un LF strict. Le problème de la paracompacité des espaces LF non stricts reste ouvert.

Signalons d'autre part qu'on aurait pu introduire une notion de paracompacité plus forte que le caractère paracompact et qui aurait été, de façon évidente, stable par quotients pour des relations d'équivalence uniformément ouvertes, à savoir la suivante: pour tout recouvrement ouvert $\mathfrak{R}$, il existe un écart uniformément continu et un recouvrement ouvert pour cet écart qui soit plus fin que $\mathfrak{R}$. Malheureusement, cette notion n'est pas stable par limites inductives strictes de suites; en effet, si M est un espace métrique non séparable (par exemple un ensemble discret non dénombrable), la limite inductive de la suite $(M\times \mathbf{R}^n)$ ne vérifie pas la condition indiquée.

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Light fiber maps *

by

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1. Introduction. Most of the literature which deals with fiber maps assumes that the fibers have some nice connectivity properties. These assumptions force the most interesting light fiber maps to be overlooked. In this paper, light fiber maps are studied with two purposes in mind. The first purpose is to obtain methods for handling more general mappings. The second purpose is that the study of light fiber maps might yield a method of attacking the following unsolved problems involving light open mappings:

I. *Does there exist a light open mapping of a manifold onto a metric space such that the inverse of some point is uncountable?*

II. *Is there a light open mapping f which is not a homeomorphism of the n cube I^n onto itself such that f is the identity on the boundary of I^n(brdy I^n) and $f^{-1}f(\mathrm{brdy}\, I^n) = \mathrm{brdy}\, I^n$?*

If the answer to I is no, then a p-adic group cannot act freely on an n-manifold.

It is shown in Section 3 that there are no Serre fibrations with the properties of I or II. Hence, a method of attacking these problems might be the following: Assume that such a mapping exists and prove that it must be a fibration.

The main theorem of this paper is: *If f is a light compact mapping from a space X onto a space Y such that paths could be lifted uniquely, given the initial endpoint of the lifting then f has the absolute covering homotopy property.* This gives a partial answer to the following conjecture raised by the author: *If f is a light mappings from a space X onto a space Y such that paths could be lifted given the initial endpoint of the lifting and such that arcs could be lifted uniquely given the initial endpoint of the lifting then f is a Serre fibration.*

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In Section 3 the relation between local section and fiber maps is studied. In Section 4 sufficient conditions are given so that fiber maps will be bundle maps. The main theorem is proved in Section 5. Related papers are [1], [4], [9].

All spaces will be Hausdorff and the notation $f\colon X \twoheadrightarrow Y$ will mean that f is a *continuous function* (*map*) of X *onto* Y.

2. Definitions. The following definitions will be needed. In all of them p will be a continuous mapping of a topological space E onto a topological space B.

(2.1) DEFINITION. The mapping p is *a-light* iff $p^{-1}(b)$ contains no arc for every b in B.

(2.2) DEFINITION. The mapping p has *weak local cross sections at every point* iff given any b in B and any y in $p^{-1}(b)$ there exists a neighborhood $U(y)$ of b and a mapping $\varphi_{U(y)}$ of $U(y)$ into E such that $\varphi_{U(y)}(b) = y$ and $p\varphi_{U(y)}$ is the identity on $U(y)$.

(2.3) DEFINITION. The mapping p has *strong local cross sections at every point* iff, given any b in B, there exists a neighborhood U of b such that if y is in $p^{-1}(U)$ there exists a map $\varphi_y\colon U \to E$ such that $p\varphi_y$ is the identity on U and $\varphi_y(p(y)) = y$.

(2.4) DEFINITION. If Σ is a class of topological spaces, then p is said to have the Σ *covering homotopy property* (ΣCHP) if given any map f of a space $X \in \Sigma$ into E and a homotopy $H\colon X \times I \to B$ such that $H(x, 0) = pf(x)$, then there exists a homotopy $K\colon X \times I \to E$ such that $K(x, 0) = f(x)$ and $pK(x, t) = H(x, t)$.

The following notation will be used. If p has the ΣCHP and

a) Σ is the class of all topological spaces, then p has the ACHP (absolute covering homotopy property, see Hu [5].)

b) Σ is the class consisting of the unit interval, then p has the ICHP.

c) Σ is the class of compact locally arcwise connected spaces, then p has the CLACHP.

d) Σ is the class of finite polyhedra, then p has the PCHP. (If p has the PCHP, then (E, p, B) is called a *fiber space in the sense of Serre*.)

e) Σ is the class consisting of the sets $p^{-1}(b)$ for every b in B, then p has the FCHP.

f) Σ is the class consisting of a one point space, then p has the 0CHP (i.e., paths can be lifted).

g) Σ is the class consisting of connected spaces, then p has the CCHP.

(2.5) DEFINITION. If p has the ΣCHP and the homotopy K in (2.4) is unique, then p has the u-ΣCHP.

(2.6) DEFINITION. p has the *path lifting property* (PLP) [5] if $r\colon E^I \to Z = \{(y, f) \in E \times B^I \mid p(y) = f(0)\}$ defined by $r(g) = (g(0), pg)$ admits a cross section. If p has the PLP, (E, P, B) is called a *Hurewicz fiber space*.

(2.7) DEFINITION. p has the *boundle property* (BP) [5] if there exists an open cover $\{U_a\}$ of B, a collection $\{\varphi_a\}$ of homeomorphisms and a space F such that F is homeomorphic to $p^{-1}(b)$ for all b in B and $\varphi_a\colon U_a \times F \twoheadrightarrow p^{-1}(U_a)$ satisfies $p\varphi_a(u, f) = u$.

(2.8) DEFINITION. p has the *slicing structure* property (SSP) [5] if there exists an open cover $\{U_a\}$ of B and a collection of maps $\{\varphi_a\}$ such that $\varphi_a\colon U_a \times p^{-1}(U_a) \to p^{-1}(U_a)$, $p\varphi_a(u, y) = u$ and $\varphi_a(p(y), y) = y$.

3. a-light fiber maps. As in Section 2, p will be a continuous map from a topological space E onto a topological space B. Conditions will be placed on E, p or B as needed.

(3.1) THEOREM. *The following conditions are equivalent*:

(A) *If σ is a loop in B which is homotopic to a constant and $y \in p^{-1}\sigma(0)$, then there is a lifting τ of σ such that $\tau(0) = y$ and any lifting of σ is a loop.*

(B) *If σ is a loop in B which is homotopic to a constant, and $y \in p^{-1}\sigma(0)$, then there exists a unique loop τ which covers σ such that $\tau(0) = y$.*

(C) *If a is a path in B and $y \in p^{-1}a(0)$, then there exists a path β in E which covers a such that $\beta(0) = y$. Also if a and γ are homotopic paths in B and β and δ are liftings of a and γ, respectively, such that $\beta(0) = \delta(0)$, then $\beta(1) = \delta(1)$.*

(D) *Same as* (C) *except for the fact that β is assumed to be unique.*

(E) *p is a-light and has the* ICHP.

(F) *p is a-light and has the* PCHP.

It should be noted that the first condition of (C) is equivalent to the 0CHP and the first condition of (D) is equivalent to the u-0CHP. This gives rise to the following conjecture which will be partially answered in Sections 4 and 5.

(3.2) CONJECTURE. *The mapping p has the* u-0CHP *iff p is a-light and has the* PCHP.

Proof of (3.1). The following chain of implications will be proved: (B) $\Rightarrow$ (A) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (F) $\Rightarrow$ (E) $\Rightarrow$ (D) $\Rightarrow$ (B).

1) (F) $\Rightarrow$ (E) is trivial.

2) (B) $\Rightarrow$ (A). To prove this it will be shown that every loop which is homotopic to a constant has exactly one lifting given the initial point of the lifting. By the hypothesis this lifting must be a loop. Let σ be a loop in B and let $y \in p^{-1}\sigma(0)$ and let τ be the unique loop which covers σ such

that $\tau(0) = y$. Assume that τ' is another lifting of σ such that $\tau'(0) = y$. Let $0 \leqslant s \leqslant 1$ and define σ_s, τ_s and τ'_s as follows:

$$\sigma_s(t) = \begin{cases} \sigma(2st), & 0 \leqslant t \leqslant 1/2, \\ \sigma(-2st+2s), & 1/2 \leqslant t \leqslant 1; \end{cases}$$

$$\tau_s(t) = \begin{cases} \tau(2st), & 0 \leqslant t \leqslant 1/2, \\ \tau(-2st+2s), & 1/2 \leqslant t \leqslant 1; \end{cases}$$

$$\tau'_s(t) = \begin{cases} \tau'(2st), & 0 \leqslant t \leqslant 1/2, \\ \tau'(-2st+2s), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then τ_s and τ'_s are loops which cover σ_s. $\tau_s(0) = \tau'_s(0) = y$ and σ_s is homotopic to a constant map. Therefore, $\tau_s = \tau'_s$ by hypothesis. Hence $\tau(s) = \tau_s(1/2) = \tau'(s)$ and since s was arbitrary, $\tau = \tau'$ as desired.

3) (E) $\Rightarrow$ (D) is essentially Lemmas 15.1 and 15.2, Chapter 3 of [5]. Although stated quite differently the proofs still hold.

4) (A) $\Rightarrow$ (C). Let α be a path in B and let $y \in p^{-1}\alpha(0)$. Let α^* be a path in B defined by $\alpha^*(t) = \alpha(1-t)$. Then $\alpha^*\alpha$ is a loop in B at $\alpha(0)$ and $\alpha^*\alpha$ is homotopic to the constant map at $\alpha(0)$. Therefore by condition (A) there exists a lifting τ of $\alpha^*\alpha$ such that $\tau(0) = y$. Define β: $I \to E$ by $\beta(t) = \tau(1/2t)$. It is easily seen that β is a lifting of α such that $\beta(0) = y$. To prove the second part of (C) let α and γ be homotopic paths in B and β and δ liftings of α and γ respectively such that $\beta(0) = \delta(0)$. Then $\delta\beta^*$ is a covering of $\gamma\alpha^*$ and $\gamma\alpha^*$ is homotopic to a constant. Therefore $\delta\beta^*$ is a loop so that $\beta(1) = \delta\beta^*(0) = \delta\beta^*(1) = \delta(1)$.

5) (C) $\Rightarrow$ (D). Let α be a path in B and let β and β' be paths in E which cover α such that $\beta(0) = \beta'(0)$. Let $0 \leqslant s \leqslant 1$ and define $\alpha_s(t) = \alpha(st)$, $\beta_s(t) = \beta(st)$ and $\beta'_s(t) = \beta'(st)$. β_s and β'_s are paths in E which cover α_s and $\beta_s(0) = \beta'_s(0)$. Since α_s is homotopic to itself, condition (C) implies that $\beta_s(1) = \beta'_s(1)$ and hence $\beta(s) = \beta_s(1) = \beta'_s(1) = \beta'(s)$. Since s was arbitrary, $\beta = \beta'$ as desired.

6) (D) $\Rightarrow$ (B). Let σ be a loop in B which is homotopic to a constant and let $y \in p^{-1}\sigma(0)$. Define η and ξ: $I \to B$ as follows:

$$\eta(t) = \sigma(1/2t), \qquad \xi(t) = \sigma(-1/2t+1).$$

η and ξ are homotopic paths in B and hence they have unique liftings θ and φ respectively such that $\theta(0) = \varphi(0) = y$ and $\theta(1) = \varphi(1)$. Then it is easily seen that $\varphi^*\theta$ is the unique loop which covers σ and has base point y.

7) (D) $\Rightarrow$ (F). By Theorem 3.1, Chapter 3 of [5] proving condition F is equivalent to proving that p has the CHP for Δ^n (the standard n-simplex) $n = 0, 1, \ldots$ and that p is a-light.

The mapping p is a-light since if not we could get two different liftings with the same initial point of some constant path. p has the CHP for Δ^0 by hypothesis. Hence let $n > 0$, H: $\Delta^n \times I \to B$ and g: $\Delta^n \to E$ be

such that $H(x, 0) = pg(x)$. For any x in Δ^n define H_x: $I \to B$ by $H_x(t) = H(x, t)$. H_x is a path in B and hence there exists a unique lifting K_x: $I \to E$ such that $K_x(0) = g(x)$ and $pK_x = H_x$. Finally define K: $\Delta^n \times I \to E$ by $K(x, t) = K_x(t)$.

The function K is well defined, $K(x, 0) = K_x(0) = g(x)$. $pK(x, t) = H_x(t) = H(x, t)$. Therefore to complete the proof all that must be shown is that K is continuous. Toward this end let $\{(x_i, t_i)\}_{i=1}^{\infty}$ be a sequence in $\Delta^n \times I$ which converges to (x, t) in $\Delta^n \times I$. Without loss of generality it may be assumed that $t_1 = 0$. Let σ_i be a path from (x_i, t_i) to (x_{i+1}, t_{i+1}). Again it could be assumed without loss of generality that $\lim\limits_{i\to\infty} \sigma_i(I) = (x, t)$. Define σ, α_i and β_i: $I \to B$ as follows:

$$\sigma(s) = \begin{cases} H\sigma_i[(i^2+i)s + (1-i^2)], & \dfrac{i-1}{i} \leqslant s \leqslant \dfrac{i}{i+1}, \quad i = 1, 2, \ldots, \\ H(x, t), & s = 1; \end{cases}$$

$$\alpha_i(s) = \begin{cases} H\big[\big((1-2s)x_1 + 2sx_i, 0\big)\big], & 0 \leqslant s \leqslant 1/2, \\ H[(x_i, 2t_is - t_i)], & 1/2 \leqslant s \leqslant 1; \end{cases}$$

$$\beta_i(s) = \sigma\left(\frac{i-1}{i}s\right), \quad i = 1, 2, \ldots$$

Note that α_i and β_i are homotopic (rel. end-points).

σ can be lifted to a unique path τ such that $\tau(0) = g(x_1)$. If τ_i: $I \to E$ is defined by $\tau_i(s) = \tau\big((i-1/i)s\big)$, then $\tau_i(0) = \tau(0) = g(x_1)$ and hence τ_i is the unique lifting of β_i such that $\tau_i(0) = g(x_1)$. α_i can also be lifted to a unique path γ_i such that $\gamma_i(0) = g(x_1)$. By the construction it is easily seen that $\gamma_i(1) = K(x_i, t_i)$ and by the hypothesis $\tau_i(1) = \gamma_i(1)$. Hence we have that $\tau\big((i-1)/i\big) = K(x_i, t_i)$. In like manner $\tau(1) = K(x, t)$ and since τ is continuous, $\{\tau\big((i-1)/i\big)\}_{\infty}^{i=1}$ converges to $K(x, t)$ so that K is continuous and this completes the proof.

If E and B are locally connected metric compacta and p is a light open map, it is known that p has the 0CHP ([3], Theorem 3). Combining this with Theorem (3.1) we get the following corollary.

(3.3) Corollary. *The following conditions are equivalent if p is a light open map and E and B are locally connected metric compacta:*

(A) *If τ is a lifting of a loop σ where σ is homotopic to a constant map, then τ is a loop.*

(B) *If β and δ are liftings of homotopic paths α and γ respectively such that $\beta(0) = \delta(0)$, then $\beta(1) = \delta(1)$.*

(C) *p has the* ICHP.

(D) *p has the* PCHP.

The following is a lemma which will have as an easy consequence another corollary similar to (3.3).

(3.4) LEMMA. *If p has strong local cross sections at every point, then p has the* 0CHP.

Proof. Let f be a path in B and let $y \in p^{-1}f(0)$. Since $f(I)$ is compact, there exists a finite collection of neighborhoods $\{U_1, \ldots, U_n\}$ which cover $f(I)$ such that $f[i/n, (i+1)/n] \subset U_{i+1}$, $i = 0, \ldots, n-1$, and there exists a collection of maps φ_i: $U_i \to T$ such that $p\varphi_i$ is the identity on U_i and $\varphi_i(f(0)) = y$ and $\varphi_{i+1}f(i/n) = \varphi_i f(i/n)$. Now define g: $I \to E$ as follows: $g(t) = \varphi_i f(i/n)$ if $(i-1)/n \leqslant t \leqslant i/n$. It is easily seen that g is the desired function.

(3.5) COROLLARY. *If p is an a-light mapping with strong local cross sections at every point, then* (A), (B), (C), *and* (D) *of* (3.3) *are equivalent.*

(3.6) EXAMPLE. A map p which has weak local cross sections at every point but does not have the 0CHP. Let E be the following subset of the plane: $E = \{(x, y) | y = 0$ and $0 \leqslant x \leqslant 1$ or $y = 1$ and $0 \leqslant x < 1/2\}$. Let B be the unit interval and define p: $E \to B$ by $p(x, y) = x$.

The following lemma can be found in [1].

(3.7) LEMMA. *If p has the* u-0CHP *and f and g are maps of an arcwise connected space X into E such that $pf = pg$ and $f(x_0) = g(x_0)$ for some x_0 in X, then $f = g$.*

The proof of the next theorem closely follows that of Theorem 3 [1].

(3.8) THEOREM. *If p has weak local cross sections at every point and p has the* u-0CHP, *then p has the* CLACHP.

Proof. Let X be compact locally arcwise connected space and let f: $X \to E$ and g: $X \times I \to B$ be such that $g(x, 0) = pf(x)$. Define f': $X \times I \to E$ by $f'(x, s) = \sigma_x(s)$ where σ_x is the unique lifting of $g|x \times I$ such that $\sigma_x(0) = f(x)$. f' is a well-defined function from $X \times I$ into E, $pf'(x, t) = p\sigma_x(t) = g(x, t)$, and $f'(x, 0) = \sigma_x(0) = f(x)$. Therefore to complete the proof it need only be shown that f' is continuous.

Let K be the set of all t in I such that f' is continuous at (x, t_1) for every x in X and $t_1 < t$. Let k be the supremum of K if K is non-empty, or 0 if K is empty. Then f' is continuous at (x, t) for every x in X and $t < k$. To complete the proof it is sufficient to show that f' is continuous in some neighborhood of (x_0, k) for every x_0 in X. In that case, since X is compact, $X \times k$ could be covered by a finite number of such neighborhoods each of which may be taken as the product of open sets of X and subintervals of I. Letting k_1 be the least of the upper bounds of the subintervals of I involved in the definition of these neighborhoods f' will be continuous at (x, t) for all x in X and $t < k_1$. If $k \neq 1$, this would be a contradiction since $k < k_1$. If $k = 1$, it would follow that f' is continuous on $X \times I$.

Let $f(x_0, k) = y_0$; since p has weak local cross sections at every point, there exists a neighborhood U of $p(y_0)$ and a homeomorphism φ_U: $U \to T$ such that $\varphi_U(p(y_0)) = y_0$ and $p\varphi_U$ is the identity on U. $p(y_0) = g(x_0, k)$ and hence by the continuity of g there exists an open neighborhood N of (x_0, k) such that $g(N) \subset U$. It may be assumed that $N = V \times I_0$ where V is an open arcwise connected neighborhood of x_0 and I_0 is an open interval in I. Note that $p\varphi_U(g|N) = g|N$.

It will now be shown that $f'|N = \varphi_U g|N$. Once this is done we shall have that f is continuous on N and hence, by the previous remarks, the proof will be complete.

If $k = 0$, then we are assuming that $N = V \times [0, t)$ and we have that $f'|V \times 0 = \varphi_U g|V \times 0$ by Lemma (3.7) since $V \times 0$ is arcwise connected and $pf|V \times 0 = p\varphi_U g|V \times 0 = g|V \times 0$ and $f'(x_0, 0) = f(x_0) = \varphi_U g(x_0, 0)$. If $(v, r) \in N$ and $r > 0$, then $f'(v, r) = \varphi_U g(v, r)$ or else α and β: $I \to E$ defined by $\alpha(s) = \sigma_v(sr) = f'(v, sr)$ and $\beta(s) = \varphi_U g(v, sr)$ will be distinct liftings with the same initial point (namely $f'(v, 0) = \varphi_U(g(v, 0))$ which cover γ: $I \to B$ defined by $\gamma(s) = g(v, sr)$. Therefore if $k = 0$, then $f'|N = \varphi_U g|N$ as desired.

If $k > 0$, let $(x_0, t) \in N$, $t < k$ and define $\alpha(s) = g(x_0, (t-k)s + k)$. Define β: $I \to T$ by $\beta(s) = \sigma_{x_0}[(t-k)s + k]$. Then β and $\varphi_U \alpha$ are liftings of α such that $\beta(0) = \varphi_U \alpha(0) = y_0$. Therefore, by hypothesis, $f'|[V \times [0, k)] \cap N = \varphi_U g|[V \times [0, k)] \cap N$ since these are liftings of $g|[V \times [0, k)] \cap N$ is arcwise connected. If $(v, r) \in N$ and $r \geqslant k$, choose a point $(v, s) \in N$ with $s < k$ and continue in a manner similar to that of the last part of the last paragraph to get $f'(v, r) = \varphi_U g(v, r)$. Therefore $f'|N = \varphi_U g|N$ and the proof is complete.

(3.9) Remark. Theorem (3.8) could be reworded as follows: *If p is a-light and has weak local cross sections at every point, then p has the* CLACHP *iff p has the* u-0CHP.

The following example will show that the conclusion of Theorem (3.8) cannot be strengthened without changing the hypothesis.

(3.10) EXAMPLE. Let E be the following subset of the plane:

$$\begin{aligned} E = \{(x, y) | y = x + 1/n\,,\ n = 1, 2, \ldots,\ 0 \leqslant x \leqslant 1\} \\ \cup \{(x, y) | y = 0 \text{ and } 0 \leqslant s \leqslant 1\} \\ \cup \{(x, y) | y = -1/2^n\,,\ n = 1, 2, \ldots \text{ and } 1/2^n \leqslant x \leqslant 1\} \\ \cup \{(x, y) | y = 2^{n-1}x - (2^{n-1}+1)/2^n,\ n = 1, 2, \ldots, 0 \leqslant x \leqslant 1/2^n\}\,. \end{aligned}$$

Let B be the unit interval and define p: $E \to B$ by $p(x, y) = x$. (See Figure 1.) p has strong local cross sections at every point. The mapping p has the ICHP and hence by (3.1) and (3.8) p has the CLACHP. The mapping p does not have the ACHP or SSP since it is easily seen that p does not have the FCHP. It should also be noted that all fibres are homeomorphic.

In most interesting cases, if p is an a-light map with the PCHP, then p admits strong local cross sections at every point.

(3.11) THEOREM. *If p is a-light and has the* PCHP *and B is first countable, locally arcwise connected and semi-locally simply connected, then p has strong local cross sections at every point.*

Proof. Let $b \in B$ and let U be a neighborhood such that any loop in U is homotopic in B to the constant map at b. It may be assumed that U is arcwise connected and locally arcwise connected since B is locally arcwise connected. Let $y \in p^{-1}(b)$ and define $\varphi_U \colon U \to T$ by $\varphi_U(x) = \tau_x(1)$ where τ_x is a lifting of a path in U from b to x such that $\tau_x(0) = y$. φ_U is well defined by (3.1) and $p\varphi_U(x) = x$. Hence to complete the proof it only has to be shown that φ_U is continuous.

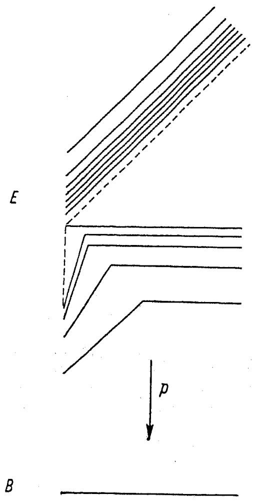

Fig. 1

To prove that φ_U is continuous let $\{x_i\}$ be a sequence in U which converges to x in U (it will be assumed that $x_1 = b$). Let σ_i be a path from x_i to x_{i+1}. Since U is first countable and locally arcwise connected, the σ_i could be chosen such that $\lim\limits_{i\to\infty} \sigma_i(I) = x$. Define $\beta\colon I \to B$ as follows:

$$\beta(t) = \begin{cases} \sigma_i(2^i t - 2^i + 2) & \text{if} \quad \dfrac{2^{i-1}-1}{2^{i-1}} \leqslant t \leqslant \dfrac{2^i - 1}{2^i}, \qquad i = 1, \ldots, \\ x & \text{if} \quad t = 1\,. \end{cases}$$

β is a path in U from b to x and hence by Theorem (3.1) β can be lifted uniquely to a path γ in T such that $\gamma(0) = y$. By the methods of Theorem (3.1)

$$\gamma(1) = \varphi_U(x) \quad \text{and} \quad \gamma\left(\frac{2^i-1}{2^i}\right) = \varphi_U(x_i)$$

and since γ is continuous, $\gamma\left(\dfrac{2^i-1}{2^i}\right)$ converges to $\varphi_U(x)$. Therefore φ_U is continuous and the proof is complete.

(3.12) COROLLARY. *If p is as in* (3.11) *and B is assumed to be first countable, locally arcwise connected, arcwise connected and simply connected, then p has global cross sections at every point.*

Proof. By the hypothesis the neighborhood U in (3.11) can be chosen to be all of B and the proof then follows in the same manner.

4. The FCHP and BP. In this section, some conditions for a light mapping to have the BP will be given. Browder [1] has shown the following: Let X and Y be Hausdorff spaces with Y connected, locally pathwise connected and semi-locally simply connected. Suppose that f is a local homeomorphism of X into Y and that f is a closed mapping of X into Y. Then X is a covering space of Y with f as a covering mapping. Browder [1] has several other theorems along this line. Lelek and Mycielski [8] also have some theorems along the same line. Although the theorems of Lelek and Mycielski seem to follow from those of Browder, they are concerned with a different class of topological spaces. All of the theorems referred to above give conditions so that a local homeomorphism is a bundle map. However, there are light bundle maps which are not local homeomorphisms. An example is the natural projection of the Cantor set crossed with the unit interval onto the unit interval. The theorems given in this section will cover examples such as the above. The following lemma, essentially from [9], will be needed.

(4.1) LEMMA. *If p is a-light and has the* PCHP *and* FCHP *and B is arcwise connected, then all of the fibers are homeomorphic.*

(4.2) THEOREM. *If p is an a-light map with strong local cross sections at every point and p has the* PCHP *and* FCHP *and if B is an arcwise connected, locally compact, uniformly locally arcwise connected metric space and if $p^{-1}(b)$ is compact for all b in B, then p has the* BP.

Proof. By Lemma (4.1), $p^{-1}(b)$ is homeomorphic to $p^{-1}(b_0)$ for all b_0, b, in B. Let $b_0 \in B$; then there exists a neighborhood U of b_0 such that U is compact, connected and locally connected and there exists a collection of maps $\{\varphi_y\}$ such that if $y \in p^{-1}(b_0)$ $\varphi_y\colon U \to E$, $p\varphi_y$ is the identity on U and $\varphi_y(b_0) = y$. From the above conditions on U we know that U is a Peano continuum and hence there exists a map $f\colon I \twoheadrightarrow U$, and it may be assumed that $f(0) = b_0$. Let $g\colon p^{-1}(b_0) \twoheadrightarrow p^{-1}(b_0)$ be the identity map and define $H\colon p^{-1}(b_0)\times I \to U$ by $H(y, t) = f(t)$. $H(y, 0) = f(0) = b_0 = pg(y)$. Hence by the FCHP there exists a map $K\colon p^{-1}(b_0)\times I \to p^{-1}(U)$ such that $pK(y, t) = H(y, t)$ and $K(y, 0) = g(y) = y$. By (3.1) and (3.7), $K(y, t) = \varphi_y f(t)$ and, by the construction in (4.1), it is seen that K maps $p^{-1}(b_0)\times I$ onto $p^{-1}(U)$.

Consider the following commutative diagram:

$$\begin{tikzcd} p^{-1}(b_0)\times I \arrow[rr, two heads, "h"] \arrow[dr, "K"'] & & p^{-1}(b_0)\times U \arrow[dl, "L"] \\ & p^{-1}(U) & \end{tikzcd}$$

where $h(y,t) = (y, f(t))$ and $L(y,u) = \varphi_y(u)$. h and K are continuous and closed and it will be shown that L is a homeomorphism and $pL(y,u) = u$.

$pL(y,u) = p\varphi_y(u) = u$ so that the second condition is true. The mapping L is continuous since, if C is closed in $p^{-1}(U)$, then $L^{-1}(C) = hK^{-1}(C)$, which is closed. The mapping L is closed since $p^{-1}(b_0)\times U$ is compact. The mapping L is onto since h and K are. Finally, L is one-to-one since if $L(y,u) = L(y',u')$ then $\varphi_y(u) = L(y',u')$ then $\varphi_y(u) = \varphi_{y'}(u')$ and $u = p\varphi_y(u) = p\varphi_{y'}(u') = u'$. Then there exists $t \in I$ such that $f(t) = u$ and hence $K(y,t) = K(y',t)$ but this is a contradiction to the methods of (4.1). Therefore L is a homeomorphism and, since b_0 was arbitrary, p has the BP.

(4.3) THEOREM. *If p is a-light and has the* FCHP *and* PCHP *and B is an arcwise connected, locally compact, uniformly locally arcwise connected and semi-locally simply connected metric space and if $p^{-1}(b)$ is compact for all b in B, then p has the* BP.

Proof. The proof follows easily from (3.11) and (4.2).

An easy corollary to (3.11) and (4.1) is the following theorem which is the only one to put a condition on E.

(4.5) THEOREM. *If E is locally arcwise connected and p is an a-light map with the* PCHP *and* FCHP *and B is first countable, locally arcwise connected and semi-locally simply connectod, then p has the* BP.

Proof. Let $b \in B$; then by (3.11) there exists a neighborhood U (which may be assumed to be connected by the hypothesis) of b and a collection φ_y of maps φ_y: $U\to E$ such that $\varphi_y(b) = y$ and $p\varphi_y$ is the identity on U. By the construction involved in (4.1) it is seen that $\{\varphi_y(U)\}$ is pairwise disjoint and the rest of the proof follows from the local arcwise connectivity of $p^{-1}(U)$.

5. Light compact mappings.

(5.1) DEFINITION. A mapping p: $X \twoheadrightarrow Y$ is *compact* if, given any compact set C in Y, $f^{-1}(C)$ is compact.

(5.2) Remark. If f is a pseudo regular map [10] or a closed map such that f^{-1} (point) is compact, then f is compact.

(5.3) THEOREM. *If p is a light compact map from metric space E onto a metric space B, then, given any $\varepsilon > 0$ and any b in B, there exists $\delta > 0$ such that if C is a compact continuum contained in $N_\delta(b)$, then each component of $p^{-1}(C)$ has diameter less than ε.*

Proof. Assume that the theorem is false. Then there exists a positive number ε and a point b in B and a sequence of compact continua $\{C_i\}$ in B such that $C_i \subset N_{1/i}(b)$ and $p^{-1}(C_i)$ has a component of diameter greater than or equal to ε. Let $A = (\bigcup C_i) \cup b$. A is compact. Therefore $p|p^{-1}(A)$ is a light map from the compact metric space $p^{-1}(A)$ to the compact metric space A and it is known that $p|p^{-1}(A)$ then has the desired property, but this is a contradiction.

(5.4) THEOREM. *Let p be a light compact mapping with the u-0CHP from a metric space E onto a metric space B; then p has the* PLP.

Proof. Let $Z = \{(y,f) \in E\times B^I |\ p(y) = f(0)\}$. Define p^*: $E^I \to Z$ by $p^*(g) = (g(0), pg)$. The mapping p^* is one-to-one and onto by the u-0CHP and hence it must be shown that p^* is closed. This will involve several lemmas in which the following notation will be used. Let C be a closed set in E^I and let $\{(y_i, g_i)\}$ be a sequence of points in $p^*(C)$ which converges to $(y,g) \in Z$. Let $f_i \in C$ be such that $p^*(f_i) = (y_i, g_i)$. The object of the remainder of the proof is to show that $(y,g) \in p^*(C)$.

(5.5) STEP 1. *$\{g_i\}$ is equicontinuous* (i.e., given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $d(g_i(x), g_i(y)) < \varepsilon$ for all i).

Proof. Trivial.

(5.6) STEP 2. *$\{f_i\}$ is equicontinuous.*

Proof. Assume that the above is false. Then there exist $\varepsilon > 0$ and points x_i, y_i in I and maps f_{j_i} such that $|x_i - y_i| < 1/i$ and $d(f_{j_i}(x_i), f_{j_i}(y_i)) > \varepsilon$. The points x_i, y_i can be chosen such that $\{f_{j_i}\}$ is infinite and $\{j_i\}$ is a monotone increasing sequence, and $\lim x_i = \lim y_i = x$. By (5.3) there exists $\delta_1 > 0$ such that if C is a compact continuum in $N_{\delta_1}((g(x))$, then each component of $p^{-1}(C)$ has diameter less than ε, and there exists $\delta_2 > 0$ such that if $|t-t'| < \delta_2$, then $d(g_i(t), g_i(t')) < \delta_1/3$ and $d(g(t), g(t')) < \delta_1/3$, and there exists an integer N such that if $n > N$ then $d(g_n(t), g(t)) < \delta_1/3$. There also exists $i > N$ such that x_i and y_i are in $N_{\delta_2}(x)$ and $|x_i - y_i| < \delta_2$. It will now be shown that $g((x_i, y_i)) \subset N_{\delta_1} g(x)$ and this will yield a contradiction. Let $t \in [x_i, y_i]$. Then

$$d(g_i(t), g(x)) \leqslant d(g_i(t), g(t)) + d(g(t), g(x)) < \delta_1/3 + \delta_1/3 < \delta_1 .$$

The contradiction comes from the fact that $f_i([x_i, y_i])$ is a connected set of diameter greater than or equal to ε which is contained in $p^{-1}g([x_i, y_i]) \subset p^{-1}N_{\delta_1}g(x)$ contrary to the choice of δ_1.

(5.7) STEP 3. *A subsequence of the sequence $\{f_i\}$ converges pointwise to a function f.*

Proof. It could be assumed that $f_i(t)$ converges on a dense countable subset $\{t_i\}$ of I, since for each t, $\{g_i(t)\}$ converges to $g(t)$ and hence $(\bigcup g_i(t)) \cup g(t)$ is compact so $p^{-1}[(\bigcup g_i(t)) \cup g(t)]$ is compact and contains $f_i(t)$, and therefore there exists a convergent subsequence of $\{f_i(t)\}$. Hence using a standard diagonal process the above assumption can be obtained.

Let $\varepsilon > 0$ be given and let $t \in I$. By (5.6) there exists $\delta > 0$ such that if $|x-t| < \delta$ then $d(f_i(x), f_i(t)) < \varepsilon/4$ and there exists $t_i \in \{t_j\}$ such that $|t_i - t| < \delta$ and there exists an integer N such that if $n > N$, then $d(f_n(t_i), \lim_m f_m(t_i)) < \varepsilon$. Therefore, if $n, m > N$,

$$d(f_n(t), f_m(t)) \leqslant d(f_n(t), f_n(t_i)) + d(f_n(t_i), \lim_k f_k(t_i)) + d(\lim_k f_k(t_i), f_m(t_i))$$
$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon .$$

Therefore, $\{f_n(t)\}$ is a Cauchy sequence which is contained in the compact metric space $p^{-1}((\bigcup g_n(t) \cup g(t))$. Therefore, $f_n(t)$ converges to some point, call it $f(t)$.

Define $f(t) = \lim_n f_n(t)$. From now it will be assumed that $\{f_n\}$ is a pointwise convergent sequence. Note by the continuity of p, $pf(t) = g(t)$ and $f(0) = y$.

(5.8) STEP 4. *The map f is continuous.*

Proof. Let $t \in I$ and let $\varepsilon > 0$. By (5.6) there exists $\delta > 0$ such that if $|x-t| < \delta$, then $d(f_i(x), f_i(t)) < \varepsilon/3$ and by (5.7) there exists an integer $N_{x,t}$ such that if $n > N_{x,t}$ then

$$d(f_n(x), f(x)) < \varepsilon/3 \quad \text{and} \quad d(f_n(t), f(t)) < \varepsilon/3 .$$

Let $|x-t| < \delta$; then

$$d(f(x), f(t)) \leqslant d(f(x), f_{N_{x,t}}(x)) + d(f_{N_{x,t}}(x), f_{N_{x,t}}(t)) + d(f_{N_{x,t}}(t), f(t))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Hence, f is continuous.

(5.9) *Completion of the proof of* (5.4). By Theorem 15, page 232 [7], $\{f_i\}$ converges uniformly to f. Therefore $f \in C$ and $p^*f = (y, g)$ so that $(y, g) \in p^*(C)$ and hence p^* is a closed mapping.

(5.10) A restatement of (5.4) is: *A light compact mapping from a metric space onto a metric space has the* PLP *iff it has the* u-0CHP.

Proof. The proof follows from (5.4) and (3.1).

(5.11) DEFINITION. A map $p: X \to Y$ will be called *pseudo compact* if p restricted to every component of X is a compact map.

(5.12) COROLLARY. *If p is a light pseudo compact mapping with the* u-0CHP *from a metric space E onto a metric space B, then p has the* CCHP.

Proof. The map p restricted to each component of E has the PLP by (5.4) which is equivalent to the ACHP (Theorem 12. 1, page 82, [5]). Hence p has the CCHP.

(5.13) DEFINITION. A map $p: X \to Y$ is *locally compact* if given $b \in B$ there exists a neighborhood U of b such that $p|p^{-1}(U)$ is a compact map.

(5.14) COROLLARY. *If p is a light locally compact mapping with the* u-0CHP *from a metric space E onto a metric space B, then p has the* PLP *or, equivalently, p has the* ACHP.

Proof. By (5.4), p has the local ACHP and by Theorem 4.8 [2], p has the ACHP.

(5.15) DEFINITION. A map $p: X \to Y$ is *pseudo locally compact* if given $b \in B$ there exists a neighborhood U of b such that p restricted to every component of $p^{-1}(U)$ is a compact mapping.

(5.16) COROLLARY. *If p is a light pseudo locally compact map with the* u-0CHP *from a metric space E onto a metric space B, then p has the* PCHP.

Proof. By the hypothesis there exists a cover $\{U_\alpha\}$ of B such that $p|p^{-1}(U_\alpha)$ is a pseudo compact map. Hence $p|p^{-1}(U_\alpha)$ has the CCHP and by Theorem 4.8 [2], p has the PCHP.

The reason for the new definitions is to find a condition (A) on the map p such that the following theorem is obtained: p satisfies condition (A) and has the u-0CHP iff p has the PCHP. The following example shows that pseudo local compactness is not the condition.

(5.17) EXAMPLE. *A Serre fibration which is not pseudo locally compact.* Let E be the following subset of the plane:

$$E = \left\{(x, y,)|y = \frac{1}{n} \text{ and } 0 \leqslant x < \frac{1}{n}\right\} \cup (0, 0) .$$

Let B be the following subset of 3-space:

$$B = \left\{(x, y, z)|z = \frac{1}{n},\ x = 0 \text{ and } -\frac{2}{n} \leqslant y \leqslant \frac{1}{n}\right\}$$
$$\cup \left\{(x, y, z)|z = \frac{1}{n},\ y = -\frac{2}{n} \text{ and } 0 \leqslant x \leqslant \frac{1}{n}\right\}$$
$$\cup \left\{(x, y, z)|z = \frac{1}{n},\ x = \frac{1}{n} \text{ and } -\frac{2}{n} \leqslant y \leqslant 0\right\}$$
$$\cup \left\{(x, y, z)|z = \frac{1}{n},\ 0 < x \leqslant \frac{1}{n} \text{ and } y = \frac{1}{n}\sin\frac{2\pi n}{x}\right\}$$
$$\cup \{(0, 0, 0)\} .$$

Let p be the natural map from E to B (see Figure 2 on p. 44). By (5.19), it will be seen that p has the PCHP but p is not pseudo locally compact.

(5.18) DEFINITION. A map $p: X \to Y$ is called a *componentwise pseudo locally compact map* if given any component C of X, $p|C$ is pseudo locally compact.

(5.19) COROLLARY. *If p is a light componentwise pseudo locally compact map with the* u-0CHP *from a metric space E onto a metric space B, then p has the* PCHP.

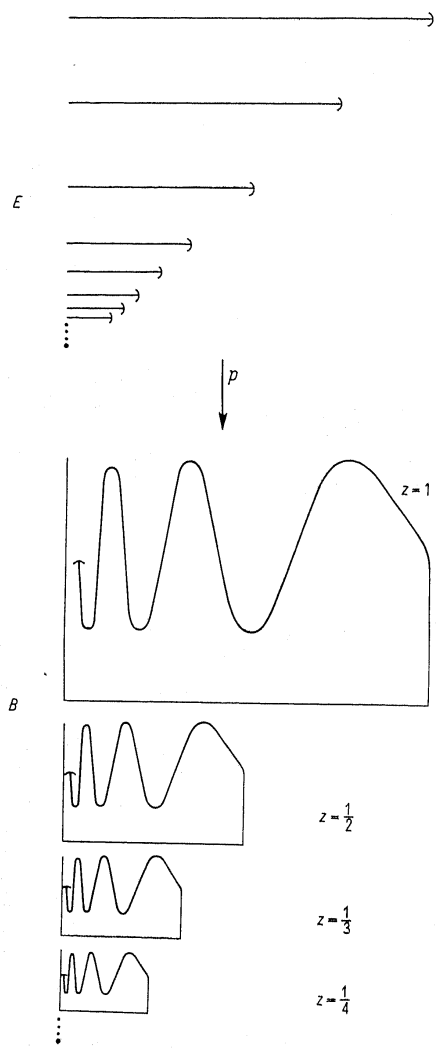

Fig. 2

Proof. By (5.18) and (5.16), p restricted to each component of E has the PCHP and hence p has the PCHP.

The author does not have an example of a light fiber map which is not componentwise pseudo locally compact.

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