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Reçu par la Rédaction le 10. 2. 1967

The topology of a partially well ordered set*

by

E. S. Wolk (Storrs, Conn.)

1. Introduction. If $(X, \leqslant)$ is a partially ordered set, there are many known ways of using the order properties of X to define certain natural or "intrinsic" topologies on X. In particular, we may define the well-known *interval topology* $\mathfrak{J}$ on X by taking all sets of the form $\{x \in X: x \leqslant a\}$ or $\{x \in X: x \geqslant b\}$ as a sub-base for the closed sets. We also define another topology $\mathfrak{D}$ on X, which we call its *Dedekind topology*, as follows. A subset A of X is said to be *up-directed* (*down-directed*) if and only if for all $x \in A$ and $y \in A$ there exists $z \in A$ with $x \leqslant z$, $y \leqslant z$ ($x \geqslant z$, $y \geqslant z$). A subset containing a greatest element is trivially up-directed, and dually. Following McShane [8], we call a subset K of X *Dedekind-closed* if and only if whenever A is an up-directed subset of K and $y = \text{l.u.b.}\,A$, or A is a down-directed subset of K and $y = \text{g.l.b.}\,A$, we also have $y \in K$. We then define $\mathfrak{D}$ as the topology whose closed sets are precisely the Dedekind-closed subsets of X. It is clear that $\mathfrak{J} \subseteq \mathfrak{D}$ for all partially ordered sets X. In [12], we called an arbitrary topology $\mathfrak{T}$ on X *order-compatible* if and only if $\mathfrak{J} \subseteq \mathfrak{T} \subseteq \mathfrak{D}$.

Let us say that a subset A of X is *totally unordered* if and only if x and y are incomparable (with respect to the order $\leqslant$) for all $x, y \in A$ with $x \neq y$. Naito [9] showed that if every totally unordered subset of X is finite, then X possesses a unique order-compatible topology (i.e., the topologies $\mathfrak{J}$ and $\mathfrak{D}$ coincide).

A partially ordered set X is called *partially well ordered* (pwo) if and only if all totally unordered subsets of X are finite and all chains in X are well ordered. The purpose of this paper is to study some of the properties of the unique order-compatible topology of a pwo-set X. We call this topology on X its *intrinsic topology*. Among our results, we characterize the convergent nets and the closure operation in this topology in "order-theoretic" terms. We show that the intrinsic topology is completely regular for any pwo-set X, and may be obtained from a certain natural proximity relation definable in terms of the ordering in X. The normal

* This research was supported by National Science Foundation Grant GP-5968.

completion of X plays a key role in this connection. We also consider continuous functions on a pwo-set X to a pwo-set X', and show that every function on X to X' is the pointwise limit of a net of continuous functions.

Since the concept of a partially well ordered set is a natural generalization of the notion of an ordinal number, some of this work can be considered as an extension of the classical theory of transfinite sequences of ordinals and of the theory of continuous ordinal-valued functions of an ordinal variable ([1], § 5). It may also be noted that since many of the standard counter-examples of general topology (such as the "Tychonoff plank") are products of spaces of ordinal numbers, they possess a natural structure as pwo-sets; and, furthermore, for such spaces the product topology coincides with the intrinsic topology. Our results can therefore be applied to these specific spaces and might provide further insight into their structure.

2. Preliminaries. In this section we collect some definitions and preliminary theorems which will be required later.

Let $(X, \leqslant)$ be a partially ordered set and $Q \subseteq X$. A subset M of Q is a *minimal subset* of Q if and only if

(i) each $m \in M$ is a minimal element of Q,

and

(ii) for all $x \in Q$, there exists $m \in M$ with $m \leqslant x$. Then X is pwo if and only if every non-empty subset of X contains a finite minimal subset. Further characterizations of pwo-sets and references to their literature are given in [13]. A mapping f of a partially ordered set $(X, \leqslant)$ into a partially ordered set $(X', \leqslant')$ is *order-preserving* if and only if $x, y \in X$ and $x \leqslant y$ imply $f(x) \leqslant' f(y)$. The image of a pwo-set under an order-preserving mapping is pwo. A 1:1 mapping of X onto X' is an *isomorphism* if and only if both f and f^{-1} are order-preserving.

A partially ordered set X is *Dedekind-complete* if and only if

(i) every up-directed subset of X has a l.u.b. in X

and

(ii) every down-directed subset of X has a g.l.b. in X.

If X is pwo, then every down-directed subset contains a least element, and hence X is Dedekind-complete if and only if condition (i) holds. An important theorem due to P. M. Cohn ([3], p. 33) states that *if every chain in a partially ordered set X has a* l.u.b. *in X, then every up-directed subset of X has a* l.u.b. *in X*. From this it follows that a pwo-set X is Dedekind-complete if and only if every chain in X has a l.u.b. in X. (It should be noted that we are not assuming that the partially ordered sets under discussion possess greatest or least elements.)

The following notation will be useful. If $A \subseteq X$, we write

$$A^* = \{x \in X\colon x \geqslant a \text{ for all } a \in A\},$$
$$A^+ = \{x \in X\colon x \leqslant a \text{ for all } a \in A\},$$
$$[a, b] = \{x \in X\colon a \leqslant x \leqslant b\},$$
$$J_a = \{x \in X\colon x \leqslant a\}.$$

We shall write A^{*+} for $(A^*)^+$. The set J_a will be called the *principal ideal generated by a*. The empty set will be denoted by $\emptyset$.

THEOREM 1. *If a partially ordered set X is compact in its interval topology $\mathfrak{J}$, then X is Dedekind-complete.*

Proof. Let A be a down-directed subset of X. The family of $\mathfrak{J}$-closed sets $\{J_a\colon a \in A\}$ has the finite intersection property, and hence $A^+ = \bigcap \{J_a\colon a \in A\} \neq \emptyset$. Now consider the family $\mathcal{F} = \{[x, y]\colon x \in A^+, y \in A\}$. This family also consists of $\mathfrak{J}$-closed sets and has the finite intersection property. If $m \in \bigcap \mathcal{F}$, then clearly $m = \text{g.l.b.}\, A$. The obvious dual argument may be applied to the up-directed subsets of X to complete the proof.

The converse of Theorem 1 does not hold in general ([4] [6]).

We shall follow the notation of Kelley [5] in regard to nets and subnets. Let $\{S_n, n \in D, \leqslant\}$ be a net on a directed set $(D, \leqslant)$ whose values lie in a partially ordered set $(X, \leqslant)$. (Note that we are using the same symbol "$\leqslant$" for the order in both D and X.) We say that $\{S_n\}$ is *monotone increasing* (*decreasing*) if and only if whenever $m, n \in D$ and $m \leqslant n$, then $S_m \leqslant S_n$ ($S_m \geqslant S_n$). A net is *monotone* if and only if it is either monotone increasing or monotone decreasing. R. W. Hansell [4] has recently proved the following theorem, which is fundamental for many of our results.

THEOREM 2 (Hansell). *If X is a partially ordered set in which every totally unordered subset is finite, then every net in X has a monotone subnet.*

In a pwo-set any monotone decreasing net is eventually constant: hence we have the following

COROLLARY. *Every net in a pwo-set has a monotone increasing subnet.*

The following theorem contains simple results concerning the convergence of nets with respect to the interval topology $\mathfrak{J}$.

THEOREM 3. *Let X be a partially ordered set and $\{S_n, n \in D\}$ a net in X.*

(i) *If $\{S_n\}$ $\mathfrak{J}$-converges to $y \in X$, and $S_n \leqslant y$ for all $n \in D$, then $y =$* l.u.b. (range S_n).

(ii) *If $\{S_n\}$ is monotone increasing and $y =$* l.u.b. (range S_n), *then $\{S_n\}$ $\mathfrak{J}$-converges to y.*

Proof of (i). Let $A = \text{range}\, S_n$. Suppose that $y \neq \text{l.u.b.}\, A$. Then there exists $z \in A^*$ with $y \notin J_z$. But then $X - J_z$ is an $\mathfrak{J}$-open set containing y; and so $\{S_n\}$ is eventually in $X-J_z$. This contradicts $z \in A^*$. The proof of (ii) may be left to the reader.

We now have

THEOREM 4. *If X is a Dedekind-complete pwo-set, then X is $\mathfrak{J}$-compact.*

Proof. If $S = \{S_n\}$ is any net in X, then S has a monotone increasing subnet T, by the corollary of Theorem 2. But T $\mathfrak{J}$-converges to the l.u.b. of its range, by Theorem 3 (ii). Hence every net S in X has an $\mathfrak{J}$-convergent subnet.

Let $\{X_a\colon a \in I\}$ be any family of partially ordered sets, and let the partial order on X_a be denoted by R_a. If $P = \Pi\{X_a\colon a \in I\}$ denotes the cartesian product of this family, then for $f, g \in P$, we may define $f \leqslant g$ if and only if $f(a) R_a g(a)$ for all $a \in I$. The partially ordered set $(P, \leqslant)$ is called the *cardinal product* of the partially ordered sets (X_a, R_a). It is a well-known result that *the cardinal product of finitely many pwo-sets is pwo* [10].

3. Basic properties of the intrinsic topology.

The intrinsic topology of a pwo-set X will be denoted by $\mathfrak{J}$. It is known that the topology $\mathfrak{J}$ is Hausdorff on any pwo-set ([12], Lemma 2). Since for a pwo-set we have $\mathfrak{J} = \mathfrak{D}$, in some of our proofs the reader will note that we use the fact that the closed sets of $\mathfrak{J}$ are precisely the Dedekind-closed subsets of X, while at other times we regard $\mathfrak{J}$ as the usual interval topology.

THEOREM 5. *If $\{S_n, n \in D, \leqslant\}$ is a net in a pwo-set X which $\mathfrak{J}$-converges to $y \in X$, then there exists $m \in D$ such that $y =$ l.u.b. $\{S_n\colon m \leqslant n\}$.*

Proof. If $J_y = X$, then Theorem 5 follows at once from Theorem 3 (i). If $X - J_y \neq \emptyset$, then it contains a finite minimal subset M such that $X - J_y = \{x \in X\colon x \geqslant b$ for some $b \in M\}$. Hence $X - J_y$ is $\mathfrak{J}$-closed, and so J_y is $\mathfrak{J}$-open. But $\{S_n\}$ $\mathfrak{J}$-converges to y implies that S_n is eventually in J_y. The theorem now follows from Theorem 3 (i).

The following theorem now characterizes the $\mathfrak{J}$-convergence of nets.

THEOREM 6. *Let $S = \{S_n, n \in D, \leqslant\}$ be a net in a pwo-set X. Then S $\mathfrak{J}$-converges to $y \in X$ if and only if for every cofinal subnet T of S, there exists an up-directed subset E of range T with $y =$ l.u.b. E.*

Proof. Suppose that S $\mathfrak{J}$-converges to y, and let T be a cofinal subnet of S. By the corollary to Theorem 2, T has a monotone increasing subnet R. Since T $\mathfrak{J}$-converges to y, so does R. If $E =$ range R, then $E \subseteq$ range T, E is up-directed, and $y =$ l.u.b. E by Theorem 5. To prove the converse, suppose that S does not $\mathfrak{J}$-converge to y. Then S is frequently in some $\mathfrak{J}$-closed set which does not contain y. Hence there is some member B of the closed base for the $\mathfrak{J}$ topology, with $y \notin B$, and such that B contains the range of some cofinal subnet R of S. But is the union of a finite number of members of the closed sub-base for $\mathfrak{J}$. Each of these sub-base members is a principal ideal or dual principal ideal in X. Furthermore R is frequently in one of them, say J_a, for some $a \in X$. This means there is a cofinal subnet T of R (and T is also a subnet of S) whose range is in J_a. But since $y \notin J_a$, there is no up-directed subset E of range T with l.u.b. $E = y$.

THEOREM 7. *If P is the cardinal product of the pwo-sets X and Y, then the intrinsic topology of P is identical with the product of the intrinsic topologies of X and Y.*

Proof. Let $\mathfrak{J}$ denote the intrinsic topology of P and $\mathfrak{T}$ the product of the intrinsic topologies of X and Y. We shall first prove that $\mathfrak{T} \subseteq \mathfrak{J}$ by showing that every $\mathfrak{J}$-convergent net in P is $\mathfrak{T}$-convergent. Let $S = \{S_n, n \in D\}$ be an $\mathfrak{J}$-convergent net in P, where $S_n = (x_n, y_n)$, and $x_n \in X$, $y_n \in Y$ for all $n \in D$; and suppose that S $\mathfrak{J}$-converges to $(a, b) \in P$. Let C be any cofinal subset of D, and consider the cofinal subnet $\{x_n, n \in C\}$ of the net $\{x_n, n \in D\}$. Since S $\mathfrak{J}$-converges to (a, b), the range of its cofinal subnet $\{S_n, n \in C\}$ must contain an up-directed subset Q with l.u.b. $Q = (a, b)$, by Theorem 6. Let $\mathrm{Pr}_1(Q)$ denote the projection of Q on the set X. From the definition of the cardinal product order it follows that $\mathrm{Pr}_1(Q)$ is an up-directed subset of the range of $\{x_n, n \in C\}$, and furthermore l.u.b. $\mathrm{Pr}_1(Q) = a$. Thus we have shown that the range of every cofinal subnet of $\{x_n, n \in D\}$ contains an up-directed subset whose l.u.b. is a. By Theorem 6, $\{x_n, n \in D\}$ converges to $a \in X$ with respect to the intrinsic topology of X. The same argument shows that the net $\{y_n, n \in D\}$ converges to $b \in Y$ in the intrinsic topology of Y. By definition of the product topology of P, the net $\{S_n, n \in D\}$ $\mathfrak{T}$-converges to (a, b).

We now complete the proof by showing that the reverse inclusion $\mathfrak{J} \subseteq \mathfrak{T}$ holds for the product P of arbitrary partially ordered sets X and Y (where $\mathfrak{T}$ is the product of the interval topologies). Let K be a typical member of the closed sub-base for the interval topology $\mathfrak{J}$ of P: for exmaple, suppose that $K = \{(x, y) \in P\colon x \leqslant a, y \leqslant b\}$. Then the projections of K on X and Y are $\mathrm{Pr}_1(K) = J_a \subseteq X$, $\mathrm{Pr}_2(K) = J_b \subseteq Y$. Since the projection mappings are continuous with respect to $\mathfrak{T}$, and since the sets J_a and J_b are closed, it follows that $K = \mathrm{Pr}_1^{-1}(J_a) \cap \mathrm{Pr}_2^{-1}(J_b)$ is $\mathfrak{T}$-closed. Hence every $\mathfrak{J}$-closed set is $\mathfrak{T}$-closed.

Remarks. (i). For arbitrary partially ordered sets X and Y, it is not true in general that the interval (or Dedekind) topology of their product is the product of their interval (or Dedekind) topologies. The complex plane, considered as the cardinal product of the real number system with itself, provides an obvious counter-example.

(ii). Let Ω_0 be the set of all ordinal numbers less than the first uncountable ordinal Ω, and let $\Omega' = \Omega_0 \cup \{\Omega\}$. Then the product topology of $\Omega_0 \times \Omega'$, which is one of the familiar counter-examples of general topology, is identical with its intrinsic topology. Theorems 5 and 6 therefore characterize convergence of nets in this space and in similar products of spaces of ordinals. This example also shows that the intrinsic topology of a pwo-set may fail to be normal ([5], p. 131).

If $A \subseteq X$, we shall use $\bar{A}$ to denote the closure of A with respect to the intrinsic topology of X.

THEOREM 8. *If X is a pwo-set and $A \subseteq X$, then $\bar{A} = \{y \in X\colon\ y = \text{l.u.b. } E$ for some up-directed $E \subseteq A\}$.*

Proof. Let us for the moment denote the set $\{y \in X\colon\ y = \text{l.u.b. } E$ for some up-directed $E \subseteq A\}$ by $A^{\wedge}$. Trivially, $A \subseteq A^{\wedge}$. Now if $y \in A^{\wedge}$, and B is any $\mathfrak{J}$-closed subset of X containing A, then $y \in B$. Hence $A^{\wedge} \subseteq \bar{A}$. Thus the theorem will be proved if we show that $A^{\wedge}$ is $\mathfrak{J}$-closed (or, equivalently, $\mathfrak{D}$-closed).

Let D be an up-directed subset of $A^{\wedge}$ possessing a l.u.b. y in X. We shall show that $y \in A^{\wedge}$. By definition of $A^{\wedge}$, for each $m \in D$ there is an up-directed set $E_m \subseteq A$ with l.u.b. $E_m = m$. For each fixed m, consider the net $S = \{S(m, n), n \in E_m\}$ defined by $S(m, n) = n$. Then we have

$$\lim_{m \in D} [\lim_{n \in E_m} S(m, n)] = y\,.$$

Now we may apply Kelley's theorem on iterated limits ([5], p. 69). Consider the set $F = D \times \Pi\{E_m\colon\ m \in D\}$, with the cardinal product order. For $(m, f) \in F$, let $R(m, f) = \big(m, f(m)\big) \in D \times E_m$. Then the set $S \circ R$ defined on F by $S\big(R(m, f)\big) = f(m)$ converges to y. Furthermore,

$$\operatorname{range}(S \circ R) \subseteq \bigcup \{E_m\colon\ m \in D\} \subseteq A\,.$$

From the corollary to Theorem 2, we may now infer that $S \circ R$ has a monotone increasing subnet T. Then T also converges to y, and range T is an updirected subset of A. By Theorem 3(i), $y = \text{l.u.b.}(\text{range } T)$. So y is the l.u.b. of an up-directed subset of A, and hence $y \in A^{\wedge}$. Thus $A^{\wedge}$ is $\mathfrak{J}$-closed, and so $A^{\wedge} = \bar{A}$.

If X is an arbitrary partially ordered set and $A \subseteq X$, then the $\mathfrak{D}$-closure of A (and thus also its $\mathfrak{J}$-closure) might be a larger set than $\{y \in X\colon\ y = \text{l.u.b. } E$ for some up-directed $E \subseteq A\}$. The reader can easily construct examples. It $\mathcal{S}$ is the relativization of the interval topology of X to the subset A, then simple examples also show (even for X pwo or linearly ordered) that $\mathcal{S}$ need not coincide with the interval topology of A. However, for pwo-sets we have the following theorem.

THEOREM 9. *Let $\mathfrak{J}$ be the intrinsic topology of a pwo-set X, and $\mathcal{S}$ the relativization of $\mathfrak{J}$ to the subset $A \subseteq X$. If $\mathcal{T}$ is the intrinsic topology of A, then $\mathcal{T} \subseteq \mathcal{S}$.*

Proof. Let B be a $\mathcal{T}$-closed subset of A, and let $\bar{B}$ denote the $\mathfrak{J}$-closure of B in X. To show that B is $\mathcal{S}$-closed, we shall prove that $B = \bar{B} \cap A$. So suppose that $y \in \bar{B} \cap A$. Then by Theorem 8 there exists an up-directed subset E of B with $y = \text{l.u.b. } E$. But $y \in A$ and B is $\mathcal{T}$-closed in A. Hence $y \in B$. This establishes that $\bar{B} \cap A \subseteq B$, and the reverse inclusion is trivial.

Following [2], we shall say that a subset A of a partially ordered set X is a *normal ideal* of X if and only if $A^{*+} = A$. The set of all normal ideals of X will be denoted by $N(X)$. With respect to the usual ordering by set inclusion, $N(X)$ is a complete lattice (with least and greatest elements $\emptyset$ and X respectively), which we call the *normal completion* of X. We denote the set of all principal ideals of X by $J(X)$. Then, for $x \in X$, the mapping $x \to J_x$ is an isomorphism of X onto the subset $J(X)$ of $N(X)$.

We wish to show next that $J(X)$ provides a homeomorphic image of X in $N(X)$. For this purpose we first need a lemma, whose proof may be found in [10], Lemma 2.

LEMMA. *Let X be any pwo-set and $\mathcal{F}(X)$ the set of all totally unordered subsets of X. For, $A, B \in \mathcal{F}(X)$, define $A \precsim B$ if and only if for all $x \in A$, there exists $y \in B$ with $x \leqslant y$. Then $(\mathcal{F}(X), \precsim)$ is a pwo-set.*

We now have

THEOREM 10. *If X is a pwo-set, then the lattice $N(X)$ is pwo.*

Proof. A partially ordered set $(P, \leqslant)$ is pwo if, for any infinite sequence $\{x_n\}$ in P, there exist i, j with $i < j$ and $x_i \leqslant x_j$. So let $\{K_n\}$ be any infinite sequence in $N(X)$. For each $n = 1, 2, \ldots$, let M_n be the minimal subset of K_n^*. Then $M_n \in \mathcal{F}(X)$ for all n. Since $\big(\mathcal{F}(X), \precsim\big)$ is pwo, there exist i, j with $i < j$ and $M_i \precsim M_j$. Thus for all $x \in M_i$, there exists $y \in M_j$ with $x \leqslant y$. This implies that $M_i^+ \subseteq M_j^+$. But by definition of normal ideal, we have $K_n = M_n^+$ for all n. So $K_i \subseteq K_j$, completing the proof.

THEOREM 11. *Let X be any pwo-set and $\mathfrak{J}$ its intrinsic topology. Let $\mathfrak{U}$ be the intrinsic topology of $N(X)$. Then the mapping $x \to J_x$ is a homeomorphism of $(X, \mathfrak{J})$ onto the subspace $J(X)$ of $(N(X), \mathfrak{U})$.*

Proof. Let $\mathcal{S}$ be the relativization of $\mathfrak{U}$ to $J(X)$, and let $\mathcal{T}$ be the intrinsic topology of $J(X)$. Since $(X, \mathfrak{J})$ and $\big(J(X), \mathcal{T}\big)$ are homeomorphic, it suffices to show that $\mathcal{S} = \mathcal{T}$. This will follow from Theorem 9 if we show that $\mathcal{S} \subseteq \mathcal{T}$. So let B be an $\mathcal{S}$-closed subset of $J(X)$. Then we have $B = \bar{B} \cap J(X)$, where $\bar{B}$ denotes the $\mathfrak{U}$-closure of B in $N(X)$. Suppose that D is an up-directed subset of B with a l.u.b. J_y in $J(X)$. But then J_y is the smallest normal ideal which contains each membre $J_x \in D$, and so *J_y is also the* l.u.b. *of D considered as a subse of $N(X)$*. Hence $J_y \in \bar{B} \cap J(X) = B$, and so B is $\mathcal{T}$-closed. Thus $\mathcal{S} \subseteq \mathcal{T}$.

THEOREM 12. *The intrinsic topology of any pwo-set X is completely regular.*

Proof. $N(X)$ is a complete lattice and hence compact in its intrinsic topology. Its subspace $J(X)$ is therefore completely regular, and the result follows by Theorem 11.

If $\bar{J}(X)$ denotes the closure of $J(X)$ in $N(X)$, then the subspace $\bar{J}(X)$ is a compactification of X. It is not necessarily true that $\bar{J}(X) = N(X)$, but we do have the following characterization of $\bar{J}(X)$.

THEOREM 13. *If X is any pwo-set, then $\bar{J}(X)$ is the set of all up-directed normal ideals of X. Furthermore, $\bar{J}(X)$ is Dedekind-complete.*

Proof. By definition of the closure operation in $N(X)$, $\bar{J}(X)$ consists of all those normal ideals of X which are unions of up-directed (with respect to set inclusion) families of principal ideals. So let $K \in \bar{J}(X)$, where K is the union of some up-directed family $\mathcal{F}$ of principal ideals of X. Let $x, y \in K$. Then there exist $J_a, J_b \in \mathcal{F}$ such that $x \in J_a$, $y \in J_b$. Also there exists $J_c \in \mathcal{F}$ with $J_a \subseteq J_c$, $J_b \subseteq J_c$. So $c \in K$, $x \leqslant a \leqslant c$, and $y \leqslant b \leqslant c$. Thus K is an up-directed subset of X. Conversely, any up-directed normal ideal trivially is a union of an up-directed family of principal ideals, and hence is a member of $\bar{J}(X)$. The second assertion of the theorem follows from Theorem 1.

We show next that the intrinsic topology of a pwo-set may be obtained as the topology of a certain natural proximity relation (for definitions and references on proximity spaces see [7]). If X is a pwo-set and A, B are subsets of X, let us define $A\delta B$ if and only if there exist up-directed sets D, E with $D \subseteq A$, $E \subseteq B$, and $D^{*+} = E^{*+}$.

THEOREM 14. *If X is any pwo-set, then the relation δ on 2^X is a proximity relation whose topology is the intrinsic topology of X.*

Proof. Since $\bar{J}(X)$ is a compact Hausdorff space, its topology is obtained from a unique proximity relation δ' which is defined as follows. For any subsets $F_1, F_2 \subseteq \bar{J}(X)$, define $F_1\delta' F_2$ if and only if the closures of F_1 and F_2 have a non-empty intersection in $\bar{J}(X)$. More explicitly, $F_1\delta' F_2$ if and only if there exist up-directed sets $E_1 \subseteq F_1$, $E_2 \subseteq F_2$ with l.u.b. $E_1 =$ l.u.b. E_2 in $\bar{J}(X)$. Now let us identify X with its image $J(X)$ in $\bar{J}(X)$. Then, for $A, B \subseteq J(X)$, it is clear that $A\delta B$ if and only if $A\delta' B$. Thus $(J(X), \delta)$ is a proximity space which is a subspace of $(\bar{J}(X), \delta')$. The proximity topology of $(J(X), \delta)$ is the relative topology which it inherits from $(\bar{J}(X), \delta')$, and this is identical with its intrinsic topology by Theorem 11.

The reader will note that the above proof also shows that the proximity space $(\bar{J}(X), \delta')$ is the *Smirnov compactification* [7] of the proximity space (X, δ).

4. Continuous functions. In this section X and X' denote pwo-sets with their associated intrinsic topologies.

LEMMA. *Let f be a function on X into X'. Then f is continuous on X if and only if whenever D is an up-directed subset of X with l.u.b. $D = y \in X$, the set $f[D]$ contains an up-directed subset E whose l.u.b. exists in X' and $= f(y)$.*

Proof. Suppose that f is continuous, D is up-directed in X, and l.u.b. $D = y$. Define a net S by $S_n = n$ for all $n \in D$. By continuity of f, the net $\{f(S_n), n \in D\}$ converges to $f(y)$. By Theorem 6, the range of the net $\{f(S_n), n \in D\}$ contains an up-directed subset E with l.u.b. $E = f(y)$.

To prove the converse, assume that the given condition holds for the function f. Let B be a closed subset of X'. We show that $A = f^{-1}[B]$ is closed in X. So let D be an up-directed subset of A with l.u.b. $D = y$. By hypothesis, $f[D]$ contains an up-directed subset E with l.u.b. $E = f(y)$. Since B is closed, we have $f(y) \in B$, and hence $y \in A$.

The following theorem generalizes a result of Sierpiński [11].

THEOREM 15. *If f is any function on X into X', then f is the pointwise limit of a net of continuous functions on X into X'.*

Proof. Let Δ be the set of all finite subsets of X, and let $G \in \Delta$. If $x \in X$ and $a \in G$, we shall say that a is a *G-cover* of x if and only if

(i) $x = a$

or

(ii) $x \notin G$, $x < a$, and there exists no $b \in G$ with $x < b < a$.

We consider Δ as up-directed by set inclusion. We define a net of functions on Δ as follows. Select an arbitrary element of X' and denote it by z. For each $G \in \Delta$, define f_G on X into X' by

(i) $f_G(x) = z$ if x has no G-cover,

(ii) $f_G(x) = z$ if x has more than one G-cover,

(iii) if x has precisely one G-cover a, then $f_G(x) = f(a)$.

We first note that for each $x_0 \in X$, we have $\lim\{f_G(x_0), G \in \Delta\} = f(x_0)$. For let G_0 be any member of Δ with $x_0 \in G_0$. Then for all $G \supseteq G_0$, we have $f_G(x_0) = f(x_0)$ by (iii) above.

Now we must show that each f_G is continuous on X. We shall use the preceding lemma. Let $G \in \Delta$, and D any up-directed subset of X with l.u.b. $D = y \in X$. By a *residual* subset of D we shall mean any subset of the form $\{x \in D: x \geqslant n\}$ for some fixed $n \in D$.

Case 1. Suppose that y has no G-cover. Let $H = \{x \in D: x$ has at least one G-cover$\}$. We claim that H is not cofinal in D. For suppose it is. Then, since G is finite, for some $a \in G$ the set $H_a = \{x \in H: x \leqslant a\}$ is cofinal in H, and hence in D. But then $y =$ l.u.b. H_a, and hence $y \leqslant a$. Since y has no G-cover, this is a contradiction. Hence $D - H$ contains a residual subset E of D; and, by definition of f_G, for all $x \in E$ we have $f_G(x) = f_G(y) = z$.

Case 2. Suppose that y has more than one G-cover. Then each $x \in D$ has at least one G-cover. Let $K = \{x \in D: x$ has precisely one G-cover$\}$. We assert that K is not cofinal in D. For if K is cofinal, then for some $a \in G$ the set $K_a = \{x \in K: a$ is a G-cover of $x\}$ is cofinal in K and hence in D. But then $y = \text{l.u.b.} K_a$, so that a is a G-cover of y. If b is another G-cover of y, then b is incomparable with a. But $b \geqslant x$ for all $x \in K_a$, so that b is another G-cover for each $x \in K_a$: contradiction. Hence $D-K$ contains a residual subset E of D; and, as in Case 1, for all $x \in E$ we have $f_G(x) = f_G(y) = z$.

Case 3. Suppose that y has precisely one G-cover a. If $R = \{x \in D$: a is the only G-cover of $x\}$, then a simple argument (similar to the cases above) shows that R contains a residual subset E of D. Also, for all $x \in E$ we have $f_G(x) = f_G(y) = f(a)$, by definition of f_G.

Thus in each of the above three cases we have a residual (and hence cofinal) subset E of D such that $f_G(y) = \text{l.u.b.} f_G[E]$. The continuity of f_G now follows by the lemma.

Sierpiński has shown in [11] that Theorem 15 does not remain valid, even when X and X' are *well ordered*, if "net" is replaced by "transfinite sequence".

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UNIVERSITY OF CONNECTICUT
Storrs, Connecticut

Reçu par la Rédaction le 25. 3. 1967

On the hyperspace of subcontinua of a finite graph, I

by

R. Duda (Wrocław)

§ 1. Introduction. Let X be a compact metric continuum with a metric ϱ. Throughout the paper $C(X)$ will denote the hyperspace of all non-empty subcontinua of X metrized by the Hausdorff metric ϱ^1 (shortly, the hyperspace for X):

$$\varrho^1(A, B) = \max[\sup_{a \in A} \varrho(a, B), \sup_{b \in B} \varrho(A, b)].$$

It has been known for a long time that $C(X)$ with the metric ϱ^1 is also a compact metric continuum, and some other properties of $C(X)$ have also been proved (cf. for instance Wojdysławski [10], Kelley [4], Duda [1], and Segal [6]). However, no characterization of spaces $C(X)$ has as yet appeared. Even what $C(X)$ is like is so far known for only a few and very simple continua X (after all, mainly in folk-lore).

The aim of the present paper is to inquire into the structure of spaces $C(X)$ in the case which seems to be natural to start with, that is in the case of spaces $C(X)$ which are locally connected and have finite dimension. The results obtained here uncover some features of their polyhedral structure and may eventually lead to their topological characterization (cf. remark following corollary 9.2).

As Vietoris [8] and Ważewski [9] have proved, continuum $C(X)$ is locally connected if and only if continuum X is locally connected, and it is fairly easy to show (cf. Kelley [4]) that the dimension of a locally connected continuum $C(X)$ is finite if and only if continuum X is a finite connected graph. Hence

1.1. *Continuum $C(X)$ is locally connected and of finite dimension if and only if continuum X is a finite connected graph.*

To gain our aims we shall proceed as follows. We start with a finite connected graph X dividing its hyperspace $C(X)$ into finitely many closed subsets $\mathfrak{M}_\mu$ which turn to be topological balls. Moreover, the decomposition of $C(X)$ into these balls (cells) is a good one (for X acyclic, cellular), and so in this way we come first to theorem 6.4 stating that $C(X)$ is a polyhedron if and only if X is a finite graph. This polyhedron is then subjected to an analysis resulting in formulas for its dimension (theo-