrak{A}_{p,q})$ the relation*

$$f(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\} \tag{2.8}$$

holds.

Proof. The class of all $(q+1)$-ary algebraic operations in $\mathfrak{R}_{q-1}(\mathfrak{A}_{p,q})$ is the union $\bigcup_{k=0}^{\infty} \mathbf{A}_k^{(q+1)}$, where the classes $\mathbf{A}_k^{(q+1)}$ are defined recursively as follows

$$\mathbf{A}_0^{(q+1)} = \{e_1^{(q+1)}, e_2^{(q+1)}, \ldots, e_{q+1}^{(q+1)}\},$$

$$\mathbf{A}_{k+1}^{(q+1)} = \mathbf{A}_k^{(q+1)} \cup \{g(f_1, f_2, \ldots, f_{q-1}) \colon g \in \mathbf{A}^{(q-1)}(\mathfrak{A}_{p,q}),\ f_j \in \mathbf{A}_k^{(q+1)},\ j = 1, 2, \ldots, q-1\}$$

(see [1], p. 47). Let f be an operation from $\mathbf{A}_k^{(q+1)}$. We shall prove the lemma by induction with respect to k. If $k = 0$, then the operation f is trivial and, consequently, relation (2.8) is obvious. Suppose that $k \geqslant 1$ and relation (2.8) holds for all operations from $\mathbf{A}_{k-1}^{(q+1)}$. Moreover, we may assume that the operation f does not belong to $\mathbf{A}_{k-1}^{(q+1)}$. Then it can be written in the form

$$\begin{aligned} f(x_1, x_2, \ldots, x_{q+1}) \\ = g\big(f_1(x_1, x_2, \ldots, x_{q+1}), f_2(x_1, x_2, \ldots, x_{q+1}), \ldots, f_{q-1}(x_1, x_2, \ldots, x_{q+1})\big), \end{aligned} \tag{2.9}$$

where $g \in \mathbf{A}^{(q-1)}(\mathfrak{A}_{p,q})$ and $f_j \in \mathbf{A}_{k-1}^{(q+1)}$ $(j = 1, 2, \ldots, q-1)$. Consequently, by the inductive assumption,

$$f_j(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\} \quad (j = 1, 2, \ldots, q-1).$$

Hence and from (2.9) it follows that

$$f(a_1, a_2, \ldots, a_{q+1}) = g(a_{i_1}, a_{i_2}, \ldots, a_{i_{q-1}}), \tag{2.10}$$

where $\{i_1, i_2, \ldots, i_{q-1}\} \subset \{1, 2, \ldots, q+1\}$. Consequently, there exist two indices $s \neq r$ $(1 \leqslant s, r \leqslant q+1)$ which do not belong to the set $\{i_1, i_2, \ldots, i_{q-1}\}$. Thus we have the inclusion

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_{q-1}}\} \subset B_s \cap B_r$$

which implies the equation $D_p(a_{i_1}, a_{i_2}, \ldots, a_{i_{q-1}}) = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{q-1}}\}$. Hence and from (2.10) we get the relation $f(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\}$ which completes the proof of the lemma.

LEMMA 2.3. *Let $v_1, v_2, \ldots, v_n$ be an arbitrary n-tuple of elements of A_q. Every operation f, satisfying the conditions*

$$f(v_1, v_2, \ldots, v_n) \in D_p(v_1, v_2, \ldots, v_n) \quad \text{and} \quad f(x_1, x_2, \ldots, x_n) = x_1$$

for $[x_1, x_2, \ldots, x_n] \neq [v_1, v_2, \ldots, v_n]$, is algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$.

Proof. We shall prove the lemma by induction with respect to n. It is evident that each operation satisfying the condition of the lemma is algebraic in the algebra $\mathfrak{A}_{p,q}$. Consequently, our statement is true for $n \leqslant q$. Suppose that $n > q$ and that the lemma is true for $(n-1)$-ary operations. Put $u = f(v_1, v_2, \ldots, v_n)$. By Lemma 2.1 there exists an index i $(1 \leqslant i \leqslant n)$ such that $u \in D_p(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. First consider the case $i > 1$. By the inductive assumption the operation g, defined by the formula $g(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) = u$ and $g(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = x_1$ otherwise, is algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. Further, the ternary operation h, defined as $h(v_1, v_i, u) = u$ and $h(x_1, x_2, x_3) = x_1$ otherwise, is algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$ because of the inequality $q \geqslant p+2 \geqslant 3$. Thus the composition $f_0(x_1, x_2, \ldots, x_n) = h\big(x_1, x_i, g(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\big)$ is algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. It is easy to verify that $f_0(v_1, v_2, \ldots, v_n) = u$ and $f_0(x_1, x_2, \ldots, x_n) = x_1$ otherwise. Thus $f = f_0$ and, consequently, the operation f is algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$.

Now consider the case $i = 1$. Of course, we may assume that all elements $v_1, v_2, \ldots, v_n$ are different, because in the opposite case as the index i an integer greater than 1 can be taken. Since $n > q \geqslant p+2 \geqslant 3$, there exists an index r $(2 \leqslant r \leqslant n)$ such that $v_r \neq u$. Setting $g_0(v_2, v_3, \ldots, v_n) = u$, $h_0(v_1, v_r, u) = u$ and $g_0(x_2, x_3, \ldots, x_n) = x_r$, $h_0(x_1, x_2, x_3) = x_1$ otherwise we get, by the inductive assumption, algebraic operations in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. Consequently, the composition $f_1(x_1, x_2, \ldots, x_n) = h_0\big(x_1, x_r, g_0(x_2, x_3, \ldots, x_n)\big)$ is also algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. Since $f_1(v_1, v_2, \ldots, v_n) = u$ and $f_1(x_1, x_2, \ldots, x_n) = x_1$ otherwise, we have the equation $f = f_1$. Thus the operation f is algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$, which completes the proof.

LEMMA 2.4. *Let $v_1, v_2, \ldots, v_n$ be an arbitrary n-tuple of elements of the set A_q and let g be an arbitrary n-ary algebraic operation in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. Every operation f satisfying the condition*

$$f(v_1, v_2, \ldots, v_n) \in D_p(v_1, v_2, \ldots, v_n) \quad \text{and} \quad f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

for $[x_1, x_2, \ldots, x_n] \neq [v_1, v_2, \ldots, v_n]$ is algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$.

Proof. Put $u = f(v_1, v_2, \ldots, v_n)$ and $v = g(v_1, v_2, \ldots, v_n)$. Of course, $u \in D_p(v, v_1, v_2, \ldots, v_n, u)$. We define two auxiliary operations h and h_0 as follows: $h(v_1, v_2, \ldots, v_n) = h_0(v, v_1, v_2, \ldots, v_n, u) = u$ and $h(x_1, x_2, \ldots, x_n) = h_0(x_1, x_2, \ldots, x_{n+2}) = x_1$ otherwise. By Lemma 2.3 both operations h

and h_0 are algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. Consequently, the composition

$$f_0(x_1, x_2, \ldots, x_n) = h_0\big(g(x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n, h(x_1, x_2, \ldots, x_n)\big)$$

is algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$ too. From the equations

$$f_0(v_1, v_2, \ldots, v_n) = h_0(v, v_1, v_2, \ldots, v_n, u) = u$$

and

$$f_0(x_1, x_2, \ldots, x_n) = h_0\big(g(x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n, x_1\big) = g(x_1, x_2, \ldots, x_n)$$

for all n-tuples $x_1, x_2, \ldots, x_n$ different from the n-tuple $v_1, v_2, \ldots, v_n$ we get the formula $f = f_0$ which shows that the operation f is algebraic in the algebra $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. The lemma is thus proved.

THEOREM 2.1. *For any pair p, q of positive integers satisfying the inequality $q \geqslant p+2$ the formula $\varrho(\mathfrak{A}_{p,q}) = q$ holds.*

Proof. The algebra $\mathfrak{A}_{p,q}$ is finite. Consequently, by consecutive application of Lemma 2.4 we obtain that each algebraic operation in $\mathfrak{A}_{p,q}$ is also algebraic in $\mathfrak{R}_q(\mathfrak{A}_{p,q})$. In other words, $\mathfrak{A}_{p,q} = \mathfrak{R}_q(\mathfrak{A}_{p,q})$ which, by proposition (ii) in Section 1, implies the inequality $\varrho(\mathfrak{A}_{p,q}) \leqslant q$.

Put $h(a_1, a_2, \ldots, a_{q+1}) = b_1$ and $h(x_1, x_2, \ldots, x_{q+1}) = x_1$ otherwise. Since $D_p(a_1, a_2, \ldots, a_{q+1}) = A_q$, the operation h is algebraic in $\mathfrak{A}_{p,q}$. On the other hand, by Lemma 2.2, it is not algebraic in $\mathfrak{A}_{q-1}(\mathfrak{R}_{p,q})$. Thus $\mathfrak{A}_{p,q} \neq \mathfrak{R}_{q-1}(\mathfrak{A}_{p,q})$, whence, by proposition (ii) in Section 1 the inequality $\varrho(\mathfrak{A}_{p,q}) \geqslant q$ follows.

THEOREM 2.2. *For any pair p, q of positive integers satisfying the inequality $q \geqslant p+2$ the formula $\varepsilon(\mathfrak{A}_{p,q}) = p$ holds.*

Proof. Let n be an integer greater than p and f an arbitrary n-ary algebraic operation in the p-enlargement $\mathfrak{E}_p(\mathfrak{A}_{p,q})$. We shall prove that the operation f is algebraic in the algebra $\mathfrak{A}_{p,q}$, i.e. for every n-tuple $v_1, v_2, \ldots, v_n$ of elements of A_q the relation

$$f(v_1, v_2, \ldots, v_n) \in D_p(v_1, v_2, \ldots, v_n) \tag{2.11}$$

holds. If $D_p(v_1, v_2, \ldots, v_n) = A_q$, then the above relation is obvious. Suppose that $\operatorname{card}\{v_1, v_2, \ldots, v_n\} \leqslant p$. Then we can choose a system $i_1, i_2, \ldots, i_p$ of indices for which the equation

$$\{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\} = \{v_1, v_2, \ldots, v_n\} \tag{2.12}$$

holds. For any k $(1 \leqslant k \leqslant n)$, let j_k denote the least index i_s for which $v_k = v_{i_s}$. Put $f_0(x_{i_1}, x_{i_2}, \ldots, x_{i_p}) = f(x_{j_1}, x_{j_2}, \ldots, x_{j_n})$. Of course, the operation f_0 is algebraic in the algebra $\mathfrak{A}_{p,q}$ and

$$f(v_1, v_2, \ldots, v_n) = f_0(v_{i_1}, v_{i_2}, \ldots, v_{i_p}) .$$

Consequently, $f(v_1, v_2, \ldots, v_n) \in D_p(v_{i_1}, v_{i_2}, \ldots, v_{i_p})$ which, by (2.2) and (2.12), implies relation (2.11).

It remains the case $\{v_1, v_2, \ldots, v_n\} \subset B_k$ for an index k $(1 \leqslant k \leqslant q+1)$ and $\operatorname{card}\{v_1, v_2, \ldots, v_n\} > p$. Let $k_1, k_2, \ldots, k_r$ be the system of all indices k_s for which the inclusion $\{v_1, v_2, \ldots, v_n\} \subset B_{k_s}$ holds. Of course,

$$D_p(v_1, v_2, \ldots, v_n) = B_{k_1} \quad \text{if} \quad r = 1 \tag{2.13}$$

and

$$D_p(v_1, v_2, \ldots, v_n) = \{v_1, v_2, \ldots, v_n\} = \bigcap_{s=1}^{r} B_{k_s} \quad \text{if} \quad r > 1 . \tag{2.14}$$

Since $\operatorname{card}\{v_1, v_2, \ldots, v_n\} > p$, we may assume without loss of generality that the elements $v_1, v_2, \ldots, v_{p-1}$ are different and do not belong to the set $\{b_1, b_2, \ldots, b_{q+1}\}$. Consequently,

$$D_p(v_1, v_2, \ldots, v_{p-1}, b_{k_s}) = B_{k_s} \quad (s = 1, 2, \ldots, r) . \tag{2.15}$$

Setting $g_{j,s}(v_1, v_2, \ldots, v_{p-1}, b_{k_s}) = v_j$ and $g_{j,s}(x_1, x_2, \ldots, x_p) = x_1$ otherwise $(j = 1, 2, \ldots, n;\ s = 1, 2, \ldots, r)$, we get algebraic operations in $\mathfrak{A}_{p,q}$. Thus the compositions

$$f_s(x_1, x_2, \ldots, x_p) = f\big(g_{1,s}(x_1, x_2, \ldots, x_p), g_{2,s}(x_1, x_2, \ldots, x_p), \ldots, g_{n,s}(x_1, x_2, \ldots, x_p)\big) \quad (s = 1, 2, \ldots, r)$$

are algebraic in $\mathfrak{A}_{p,q}$ too. Consequently, by (2.3) and (2.15), we have the relation $f_s(v_1, v_2, \ldots, v_{p-1}, b_{k_s}) \in B_{k_s}$ $(s = 1, 2, \ldots, r)$. Hence in view of the equations $f(v_1, v_2, \ldots, v_n) = f_s(v_1, v_2, \ldots, v_{p-1}, b_{k_s})$ $(s = 1, 2, \ldots, r)$ we get the relation $f(v_1, v_2, \ldots, v_n) \in \bigcap_{s=1}^{r} B_{k_s}$ which, by (2.13) and (2.14), implies relation (2.11). Thus we have proved that each operation algebraic in $\mathfrak{E}_p(\mathfrak{A}_{p,q})$ is algebraic in $\mathfrak{A}_{p,q}$. Hence the equation $\mathfrak{A}_{p,q} = \mathfrak{E}_p(\mathfrak{A}_{p,q})$ follows. Consequently, by proposition (i) in Section 1, we have the inequality

$$\varepsilon(\mathfrak{A}_{p,q}) \leqslant p . \tag{2.16}$$

From the definition of the sets $D_p(v_1, v_2, \ldots, v_n)$ and algebraic operations in $\mathfrak{A}_{p,q}$ it follows that each subalgebra of $\mathfrak{A}_{p,q}$ generated by p elements either is equal to B_k $(k = 1, 2, \ldots, q+1)$ or consists of p elements. Thus the minimal number of generators of $\mathfrak{A}_{p,q}$ is at least $p+1$. Consequently, $\gamma_0(\mathfrak{A}_{p,q}) \geqslant p+1$ and, by Theorem 6.1 in [4], $\varepsilon(\mathfrak{A}_{p,q}) \geqslant \gamma_0(\mathfrak{A}_{p,q}) - 1 \geqslant p$ which together with (2.16) implies the assertion of the theorem.

3. A class of rigid algebras with infinite arity. In the sequel we shall use the following analogue of Theorem 12.2 in [4].

THEOREM 3.1. *Suppose that the algebra $\mathfrak{A}$ contains an algebraic constant c such that*

(i) *if* $f \in \mathbf{A}(\mathfrak{A})$ *and* $f(x, c, c, \ldots, c) = x$, *then the operation* f *is trivial,*

(ii) *for every system* $f_1, f_2, \ldots, f_n \in \mathbf{A}^{(1)}(\mathfrak{A})$ $(n \geqslant 2)$ *of non-trivial operations satisfying the condition* $f_1(c) = f_2(c) = \ldots = f_n(c)$ *there exists one and only one operation* $h \in \mathbf{A}^{(n)}(\mathfrak{A})$ *for which the equations*

$$h(u_{1j}, u_{2j}, \ldots, u_{nj}) = f_j(x) \qquad (j = 1, 2, \ldots, n)$$

hold, where

$$u_{jj} = x \quad \text{and} \quad u_{ij} = c \quad \text{if} \quad i \neq j \quad (i, j = 1, 2, \ldots, n). \tag{3.1}$$

Then $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ *if* $\gamma_0(\mathfrak{A}) \geqslant 2$ *and* $\varepsilon(\mathfrak{A}) \leqslant 2$ *if* $\gamma_0(\mathfrak{A}) \leqslant 1$.

Proof. Since, by the assumption, each n-ary algebraic operation in $\mathfrak{A}$ is uniquely determined by its values on the n-tuples $u_{1j}, u_{2j}, \ldots, u_{nj}$ $(j = 1, 2, \ldots, n)$ defined by formula (3.1), we infer that the algebra $\mathfrak{A}$ has property $(\ast)$ defined in [4], Chapter 7. Since both exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ defined in [4], Chapter 8 do not satisfy the conditions of the theorem, we infer, by Theorem 9.2 in [4], that

$$\varepsilon(\mathfrak{A}) \geqslant \gamma_0(\mathfrak{A}). \tag{3.2}$$

Now we shall prove that the algebra $\mathfrak{A}$ has property $(\ast\ast)$ defined in [4], Chapter 12. Let g be an n-ary operation $(n > 3)$ such that for all operations $g_1, g_2, \ldots, g_n \in \mathbf{A}^{(n-1)}(\mathfrak{A})$ the composition $g(g_1, g_2, \ldots, g_n)$ belongs to $\mathbf{A}^{(n-1)}(\mathfrak{A})$. First, let us suppose that $g(x, c, c, \ldots, c) = x$. Then, by property (i) of the algebra $\mathfrak{A}$, we have the equation $g(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = x_1$ for all indices i, j satisfying the inequality $2 \leqslant i < j \leqslant n$. Hence and from the inequality $n > 3$ it follows that $g(x_1, x_2, \ldots, x_n) = e_1^{(n)}(x_1, x_2, \ldots, x_n)$ whenever the n-tuple $x_1, x_2, \ldots, x_n$ contains at most two different elements.

Now it remains the case where all operations

$$f_j(x) = g(u_{1j}, u_{2j}, \ldots, u_{nj}) \qquad (j = 1, 2, \ldots, n), \tag{3.3}$$

where u_{ij} are defined by (3.1), are non-trivial. Of course, the operations $f_1, f_2, \ldots, f_n$ are algebraic in $\mathfrak{A}$ and $f_1(c) = f_2(c) = \ldots = f_n(c)$. Consequently, by property (ii) of $\mathfrak{A}$, there exists an operation $h \in \mathbf{A}^{(n)}(\mathfrak{A})$ such that

$$h(u_{1j}, u_{2j}, \ldots, u_{nj}) = f_j(x) \qquad (j = 1, 2, \ldots, n) \tag{3.4}$$

for n-tuples $u_{1j}, u_{2j}, \ldots, u_{nj}$ defined by (3.1). Given $1 \leqslant i < j \leqslant n$, we put

$$v_{ij}(x, y) = g(z_1, z_2, \ldots, z_n), \qquad w_{ij}(x, y) = h(z_1, z_2, \ldots, z_n),$$

where $z_i = x$, $z_j = y$ and $z_k = c$ if $k \neq i, j$. Since $n > 3$, the operations v_{ij} and w_{ij} are algebraic. Moreover, by (3.3) and (3.4), $v_{ij}(x, c) = w_{ij}(x, c)$ and $v_{ij}(c, y) = w_{ij}(c, y)$ which implies the equation

$$v_{ij} = w_{ij} \qquad (i, j = 1, 2, \ldots, n;\ i < j). \tag{3.5}$$

Further, setting

$$g_{rs}(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_{r-1}, x_s, x_{r+1}, \ldots, x_n),$$
$$h_{rs}(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_{r-1}, x_s, x_{r+1}, \ldots, x_n)$$

for each pair of indices $r < s$ $(r, s = 1, 2, \ldots, n)$ we get algebraic operations satisfying the conditions

$$g_{js}(u_{1j}, u_{2j}, \ldots, u_{nj}) = g(c, c, \ldots, c),$$
$$h_{js}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h(c, c, \ldots, c),$$
$$g_{rj}(u_{1j}, u_{2j}, \ldots, u_{nj}) = v_{rj}(x, x),$$
$$h_{rj}(u_{1j}, u_{2j}, \ldots, u_{nj}) = w_{rj}(x, x)$$

and for $j \neq r, s$ the conditions

$$g_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = g(u_{1j}, u_{2j}, \ldots, u_{nj}),$$
$$h_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h(u_{1j}, u_{2j}, \ldots, u_{nj}),$$

where the n-tuples $u_{1j}, u_{2j}, \ldots, u_{nj}$ are defined by (3.1). Hence and from (3.3), (3.4) and (3.5) we get the equations

$$g_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) = h_{rs}(u_{1j}, u_{2j}, \ldots, u_{nj}) \qquad (j, r, s = 1, 2, \ldots, n;\ r < s).$$

Since both operations g_{rs} and h_{rs} are algebraic and, consequently, uniquely determined by their values on the n-tuples $u_{1j}, u_{2j}, \ldots, u_{nj}$ $(j = 1, 2, \ldots, n)$, the last equation implies the identity $g_{rs} = h_{rs}$ $(r, s = 1, 2, \ldots, n;\ r < s)$. Hence it follows that $g(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_n)$ for all n-tuples $x_1, x_2, \ldots, x_n$ containing at most two different elements. Thus the algebra $\mathfrak{A}$ has property $(\ast\ast)$.

By Theorem 12.1 in [4], we have the equation $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ if $\gamma_0(\mathfrak{A}) \geqslant 3$ and the inequality $\varepsilon(\mathfrak{A}) \leqslant 3$ if $\gamma_0(\mathfrak{A}) \leqslant 2$. Consequently, by (3.2), it remains to prove the inequality $\varepsilon(\mathfrak{A}) \leqslant 2$ for $\gamma_0(\mathfrak{A}) \leqslant 2$. To prove this inequality it suffices, by the inequality $\varepsilon(\mathfrak{A}) \leqslant 3$, to prove that each algebraic ternary operation g in $\mathfrak{E}_2(\mathfrak{A})$ is algebraic in $\mathfrak{A}$. Put

$$f_1(x) = g(x, c, c), \qquad f_2(x) = g(c, x, c), \qquad f_3(x) = g(c, c, x) \tag{3.6}$$

and

$$d_1(y, z) = g(c, y, z),\ d_2(x, z) = g(x, c, z),\ d_3(x, y) = g(x, y, c). \tag{3.7}$$

Of course, the operations f_j and d_j are algebraic in $\mathfrak{A}$. Moreover,

$$\begin{gathered} d_1(c, z) = d_2(c, z) = f_3(z), \qquad d_1(y, c) = d_3(c, y) = f_2(y), \\ d_2(x, c) = d_3(x, c) = f_1(x). \end{gathered} \tag{3.8}$$

Now we shall prove that there exists a ternary algebraic operation h in $\mathfrak{A}$ such that

$$g(x, y, z) = h(x, y, z) \quad \text{if} \quad c \in \{x, y, z\}. \tag{3.9}$$

First, let us suppose that at least one operation f_j is trivial. Without loss of generality we may assume that $f_1(x) = x$. Then, by (3.8) and condition (i), we have the equations $d_2(x, z) = d_3(x, y) = x$ and $d_1(y, z) = c$. Consequently, according to (3.7), the operation $h(x, y, z) = e_1^{(3)}(x, y, z)$ satisfies (3.9).

Suppose now that all the operations f_1, f_2 and f_3 are non-trivial. Since $f_1(c) = f_2(c) = f_3(c)$, there exists, by condition (ii), a ternary algebraic operation h such that

$$f_1(x) = h(x, c, c)\,, \quad f_2(x) = h(c, x, c)\,, \quad f_3(x) = h(c, c, x)\,.$$

Moreover, by (3.8) and condition (ii), we have the equations

$$d_1(y, z) = h(c, y, z)\,, \quad d_2(x, z) = h(x, c, z)\,, \quad d_3(x, y) = h(x, y, c)$$

which together with (3.7) imply (3.9).

Let b be an arbitrary algebraic constant in $\mathfrak{A}$. Put $v_1(x, y) = g(x, y, b)$ and $w_1(x, y) = h(x, y, b)$. The operations v_1 and w_1 are algebraic in $\mathfrak{A}$ and, by (3.9), $v_1(c, y) = w_1(c, y)$, $v_1(x, c) = w_1(x, c)$. Consequently, by condition (ii), $v_1 = w_1$, i.e.

$$g(x, y, b) = h(x, y, b) \quad \text{if} \quad b \in \mathbf{A}^{(0)}(\mathfrak{A})\,. \tag{3.10}$$

For any pair q_1, q_2 of unary algebraic operations in $\mathfrak{A}$ we put $v_2(x, y) = g\big(x, q_1(y), q_2(y)\big)$ and $w_2(x, y) = h\big(x, q_1(y), q_2(y)\big)$. Both operations v_2 and w_2 are algebraic in $\mathfrak{A}$ and, by (3.9) and (3.10), $v_2(c, y) = w_2(c, y)$, $v_2(x, c) = w_2(x, c)$. Consequently, by condition (ii), $v_2 = w_2$, i.e. $g\big(x, q_1(y), q_2(y)\big) = h\big(x, q_1(y), q_2(y)\big)$. Hence we get the formula

$$g\big(q_1(x), q_2(x), q_3(x)\big) = h\big(q_1(x), q_2(x), q_3(x)\big) \quad \text{if} \quad q_1, q_2, q_3 \in \mathbf{A}^{(1)}(\mathfrak{A})\,. \tag{3.11}$$

Further, for any triple g_1, g_2 and g_3 of binary algebraic operations in $\mathfrak{A}$ we put

$$v_3(x, y) = g\big(g_1(x, y), g_2(x, y), g_3(x, y)\big) \quad \text{and}$$
$$w_3(x, y) = h\big(g_1(x, y), g_2(x, y), g_3(x, y)\big)\,.$$

Both operations v_3 and w_3 are algebraic in $\mathfrak{A}$ and, by (3.11), $v_3(c, y) = w_3(c, y)$, $v_3(x, c) = w_3(x, c)$, which, by condition (ii), implies the identity $v_3 = w_3$. Thus

$$g\big(g_1(x, y), g_2(x, y), g_3(x, y)\big) = h\big(g_1(x, y), g_2(x, y), g_3(x, y)\big) \quad \text{for all} \quad g_1, g_2, g_3 \in \mathbf{A}^{(2)}(\mathfrak{A})\,. \tag{3.12}$$

Let a_1, a_2, a_3 be an arbitrary triple of elements of $\mathfrak{A}$. Since $\gamma_0(\mathfrak{A}) \leqslant 2$, there exist elements $b_1, b_2 \in \mathfrak{A}$ and operations $g_1, g_2, g_3 \in \mathbf{A}^{(2)}(\mathfrak{A})$ such that $a_j = g_j(b_1, b_2)$. Taking into account (3.12) we obtain the equation $g(a_1, a_2, a_3) = h(a_1, a_2, a_3)$ which implies the identity $g = h$. Thus the operation g is algebraic in $\mathfrak{A}$ which completes the proof.

Let A_1 be the set of all rationals and $\mathbf{F}_1$ the family of all operations f of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j + a \qquad (n = 1, 2, \ldots)\,,$$

where $c_1, c_2, \ldots, c_n$ are non-negative even integers and $a \in A_1$. We denote the algebra $(A_1; \mathbf{F}_1)$ by $\mathfrak{A}_{1,\infty}$.

For any p satisfying the inequality $2 \leqslant p \leqslant \infty$ we put $A_p = \{0, 1, \ldots$ $\ldots, p+1\}$. Let $\mathbf{F}_p$ be the family of all operations f_n $(n \geqslant 1)$ defined as $f_n(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} 2x_j \pmod 4$ if $\{x_1, x_2, \ldots, x_n\} \subset \{0, 1, 2\}$ and $f_n(x_1, x_2, \ldots, x_n) = 0$ otherwise. The algebra $(A_p; \mathbf{F}_p)$ will be denoted by $\mathfrak{A}_{p,\infty}$.

THEOREM 3.2. *For any p satisfying the inequality $1 \leqslant p \leqslant \infty$ the formulas $\varepsilon(\mathfrak{A}_{p,\infty}) = p$ and $\varrho(\mathfrak{A}_{p,\infty}) = \infty$ hold.*

Proof. First consider the algebra $\mathfrak{A}_{1,\infty}$. Since all its elements are algebraic constants, we have the formula $\gamma_0(\mathfrak{A}_{1,\infty}) = 0$. Moreover, the algebra $\mathfrak{A}_{1,\infty}$ satisfies conditions of Theorem 3.1. Consequently, $\varepsilon(\mathfrak{A}_{1,\infty}) \leqslant 2$. Obviously, $\mathfrak{A}_{1,\infty}$ is not the complete algebra over the set A_1. Therefore we have the inequality $\varepsilon(\mathfrak{A}_{1,\infty}) \geqslant 1$. Consequently, to prove the formula $\varepsilon(\mathfrak{A}_{1,\infty}) = 1$ it suffices to prove that each binary algebraic operation f in $\mathfrak{E}_1(\mathfrak{A}_{1,\infty})$ is algebraic in $\mathfrak{A}_{1,\infty}$. For such operation f and for arbitrary elements a and b from A_1 we have the formulas

$$f(a, y) = c_a y + d_a\,, \quad f(x, b) = e_b x + g_b\,,$$

where either c_a, e_b are non-negative even integers and $d_a, g_b \in A_1$ or $c_a = 1$, $d_a = 0$ or $e_b = 1$, $g_b = 0$. Hence we get the equation

$$c_a b + d_a = e_b a + g_b \qquad (a, b \in A_1)\,.$$

Consequently,

$$d_a = e_0 a + g_0\,, \quad g_b = c_0 b + d + d_0\,, \quad c_1 b + d_1 = e_b + g_b\,, \quad e_1 a + g_1 = c_a + d_a\,.$$

Thus $e_b = (c_1 - c_0)b + e_0$ and $c_a = (e_1 - e_0)a + c_0$ for all rationals a and b. But this is possible only if $c_1 = c_0$, $e_1 = e_0$. Hence we obtain the equation $f(a, y) = e_0 a + c_0 y + d_0$ for all $a \in A_1$ and the equation $f(x, b) = e_0 x + c_0 b + d_0$ for all $b \in A_1$. Moreover, the equation $e_0 = 1$ implies the equation $c_0 b + d_0 = 0$ for all $b \in A_1$ and, consequently, the equation $c_0 = d_0 = 0$. In the same way we prove that the equation $c_0 = 1$ implies the equation $e_0 = d_0 = 0$. Thus either f is a trivial operation or e_0 and c_0 are non-negative even integers and $f(x, y) = e_0 x + c_0 y + d_0$. In both cases the operation f is algebraic in $\mathfrak{A}_{1,\infty}$ which completes the proof of the formula $\varepsilon(\mathfrak{A}_{1,\infty}) = 1$.

It is clear that the operation $\sum_{j=1}^{n} 2x_j$ is algebraic in $\mathfrak{A}_{1,\infty}$ and is not algebraic in $\mathfrak{R}_{n-1}(\mathfrak{A}_{1,\infty})$ $(n = 2, 3, \ldots)$. Hence and from proposition (ii) in Section 1 we get the formula $\varrho(\mathfrak{A}_{1,\infty}) = \infty$.

Now consider the algebras $\mathfrak{A}_{p,\infty}$ for $p \geqslant 2$. It is evident that each non-trivial n-ary algebraic operation f in $\mathfrak{A}_{p,\infty}$ is either identically equal to 0 or of the form

$$f(x_1, x_2, \ldots, x_n) = f_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}),$$

where $1 \leqslant k \leqslant n$ and $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. Hence it follows that the set $A_p \setminus \{0, 2\}$ is the only set of generators of the algebra $\mathfrak{A}_{p,\infty}$. Thus $\gamma_0(\mathfrak{A}_{p,\infty}) = p$. Moreover, the algebra $\mathfrak{A}_{p,\infty}$ satisfies conditions of Theorem 3.1 which implies the equation $\varepsilon(\mathfrak{A}_{p,\infty}) = p$. Further, it is easy to verify that for all non-trivial algebraic operations f and g in $\mathfrak{A}_{p,\infty}$ the equation

$$f\big(g(x_1, x_2, \ldots, x_n), y_2, \ldots, y_m\big) = f(0, y_2, \ldots, y_m)$$

is true. Hence it follows that the fundamental operation f_n $(n \geqslant 2)$ is not algebraic in $\mathfrak{R}_{n-1}(\mathfrak{A}_{p,\infty})$. Consequently, by proposition (ii) in Section 1, $\varrho(\mathfrak{A}_{p,\infty}) = \infty$ which completes the proof of the Theorem.

4. Algebras whose order of enlargeability is 0. The purpose of this section is to compute the arity of algebras whose order of enlargeability is 0. First we shall prove some lemmas.

LEMMA 4.1. *If* $\varepsilon(\mathfrak{A}) = 0$, *then* $\mathbf{A}^{(0)}(\mathfrak{A}) \neq \emptyset$.

Proof. Contrary to this suppose that $\varepsilon(\mathfrak{A}) = 0$ and $\mathbf{A}^{(0)}(\mathfrak{A}) = \emptyset$. Hence, in particular, it follows that the algebra in question contains at least two elements. Let a, b be a pair of elements of $\mathfrak{A}$. We define two operations f and g as follows: $f(a) = b$, $f(b) = a$, $g(b, b) = b$ and $f(x) = x$, $g(x, y) = a$ otherwise. Denoting the carrier of $\mathfrak{A}$ by A we put $\mathfrak{A}_1 = (A; \{f\})$ and $\mathfrak{A}_2 = (A; \{g\})$. Since $\mathbf{A}^{(0)}(\mathfrak{A}_1) = \mathbf{A}^{(0)}(\mathfrak{A}_2) = \emptyset$, we infer that both algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are reducts of the algebra $\mathfrak{A}$. Consequently, both operations f and g are algebraic in $\mathfrak{A}$. But the composition $g\big(x, f(x)\big)$ is a constant operation identically equal to a which contradicts the equation $\mathbf{A}^{(0)}(\mathfrak{A}) = \emptyset$. The lemma is thus proved.

LEMMA 4.2. *Let* $B_1, B_2, \ldots, B_k$ *be disjoint subsets of the carrier of an algebra* $\mathfrak{A}$ *with* $\varepsilon(\mathfrak{A}) = 0$. *Then the* $(k+2)$*-ary operation* $h_{B_1, B_2, \ldots, B_k}$ *defined by the condition* $h_{B_1, B_2, \ldots, B_k}(x_1, x_2, \ldots, x_{k+2}) = x_j$ *if* $x_{k+2} \in B_j$ $(j = 1, 2, \ldots, k)$ *and* $h_{B_1, B_2, \ldots, B_k}(x_1, x_2, \ldots, x_{k+2}) = x_{k+1}$ *if* $x_{k+2} \notin \bigcup_{j=1}^{k} B_j$ *is algebraic in the reduct* $\mathfrak{R}_2(\mathfrak{A})$.

Proof. We shall prove the lemma by induction with respect to k. First consider the case $k = 1$. By Lemma 4.1 there exists an algebraic constant, say c, in $\mathfrak{A}$. Put $g_1(x, y) = x$ if $y \in B_1$, $g_1(x, y) = c$ if $y \notin B_1$, $g_2(x, y) = c$ if $y \in B_1$, $g_2(x, y) = x$ if $y \notin B_1$, $g_3(x, c) = g_3(c, x) = x$ and $g_3(x, y) = c$ otherwise. The operations g_1, g_2 and g_3 preserve the algebraic constants in $\mathfrak{A}$ and, consequently, are algebraic in $\mathfrak{A}$ because of the formula $\varepsilon(\mathfrak{A}) = 0$. Moreover, being binary operations they are also algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Further, it is easy to verify the equation

$$h_{B_1}(x_1, x_2, x_3) = g_3\big(g_1(x_1, x_3), g_2(x_2, x_3)\big)$$

which shows that the operation h_{B_1} is algebraic in $\mathfrak{R}_2(\mathfrak{A})$.

Now suppose that $k \geqslant 2$ and the $(k+1)$-ary operation $h_{B_1, B_2, \ldots, B_{k-1}}$ is algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Since

$$h_{B_1, B_2, \ldots, B_k}(x_1, x_2, \ldots, x_{k+2}) = h_B\big(h_{B_1, B_2, \ldots, B_{k-1}}(x_1, x_2, \ldots, x_k, x_{k+2}), x_{k+1}, x_{k+2}\big),$$

where $B = \bigcup_{j=1}^{k} B_j$, we infer that the operation $h_{B_1, B_2, \ldots, B_k}$ is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ which completes the proof.

It is evident that $\varepsilon = \varrho = 0$ for one-element algebras only. Moreover, it is well known that the arity of the complete algebra over an at least two-element set is 2 (see [3] and [5]). Now we shall prove a sharpening of this theorem. We note that for infinite algebras the proof is based on the axiom of choice.

THEOREM 4.1. *If* $\mathfrak{A}$ *is an algebra over an at least two-element set and* $\varepsilon(\mathfrak{A}) = 0$, *then* $\varrho(\mathfrak{A}) = 2$.

Proof. Let A be the carrier of the algebra $\mathfrak{A}$ and $\emptyset \neq B \neq A$. Then the ternary operation h_B defined in Lemma 4.2 is algebraic in $\mathfrak{A}$ and, of course, depends on every variable. Thus $\mathfrak{A}$ cannot be a unary algebra and consequently, $\varrho(\mathfrak{A}) \geqslant 2$. To prove the converse inequality it suffices, by proposition (ii) in Section 1, to prove that each n-ary algebrac operation in A is algebraic in $\mathfrak{R}_2(\mathfrak{A})$. We shall prove this statement by induction with respect to n. For $n \leqslant 2$ it is obvious. Suppose that $n > 2$ and that the statement is true for $(n-1)$-ary operations. Let f be an arbitrary n-ary algebraic operation in $\mathfrak{A}$. We shall consider three cases.

Case 1: $\operatorname{card} A < \aleph_0$. The set $\mathbf{A}^{(0)}(\mathfrak{A})$ of algebraic constants in $\mathfrak{A}$ will be briefly denoted by $\mathbf{A}^{(0)}$. Further, we introduce the notation $\mathbf{A}^{(0)} = \{c_1, c_2, \ldots, c_r\}$ and $A \setminus \mathbf{A}^{(0)} = \{d_1, d_2, \ldots, d_s\}$. Put $B_k = \{c_k\}$ $(k = 1, 2, \ldots, r)$ and

$$\begin{aligned} g_j(x_1, x_2, \ldots, x_n) = h_{B_1, B_2, \ldots, B_r}\big(&f(x_1, x_2, \ldots, x_{j-1}, c_1, x_{j+1}, \ldots, x_n), \\ &f(x_1, x_2, \ldots, x_{j-1}, c_2, x_{j+1}, \ldots, x_n), \ldots \\ &\ldots, f(x_1, x_2, \ldots, x_{j-1}, c_r, x_{j+1}, \ldots, x_n), x_n, x_j\big) \quad (j = 1, 2, \ldots, n). \end{aligned}$$

By Lemma 4.2 and the inductive assumption, the operations $g_1, g_2, \ldots, g_n$ are algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover, $g_j(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots x_n)$ if

$x_j \in \mathbf{A}^{(0)}$. Now we define operations $h_1, h_2, \ldots, h_n$ recursively as follows

$$h_1(x_1, x_2, \ldots, x_n) = g_1(x_1, x_2, \ldots, x_n),$$

$$h_{k+1}(x_1, x_2, \ldots, x_n) = h_B\big(g_{k+1}(x_1, x_2, \ldots, x_n), g_k(x_1, x_2, \ldots, x_n), x_n\big) \quad (k = 1, 2, \ldots, n-1),$$

where $B = \mathbf{A}^{(0)}$ and the operation h_B is defined in Lemma 4.2. Obviously, the operations $h_1, h_2, \ldots, h_n$ are algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover, it is easy to verify that $h_k(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$ if $x_i \in \mathbf{A}^{(0)}$ for some undex i satisfying the inequality $1 \leqslant i \leqslant k$. In particular, we have the equation

$$\text{(4.1)} \quad h_n(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) \quad \text{if} \quad \{x_1, x_2, \ldots, x_n\} \cap \mathbf{A}^{(0)} \neq \emptyset .$$

From this equations it follows that the operation f is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ if $A = \mathbf{A}^{(0)}$. Suppose now that $A \neq \mathbf{A}^{(0)}$ and put $p_j(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_{n-1}, d_j)$ if all $x_1, x_2, \ldots, x_{n-1} \notin \mathbf{A}^{(0)}$ and $p_j(x_1, x_2, \ldots, x_{n-1}) = c_1$ otherwise ($j = 1, 2, \ldots, s$). The operations p_j preserve all algebraic constants in $\mathfrak{A}$ and, consequently, are algrebaic in $\mathfrak{A}$. By the inductive assumption they are also algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Thus the composition

$$p(x_1, x_2, \ldots, x_n) = h_{D_1, D_2, \ldots, D_s}\big(p_1(x_1, x_2, \ldots, x_{n-1}), p_2(x_1, x_2, \ldots, x_{n-1}), \ldots \\ \ldots, p_s(x_1, x_2, \ldots, x_{n-1}), x_1, x_n\big),$$

where $D_j = \{d_j\}$ ($j = 1, 2, \ldots, s$) and the operation $h_{D_1, D_2, \ldots, D_s}$ is defined in Lemma 4.2, is algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover,

$$\text{(4.2)} \quad p(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) \quad \text{if} \quad \{x_1, x_2, \ldots, x_n\} \cap \mathbf{A}^{(0)} = \emptyset .$$

Now we define auxiliary operations $q_1, q_2, \ldots$ recursively as follows

$$q_1(x_1, x_2, x_3) = h_B(x_1, x_2, x_3),$$

$$q_{j+1}(x_1, x_2, \ldots, x_{j+3}) = q_j\big(x_1, q_1(x_1, x_2, x_{j+3}), x_3, \ldots, x_{j+2}\big) \quad (j = 1, 2, \ldots),$$

where $B = \mathbf{A}^{(0)}$ and the operation h_A is defined in Lemma 4.2. Of course, all these operations are algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover, $q_1(x_1, x_2, \ldots, x_{j+2}) = x_1$ if $x_i \in \mathbf{A}^{(0)}$ for some index i satisfying the inequality $3 \leqslant i \leqslant j+2$ and $q_j(x_1, x_2, \ldots, x_{j+2}) = x_2$ otherwise. The composition

$$f_0(x_1, x_2, \ldots, x_n) = q_n\big(h_n(x_1, x_2, \ldots, x_n), p(x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n\big)$$

is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ too. If $\{x_1, x_2, \ldots, x_n\} \cap \mathbf{A}^{(0)} \neq \emptyset$, then, by (4.1),

$$f_0(x_1, x_2, \ldots, x_n) = h_n(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n).$$

Further, if $\{x_1, x_2, \ldots, x_n\} \cap \mathbf{A}^{(0)} = \emptyset$, then, by (4.2),

$$f_0(x_1, x_2, \ldots, x_n) = p(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n).$$

Thus $f = f_0$ and, consequently, the operation f is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ which completes the proof of Case 1.

Case 2: $\operatorname{card} \mathbf{A}^{(0)} < \aleph_0$ and $\operatorname{card} A \geqslant \aleph_0$. Denoting by B^k the Cartesian product of k copies of the set B we have the equation $\operatorname{card}\big(A^{n-1} \setminus (\mathbf{A}^{(0)})^{n-1}\big) = \operatorname{card}(A \setminus \mathbf{A}^{(0)})$. Let g_0 be a one-to-one mapping from $A^{n-1} \setminus (\mathbf{A}^{(0)})^{n-1}$ onto $A \setminus \mathbf{A}^{(0)}$. We note that, by Lemma 4.1, the set $\mathbf{A}^{(0)}$ is non-void. Let $\mathbf{A}^{(0)} = \{c_1, c_2, \ldots, c_r\}$. We define an auxiliary operation g as follows: $g(x_1, x_2, \ldots, x_{n-1}) = g_0(x_1, x_2, \ldots, x_{n-1})$ if $(x_1, x_2, \ldots, x_{n-1}) \in A^{n-1} \setminus (\mathbf{A}^{(0)})^{n-1}$ and $g(x_1, x_2, \ldots, x_{n-1}) = c_1$ otherwise. The operation g preserves algebraic constants in A and, consequently, is algebraic in A. Moreover, by the inductive assumption, it is also algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Further, we define a binary operation on A as follows: $h(x, y) = f(x_1, x_2, \ldots, x_{n-1}, y)$ if $x \in A \setminus \mathbf{A}^{(0)}$, $x = g_0(x_1, x_2, \ldots, x_{n-1})$ and $h(x, y) = c_1$ otherwise. Since h preserves algebraic constants in $\mathfrak{A}$, we infer that it is algebraic in A and, consequently, in $\mathfrak{R}_2(\mathfrak{A})$. Put $B_j = \{c_j\}$ ($j = 1, 2, \ldots, r$) and

$$f_0(x_1, x_2, \ldots, x_n) = h_{B_1, B_2, \ldots, B_r}\big(f(c_1, x_2, \ldots, x_n), f(c_2, x_2, \ldots, x_n), \ldots \\ \ldots, f(c_r, x_2, \ldots, x_n), h\big(g(x_1, x_2, \ldots, x_{n-1}), x_n\big), x_1\big),$$

where the operation $h_{B_1, B_2, \ldots, B_r}$ is defined by Lemma 4.2. Of course, the operation f_0 is algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover, it is easy to verify that

$$f_0(c_j, x_2, \ldots, x_n) = f(c_j, x_2, \ldots, x_n) \quad (j = 1, 2, \ldots, r)$$

and

$$f_0(x_1, x_2, \ldots, x_n) = h\big(g(x_1, x_2, \ldots, x_{n-1}), x_n\big) = h\big(g_0(x_1, x_2, \ldots, x_{n-1}), x_n\big) \\ = f(x_1, x_2, \ldots, x_n) \quad \text{if} \quad x_1 \notin \mathbf{A}^{(0)}.$$

Thus $f = f_0$ and, consequently, the operation f is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ which completes the proof of Case 2.

Case 3: $\operatorname{card} \mathbf{A}^{(0)} \geqslant \aleph_0$. We define an auxiliary $(n-1)$-ary operation as follows. If $\operatorname{card} A = \operatorname{card} \mathbf{A}^{(0)}$, then g is defined to be a one-to-one mapping from A^{n-1} onto $\mathbf{A}^{(0)}$. If $\operatorname{card} A > \operatorname{card} \mathbf{A}^{(0)}$, then as the operation g we take a one-to-one mapping from A^{n-1} onto A which transforms $(\mathbf{A}^{(0)})^{n-1}$ onto $\mathbf{A}^{(0)}$. The operation g preserves algebraic constants in $\mathfrak{A}$ and, consequently, is algebraic in $\mathfrak{A}$. Moreover, by the inductive assumption, it is algebraic in $\mathfrak{R}_2(\mathfrak{A})$.

Let c be an algebraic constant in $\mathfrak{A}$. For any index j ($1 \leqslant j \leqslant n$) we put $d_j(x, y) = f(x_1, x_2, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n)$ if $y \notin \mathbf{A}^{(0)}$ or

$$x = g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \quad \text{and} \quad x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in \mathbf{A}^{(0)}.$$

Further, $d_j(x, y) = c$ in the opposite case. The operations $d_1, d_2, \ldots, d_n$ are algebraic in $\mathfrak{R}_2(\mathfrak{A})$, because they preserve algebraic constants in $\mathfrak{A}$. Moreover, for the compositions

$$s_j(x_1, x_2, \ldots, x_n) = d_j\big(g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), x_j\big) \quad (j = 1, 2, \ldots, n)$$

we have the formula

$$(4.3) \quad s_j(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$$
$$\text{if either } x_1, x_2, \ldots, x_n \in \mathbf{A}^{(0)} \text{ or } x_j \notin \mathbf{A}^{(0)} \quad (j = 1, 2, \ldots, n).$$

Setting $B = \mathbf{A}^{(0)}$ we define operations $f_1, f_2, \ldots, f_n$ resursively as follows

$$f_1(x_1, x_2, \ldots, x_n) = s_1(x_1, x_2, \ldots, x_n),$$
$$f_{j+1}(x_1, x_2, \ldots, x_n) = h_B\big(f_j(x_1, x_2, \ldots, x_n), s_{j+1}(x_1, x_2, \ldots, x_n), x_{j+1}\big)$$
$$(j = 1, 2, \ldots, n-1),$$

where the operation h_B is defined in Lemma 4.2. Of course, all operations $f_1, f_2, \ldots, f_n$ are algebraic in $\mathfrak{R}_2(\mathfrak{A})$. Moreover, taking into account (4.3), we can easily prove by induction with respect to j the formula $f_j(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$ if either $x_1, x_2, \ldots, x_n \in \mathbf{A}^{(0)}$ or $x_i \notin \mathbf{A}^{(0)}$ for some index i satisfying the inequality $1 \leqslant i \leqslant j$. Thus $f = f_n$ and, consequently, the operation f is algebraic in $\mathfrak{R}_2(\mathfrak{A})$ which completes the proof of the theorem.

5. Some reducts of Boolean algebras. Let $\mathfrak{A}$ be a Boolean algebra with a denumerable set of generators, 0 as the neutral element and $1 = 0'$. Let us introduce the notation

$$t(x) = x', \quad s_1(x, y) = x \cup y, \quad s_2(x, y) = x \cap y,$$
$$q_2(x, y, z) = x \cup (y \cap z), \quad u_n(x_1, x_2, \ldots, x_n) = \bigcup_{j=1}^{n} \bigcap_{i \neq j} x_i \quad (n = 3, 4, \ldots).$$

Moreover, by **0** and **1** we shall denote the constant operations equal to 0 and 1 respectively. Denoting the carrier of $\mathfrak{A}$ by A we put

$$\mathfrak{A}_0 = (A; \{\mathbf{0}, \mathbf{1}\}), \quad \mathfrak{A}_1 = (A; \{\mathbf{0}, t\}), \quad \mathfrak{A}_2 = (A; \{\mathbf{0}, \mathbf{1}, s_1, s_2\}),$$
$$\mathfrak{A}_q = (A; \{q_2, u_q\}) \quad (q = 3, 4, \ldots).$$

For any p satisfying the inequality $\max(2, q-1) \leqslant p \leqslant \infty$ by $\mathfrak{A}_{p,q}$ we shall denote a subalgebra of $\mathfrak{A}_q$ with $\gamma_0(\mathfrak{A}_{p,q}) = p$ containing both elements 0 and 1. Moreover, we put $\mathfrak{A}_{1,2} = (\{0, 1\}; \{\mathbf{0}, \mathbf{1}, s_1, s_2\})$.

THEOREM 5.1. *For any p and q satisfying the condition $q = 0, 1, \ldots$ and $\max(2, q-1) \leqslant p \leqslant \infty$ the formulas $\varepsilon(\mathfrak{A}_{p,q}) = p$ and $\varrho(\mathfrak{A}_{p,q}) = q$ hold. Moreover, $\varepsilon(\mathfrak{A}_{1,2}) = 1$ and $\varrho(\mathfrak{A}_{1,2}) = 2$.*

Proof. It is clear that the elements 0 and 1 form a two-element subalgebra of $\mathfrak{A}_q$ which will be denoted by $\mathfrak{B}_q$. The algebra $\mathfrak{B}_q$ is a subalgebra of $\mathfrak{A}_{p,q}$ for all indices p satisfying the inequality $\max(2, q-1) \leqslant p \leqslant \infty$. Moreover,

$$(5.1) \quad \mathfrak{A}_{1,2} = \mathfrak{B}_2.$$

Each algebraic operation in $\mathfrak{A}_{p,q}$ is uniquely determined by its restriction to $\mathfrak{B}_q$. Consequently,

$$(5.2) \quad \varrho(\mathfrak{A}_{p,q}) = \varrho(\mathfrak{B}_q)$$

and, by Theorem 4.1 in [4],

$$(5.3) \quad \varepsilon(\mathfrak{A}_{p,q}) \leqslant \max\big(\varepsilon(\mathfrak{B}_q), \gamma_0(\mathfrak{A}_{p,q})\big) = \max\big(\varepsilon(\mathfrak{B}_q), p\big) \quad (p \geqslant 2).$$

Further, the algebras $\mathfrak{A}_{p,q}$ have property (✱) defined in [4], Chapter 7 and are not isomorphic to the exceptional algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ defined in [4], Chapter 8. Thus, by Theorem 9.2 in [4],

$$(5.4) \quad \varepsilon(\mathfrak{A}_{p,q}) \geqslant \gamma_0(\mathfrak{A}_{p,q}) = p \quad (p \geqslant 2).$$

Now consider the subalgebras $\mathfrak{B}_q$ $(q = 0, 1, \ldots)$. From the results presented in [4], Chapter 8 we get the formulas

$$\mathfrak{B}_0 = \mathfrak{H}_1, \quad \mathfrak{B}_1 = \mathfrak{H}_2, \quad \mathfrak{B}_2 = \mathfrak{G}, \quad \mathfrak{B}_q = \mathfrak{R}_{2,q-1} \quad (q \geqslant 3).$$

(We note that in the definition of $\mathfrak{R}_{2,n}$ in [4], p. 273 u_{n+1} instead of u_n should be written.) Hence and from the table in [4], p. 274 we get the following formulas

$$\varepsilon(\mathfrak{B}_0) = \varepsilon(\mathfrak{B}_1) = 2, \quad \varepsilon(\mathfrak{B}_2) = 1, \quad \varepsilon(\mathfrak{B}_q) = q-1 \quad (q \geqslant 3),$$
$$\varrho(\mathfrak{B}_q) = q \quad (q = 0, 1, \ldots).$$

Now the assertion of the theorem is a direct consequence of formulas (5.1), (5.2), (5.3) and (5.4).

6. A class of unary algebras. An algebra is said to be *unary* if all its algebraic operations essentially depend on at most one variable. It was proved in [4] (Theorem 13.1) that for unary algebras with $\gamma_0(\mathfrak{A}) \geqslant 3$ the equation $\varepsilon(\mathfrak{A}) = \gamma_0(\mathfrak{A})$ holds. In this section we shall study unary algebras whose all elements are algebraic constants.

THEOREM 6.1. *For unary algebras with $\gamma_0(\mathfrak{A}) = 0$ the inequality $\varepsilon(\mathfrak{A}) \leqslant 2$ holds.*

Proof. By Theorems 3.1 and 13.1 in [4], we have the inequality $\varepsilon(\mathfrak{A}) \leqslant 3$. Consequently, by proposition (i) in Section 1, to prove the theorem it suffices to prove that each ternary algebraic operation f in $\mathfrak{E}_2(\mathfrak{A})$ is algebraic in $\mathfrak{A}$. Of course, without loss of generality we may assume that the operation f depends on the first variable. Consequently, there exist elements a_0 and b_0 and a non-constant unary operation g algebraic in $\mathfrak{A}$ such that

$$f(x, a_0, b_0) = g(x)\,. \tag{6.1}$$

Further, the binary operation $f(x, a_0, z)$ is algbraic in $\mathfrak{A}$ and, consequently, depends on at most one variable. By (6.1) it depends on the variable x which implies the formula

$$f(x, a_0, z) = g(x)\,. \tag{6.2}$$

Let c be an arbitrary element of the algebra $\mathfrak{A}$. The binary operation $f(x, y, c)$ is also algebraic in $\mathfrak{A}$ and, consequently, depends on at most one variable. Taking into account (6.2) we infer that it depends on the variable x and, consequently, $f(x, y, c) = g(x)$ for all elements c. Thus the operation f is algebraic in $\mathfrak{A}$ which completes the proof.

We note that if all unary operations are algebraic in an at least two-element algebra $\mathfrak{A}$, then $\gamma_0(\mathfrak{A}) = 0$ and $\varepsilon(\mathfrak{A}) = 2$.

THEOREM 6.2. *Suppose that the unary algebra $\mathfrak{A}$ with $\gamma_0(\mathfrak{A}) = 0$ satisfies the condition*

$$\min\{\operatorname{card} g(A):\ g \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})\} > \operatorname{card}\big(\mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})\big)+1\,, \tag{6.3}$$

where A is the carrier of $\mathfrak{A}$. Then $\varepsilon(\mathfrak{A}) = 1$.

Proof. Since $\mathfrak{A}$ is not the complete algebra over the set $\mathfrak{A}$, we have the inequality $\varepsilon(\mathfrak{A}) \geqslant 1$. Moreover, by Theorem 6.1, $\varepsilon(\mathfrak{A}) \leqslant 2$. Thus, to prove the Theorem it suffices to prove that each binary algebraic operation f in $\mathfrak{E}_1(\mathfrak{A})$ is algebraic in $\mathfrak{A}$. Without loss of generality we may assume that f depends on the first variable. Consequently, there exist an element a_0 in A and a non-constant operation $h \in \mathbf{A}^{(1)}(\mathfrak{A})$ such that

$$f(x, a_0) = h(x)\,. \tag{6.4}$$

Put

$$B = \{b:\ h(b) \neq g(a_0) \text{ for all } g \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})\}\,.$$

From inequality (6.3) it follows that

$$\operatorname{card} h(B) \geqslant 2\,. \tag{6.5}$$

For every $b \in B$ the unary operation $f(b, x)$ is algebraic in $\mathfrak{A}$ and does not depend on the variable x. Indeed, the equation $f(b, x) = g_0(x)$, where $g_0 \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})$, would imply, by (6.4), the equation $h(b) = g_0(a_0)$ and, consequently, the relation $b \in B$. Thus for every $b \in B$ the operation $f(b, x)$ is constant. Hence and from (6.4) we get the formula

$$f(b, x) = h(b) \qquad (b \in B)\,. \tag{6.6}$$

Let c be an arbitrary element of A. Suppose that the operation $f(x, c)$ is constant, say $f(x, c) \equiv c_0$. From (6.5) it follows that there exists an element $b_0 \in B$ such that $h(b_0) \neq c_0$. Further, by (6.6), $h(b_0) = f(b_0, c) = c_0$ which gives the contradiction. Thus for every element $c \in A$ the operation $f(x, c)$ depends on the variable x. In other words, for every $c \in A$ there exists an operation $g_c \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})$ such that

$$f(x, c) = g_c(x) \qquad (x, c \in A)\,. \tag{6.7}$$

Suppose now that the operation f depends on both variables. Then there exist an element $e_0 \in A$ and an operation $h_0 \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})$ such that $f(e_0, y) = h_0(y)$ $(y \in A)$. Hence and from (6.7) we get the inclusion

$$h_0(A) = \{g_c(e_0):\ c \in A\} \subset \{g(e_0):\ g \in \mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})\}\,.$$

Consequently,

$$\operatorname{card} h_0(A) \leqslant \operatorname{card}\big(\mathbf{A}^{(1)}(\mathfrak{A})\setminus \mathbf{A}^{(0)}(\mathfrak{A})\big)$$

which contradicts condition (6.3). Thus the operation f depends on one variable and, consequently, is algebraic in A which completes the proof of the theorem.

Let $A_0 = \{0, 1, \ldots, m\}$ where $m > 3$. By $\mathbf{F}_0$ we shall denote the family of all constant operations on A_0 and by h the unary operation defined by the conditions $h(0) = 1$ and $h(x) = x$ for $x \geqslant 1$. Put $\mathfrak{A}_{1,0} = (A_0; \mathbf{F}_0)$ and $\mathfrak{A}_{1,1} = (A_0; \mathbf{F}_0 \cup \{h\})$. The formulas

$$\varrho(\mathfrak{A}_{1,0}) = 0 \quad \text{and} \quad \varrho(\mathfrak{A}_{1,1}) = 1 \tag{6.8}$$

are obvious. Moreover, $\gamma_0(\mathfrak{A}_{1,0}) = \gamma_0(\mathfrak{A}_{1,1}) = 0$. Further, non-constant algebraic operations in $\mathfrak{A}_{1,0}$ are trivial and the operation h is the only non-constant and non-trivial unary operation in $\mathfrak{A}_{1,1}$. Thus for the algebra $\mathfrak{A}_{1,0}$ the left-hand side and the right-hand side of (6.3) are equal to $m+1$ and 2 respectively. The same quantities for the algebra $\mathfrak{A}_{11}$ are equal to m and 3 respectively. Thus, the both algebras satisfy the condition of theorem 6.2 and, consequently,

$$\varepsilon(\mathfrak{A}_{1,0}) = \varepsilon(\mathfrak{A}_{1,1}) = 1\,. \tag{6.9}$$

7. Description of all pairs (ε, ϱ). The algebras $\mathfrak{A}_{p,q}$ $(p = 1, 2, \ldots, \infty;\ q = 0, 1, \ldots, \infty)$ defined in the preceding sections satisfy the conditions $\varepsilon(\mathfrak{A}_{p,q}) = p$ and $\varrho(\mathfrak{A}_{p,q}) = q$ (see Theorems 2.1, 2.2, 3.2 and 5.1 and formulas (6.8) and (6.9)). Moreover, for a one-element algebra we have the formula $\varepsilon = \varrho = 0$ and for complete algebras over an at least two-element set the formulas $\varepsilon = 0$ and $\varrho = 2$. On the other hand, by Theorem 4.1, for algebras with $\varepsilon = 0$ we have either $\varrho = 0$ or $\varrho = 2$. Thus we have proved the following theorem.

THEOREM 7.1. *The set of all possible pairs* (ε, ϱ) *for abstract algebras is the set of all pairs* (p, q), *where either* $p = 1, 2, \ldots, \infty$, $q = 0, 1, \ldots, \infty$ *or* $p = 0$ *and* $q = 0, 2$.

References

[1] E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45-61.

[2] — *Independence in abstract algebras. Results and problems*, Colloq. Math. 14 (1966), pp. 169-188.

[3] W. Sierpiński, *Sur les fonctions de plusieurs variables*, Fund. Math. 23 (1945), pp. 169-173.

[4] K. Urbanik, *On some numerical constants associated with abstract algebras*, ibidem 59 (1966), pp. 263-288.

[5] D. Webb, *Generation of any n-valued logic by one binary operator*, Proc. Nat Acad. Sci. 21 (1935), pp. 252-254.

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