Maximal lattice-ordered algebras of continuous functions*

by

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Let X be a compact Hausdorff space and let $D(X)$ be the lattice of continuous functions from X to the extended real line, γR, which are real-valued on a dense subset. With a natural definition of operations, $D(X)$ becomes a partial algebra.

A Φ-algebra is a real Archimedean lattice-ordered algebra with positive 1 which is a weak order unit. A Φ-subalgebra of $D(X)$ is a subset of $D(X)$ which is a Φ-algebra under the operations defined above. By Zorn's lemma, every Φ-subalgebra of $D(X)$ is contained in a maximal Φ-subalgebra.

The purpose of this work is a study of $D(X)$ by means of these maximal Φ-subalgebras.

$D(X)$ is an object of some importance in representing certain algebraic structures as collections of functions, as the following paragraphs indicate.

M. Henriksen and D. G. Johnson have proved ([6], 2.3) that every Φ-algebra A is isomorphic to a point-separating Φ-subalgebra of $D(\mathcal{M}(A))$, where $\mathcal{M}(A)$ is the (compact) space of maximal l-ideals of A.

Likewise, each Archimedean vector-lattice is isomorphic to a point-separating l-subspace of $D(X)$ for an appropriate choice of X (see [9], 6.8).

The operations on $D(X)$ need not be everywhere defined. However, $D(X)$ can be embedded "isomorphically" in a Φ-algebra $D(X_\infty)$. This proposition leads to the following result: A subalgebra of $D(X)$ is a maximal Φ-subalgebra iff it is a maximal subalgebra.

The idea of locallity in maximal Φ-subalgebras turns out to be of some importance. Let A be a maximal Φ-subalgebra of $D(X)$. If f is locally in A, then f belongs to A.

The structure space, $\mathcal{M}(A)$, of A is obtained as a quotient space of X. The embedding of A into $D(\mathcal{M}(A))$ induced by the quotient map

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is just the Henriksen-Johnson embedding. Locallity is used in showing that the (maximal) stationary sets of A are nowhere dense, closed, and connected (or are isolated points).

The result of the previous paragraph indicates that stationary sets are small. Examples are given to show that stationary sets may be very large. For a familiar 1-dimensional space X, there is a maximal Φ-subalgebra of $D(X)$ with a 1-dimensional stationary set. If a is any cardinal number, there is a maximal Φ-subalgebra of $D([0,1]^a)$ with structure space $[0,1]$.

If X is a 1st-countable space, the intersection of all the maximal Φ-subalgebras of $D(X)$ is exactly the set of locally constant functions.

The class of uniformly closed (= closed under uniform convergence) Φ-algebras is of particular interest. Just as the Stone–Weierstrass theorem shows that $C(X)$ is the smallest uniformly closed Φ-algebra with structure space X, we prove that there is a largest uniformly closed Φ-subalgebra, $U(X)$, of $D(X)$ with structure space X. $U(X)$ is characterized as the collection of all elements of $D(X)$ whose coinfinity sets are C^*-embedded in X. Every uniformly closed maximal Φ-algebra is $U(X)$ for some space X. $U(X)$ is the intersection of all the maximal l-subspaces of $D(X)$. Conditions are given under which $U(X) = C(X)$, and necessary and sufficient conditions are given for $U(X)$ to be isomorphic to $C(Y)$ for some space Y.

Using the characterization of $U(X)$, it is show that if P is a hyper-real prime ideal of a uniformly closed maximal Φ-algebra A, then A/P has no countable cofinal subset. Hence hyper-real quotient fields of uniformly closed maximal Φ-algebras are (real closed) η_1-fields. It is known that in the absence of maximality this result need not hold.

Since $D(Y) = D(\beta Y)$ for any completely regular space Y, no particular effort has been made to eliminate compactness from the hypotheses of certain theorems.

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1. Preliminaries. An attempt has been made to keep this paper reasonably self-contained. Questions of notation and terminology can be answered by consulting [3] or [6], the latter of which is devoted to a study of the structure of Φ-algebras.

In this paper, *all given spaces are assumed to be completely regular (and Hausdorff)*. When not mentioned to the contrary, *X is always a compact space.*

The set (field, topological space) of real numbers is denoted by $\boldsymbol{R}$; and the set of natural numbers, by N. If Y is a space and $r \in \boldsymbol{R}$, then $\boldsymbol{r}$ denotes the constant function on Y taking the value r.

1.1. A space Y is *extremally disconnected* iff the closure of every open subset of Y is open. It is known ([3], 1H) that every open subset of an extremally disconnected space is C^*-embedded and extremally disconnected, and that every dense subset of an extremally disconnected space is extremally disconnected.

Let $f\colon Y \to Z$ be a continuous function. Then f is *tight* iff every non-empty open subset of Y contains the non-empty inverse image of an open subset of Z; f is *fitting* iff f is closed, onto, and the inverse image of each point of Z is compact.

THEOREM. ([13], section 2.) *Every space Y is the continuous image under a tight fitting map, τ, of an extremally disconnected space, Y_∞. If Z is another extremally disconnected space and σ is a tight fitting map of Z onto Y, then there is a homeomorphism ϱ of Y_∞ onto Z such that $\sigma \circ \varrho = \tau$.* (This is an extension of a theorem of Gleason [2].)

The pair (Y_∞, τ) is called the *minimal projective extension* of Y.

1.2. Let Y be a subset of Z. Then Y is *C^*-embedded* in Z iff every continuous function on Y to $[0,1]$ has a continuous extension over Z. When Y is dense in Z, Y is C^*-embedded in Z iff every continuous function on Y to any compact space has a continuous extension over Z. For any (completely regular) space Y, there exists a compactification, βY, of Y characterized by the condition Y is C^*-embedded in βY. For a more detailed description of βY, see [3], Chapt. 6.

1.3. A *Φ-algebra* is a real Archimedean lattice-ordered algebra with positive 1 which is a weak order unit. Equivalently, a Φ-algebra is a real Archimedean lattice-ordered algebra which as a ring is a subdirect sum of totally ordered rings, and which has an identity.

Let Y by a completely regular space. A continuous function $f\colon Y \to \gamma R$ is said to be an *extended real-valued function* iff the *coinfinity set*, $\mathfrak{R}(f) = \{y \in Y\colon f(y) \in \boldsymbol{R}\}$, of f is a dense subset of Y. $D(Y)$ is the set of all extended real-valued functions on Y. With the partial order defined pointwise, $D(Y)$ becomes a lattice.

Let f and g be extended real-valued functions. If there is a function h in $D(Y)$ such that $h(x) = f(x) + g(x)$ for all $x \in \mathfrak{R}(f) \cap \mathfrak{R}(g)$, then h is the *sum*, $f+g$, *of f and g*. Since $\mathfrak{R}(f) \cap \mathfrak{R}(g)$ is dense, this sum is unique if it exists. Similarly, fg is the extension of the pointwise product defined on the intersection of the coinfinity sets, and it is unique if it exists.

A subset A of $D(Y)$ which is a Φ-algebra under the operations on $D(Y)$ is called a *Φ-subalgebra* of $D(Y)$. *Subalgebra*, *subspace*, and *l-subspace* are defined similarly.

Note that a Φ-subalgebra of $D(Y)$ is simply an l-subalgebra with 1.

PROPOSITION. ([6], 2.2.) *$D(Y)$ is a Φ-algebra iff every dense cozero subset of Y is C^*-embedded.*

1.4. An (algebra) ideal I of a Φ-algebra A is an *l-ideal* iff $a \in A$, $b \in I$, and $|a| \leqslant |b|$ together imply $a \in I$. Let $\mathcal{M}(A)$ denote the set of all maximal l-ideals of A. If I is l-ideal of A, let $h(I) = \{M \in \mathcal{M}(A)\colon I \subseteq M\}$. If $S \subseteq \mathcal{M}(A)$, let $k(S) = \bigcap S$. The topological space with underlying set $\mathcal{M}(A)$ and closure operator $S \to hk(S)$ is called the *structure space* of A. For $a \in A$, let $\mathcal{M}(a)$ be the set of maximal l-ideals of A containing a. Then the collection $\{\mathcal{M}(a)\colon a \in A\}$ forms a base for the closed sets of $\mathcal{M}(A)$.

If I is an l-ideal of a Φ-algebra A and if $a \in A$, then $I(a)$ denotes the image of a under the natural projection of A onto A/I.

Let A be a lattice-ordered ring and I an l-ideal of A. Then A/I can be made into a lattice-ordered ring as follows: call $I(a)$ positive iff $a^- = (-a \vee 0)$ belongs to I (see [3], Chapt. 5).

The following fundamental representation theorem is due to M. Henriksen and D. G. Johnson.

THEOREM. ([6], 2.3.) *Every Φ-algebra A is isomorphic to an algebra A' of extended real-valued functions on $\mathcal{M}(A)$. Moreover,*

(a) *$\mathcal{M}(A)$ is a compact (Hausdorff) space, and*

(b) *if S and T are disjoint closed subsets of $\mathcal{M}(A)$, then there is an $a' \in A'$ such that $a'[S] = \{0\}$ and $a'[T] = \{1\}$.*

Proof. We give only the definition of the isomorphism $a \to a'$ of A into $D(\mathcal{M}(A))$. For each $0 \leqslant a \in A$, let

$$a'(M) = \inf\{r \in \mathbf{R}\colon\ M(a) \leqslant r\} \qquad (M \in \mathcal{M}(A))\,.$$

For arbitrary $a \in A$, let $a'(M) = (a^+)'(M) - (a^-)'(M)$.

1.5. THEOREM. ([5], 3.3.) *If A is a Φ-algebra and B is a subalgebra, then the least sublattice of A containing B is a subalgebra of A.*

1.6. The next theorem is a slightly weakened version of a theorem of [6], and follows immediately from the proof of [6], 5.2.

THEOREM. ([6], 5.2.) *A Φ-algebra A is isomorphic to $C(Y)$ for some space Y iff*

(a) *$\mathcal{R}(A) = \bigcap\{\mathcal{R}(f)\colon f \in A\}$ is dense and C^*-embedded in $\mathcal{M}(A)$,*

(b) *A is uniformly closed (i.e., closed under uniform convergence), and*

(c) *A is closed under inversion (i.e., if $a \in A$ with $Z(a)$ disjoint from $\mathcal{R}(A)$, then a is invertible in A).*

In this case, Y may be taken to be $\mathcal{R}(A)$.

2. Maximal Φ-algebras. A *maximal Φ-subalgebra* of $D(Y)$ is a Φ-subalgebra of $D(Y)$ which is not properly contained in any Φ-subalgebra of $D(Y)$.

In many of the arguments throughout this paper, we will want to know whether a particular function f belongs to a given maximal Φ-subalgebra A. The results of this section permit us to ignore the lattice operations in attempting to adjoin f to A, and to concentrate on "local belongingness".

2.1. Let (Y_∞, τ) be the minimal projective extension of Y. It is easy to show that the map $\tau^*\colon D(Y) \to D(Y_\infty)$ given by $\tau^*(f) = f \circ \tau$ is a lattice isomorphism which preserves sums and products when they exist. Moreover, if $\tau^*(f) + \tau^*(g)$ belongs to $\tau^*[D(Y)]$, then $f+g$ is defined in $D(Y)$; similarly for products. Since Y_∞ is extremally disconnected, it follows from 1.3 that $D(Y_\infty)$ is a Φ-algebra.

THEOREM. *A subalgebra A of $D(Y)$ is a maximal subalgebra iff A is a maximal Φ-subalgebra.*

Proof. It clearly suffices to prove that every subalgebra of $D(Y)$ is contained in a Φ-subalgebra.

Let A be a subalgebra of $D(Y)$. Then $\tau^*[A]$ is a subalgebra of $\tau^*[D(Y)]$, and $\tau^*[D(Y)]$ is a sublattice of $D(Y_\infty)$; hence the sublattice, E, of $D(Y_\infty)$ generated by $\tau^*[A]$ lies in $\tau^*[D(Y)]$. By 1.5, E is a Φ-algebra. Clearly, then, $\tau^{*\leftarrow}[E]$ is a Φ-subalgebra of $D(Y)$ containing A.

The proof of 2.1 with the Henriksen–Isbell result (1.5) replaced by its analog for vector-lattices (easily proved by induction using the well-known identities $a(a \vee b) = aa \vee ab$, $(a \vee b) + c = (a+c) \vee (b+c)$ (for $0 \leqslant a \in \mathbf{R}$ and a, b, c, in a (real) vector lattice and their duals) can be used to show that maximal subspace and maximal l-subspace of $D(Y)$ are the same.

2.2. Since maximal subalgebras of $D(Y)$ contain the constant functions, we have the following corollary.

COROLLARY. *If A is a maximal Φ-subalgebra of $D(Y)$, and $f \in D(Y) \sim A$, then for some $n \in N$, there exist $m_i \in A$ $(0 \leqslant i \leqslant n)$ such that $\sum m_i f^i$ is not defined.*

2.3. Let A be a Φ-subalgebra of $D(Y)$ and let $f \in D(Y)$. Then f is *in A at $x \in Y$* iff x has a neighborhood on which f agrees with some element of A. The Φ-subalgebra A is called *local* iff f in A at each $x \in Y$ implies that f belongs to A.

THEOREM. *Let X be a compact space. Then every maximal Φ-subalgebra of $D(X)$ is local.*

Proof. Let A be a maximal Φ-subalgebra of $D(X)$ and suppose that f is in A at each $x \in X$. By a compactness argument, there exist a finite open cover $\{U_i\colon 1 \leqslant i \leqslant n\}$ of X and a subset $\{a_i\colon 1 \leqslant i \leqslant n\}$ of A such that $f|_{U_i} = a_i|_{U_i}$ for all $1 \leqslant i \leqslant n$.

Let $g_0, \ldots, g_k \in A$. Define $h\colon X \to \gamma R$ by

$$h(x) = \sum\{g_j a_i^{\,j}(x)\colon\ 0 \leqslant j \leqslant k\} \qquad \text{for} \qquad x \in U_i\,.$$

Now, h is well-defined and $h \in D(X)$. Since h agrees with $\sum g_j f^j$ on $\mathfrak{R}(f) \cap \cap \mathfrak{R}(g_0) \cap \ldots \cap \mathfrak{R}(g_k)$, we have $h = \sum g_j f^j$. By corollary 2.2, $f \in A$.

The ideal of locallity has been considered by Arens [1] in commutative Banach algebras.

3. Structure spaces. In this section, A is assumed to be a Φ-subalgebra of $D(X)$ *containing the constant functions.*

3.1. The following proposition generalizes a result of M. Henriksen and D. G. Johnson ([6], 2.5).

PROPOSITION. *For $x \in X$, the set*

$$M_x = \{f \in A: fg(x) = 0 \text{ for all } g \in A\}$$

is a maximal l-ideal of A. Every maximal l-ideal of A is of this form. Finally, $M_x \neq M_y$ iff there is some $f \in A$ such that $f(x) \neq f(y)$.

Proof. Clearly M_x is a proper l-ideal of A.

Note that if $f \in A$ vanishes on a neighborhood of x, then $f \in M_x$, and that if $f \in M_x$, then $f(x) = 0$.

Suppose that I is an l-ideal of A such that for all $x \in X$, there exist $f \in I$ and $g \in A$ with $fg(x) > 0$. By compactness of X there are $f_1, \ldots, f_n$ in I such that $(|f_1 g_1| + \ldots + |f_n g_n|)$ is strictly positive for all $x \in X$, so $I = A$. Hence every proper l-ideal of A is contained in some M_x.

Suppose that M_x properly contains M_y. Let $f \in M_x \sim M_y$. Then there is $g \in A$ such that $fg(y) = 1$ (and $fg(x) = 0$), so $(fg \wedge \mathbf{1/2}) - \mathbf{1/2}$ belongs to $M_y \sim M_x$, a contradiction. Hence no M_x properly contains any M_y. In view of the preceeding paragraph, this implies that each M_x is a maximal l-ideal. If $M_x \neq M_y$, then there are $f \in M_x$ and $g \in A$ such that $fg(y) \neq 0$. If f belongs to A with $f(x) \neq f(y)$, we may suppose that $f(x) = 1$ and $f(y) = -1$; then $f \vee \mathbf{0} \in M_y$; clearly $f \vee \mathbf{0} \notin M_x$.

3.2. A subset S of $D(X)$ is said to *separate points* of $Y \subseteq X$ iff whenever x and y are distinct points of Y, then there exists $a \in S$ with $a(x) \neq a(y)$.

Let $\mathfrak{r}$ be the relation on X defined by $(x, y) \in \mathfrak{r}$ iff A does not separate the points of $\{x, y\}$. Clearly $\mathfrak{r}$ is an equivalence relation. The equivalence classes of $\mathfrak{r}$ are called *stationary sets* of A. (Note that here "stationary set" is used instead of the customary "maximal stationary set".)

THEOREM. *The structure space, $\mathcal{M}(A)$, of A is homeomorphic to (the quotient topological space) $X/\mathfrak{r}$.*

Proof. Write Y for $X/\mathfrak{r}$, and let ϱ be the natural projection of X onto Y. Since $M_x = M_y$ iff $(x, y) \in \mathfrak{r}$, $\tau(\varrho(x)) = M_x$ defines a bijection $\tau: Y \to \mathcal{M}(A)$. Now, Y is a continuous image of X, so Y is quasi-compact.

Let $\mathcal{M}(a) = \{M \in \mathcal{M}(A): a \in M\}$ be a basic closed subset of $\mathcal{M}(A)$. Then

$$\varrho^{\leftarrow} \circ \tau^{\leftarrow}(\mathcal{M}(a)) = \varrho^{\leftarrow}(\{\varrho(x): a \in M_x\}) = \{x \in X: a \in M_x\} = \bigcap \{Z(ab): b \in A\},$$

which is closed. Hence τ is continuous. Since $\mathcal{M}(A)$ is Hausdorff and Y is quasi-compact, τ is a homeomorphism.

This theorem can be used to obtain the structure space of a Φ-subalgebra B of $D(Y)$ for an arbitrary completely regular space Y. Every function in $D(Y)$ has a unique extension to an element of $D(\beta Y)$—this yields an isomorphism of B into $D(\beta Y)$. Let 1_B be the identity element of B. Since 1_B is idempotent, the only values that it can assume are 0, 1, or ∞. Since $1_B^{\leftarrow}(\infty)$ is thus open (and nowhere dense), 1_B can assume only the values 0 and 1. Hence the set

$$\{x \in \beta Y: 1_B(x) = 1\} = \{x \in \beta Y: 1_B(x) \neq 0\}$$

is an open-closed subset of βY on the complement of which every function in B vanishes. Hence B can be considered as a Φ-algebra of extended real-valued functions on this set.

3.3. If A is a Φ-subalgebra of $D(X)$ and ϱ is the natural projection of X onto $X/\mathfrak{r}$, then A can be considered as a Φ-subalgebra of $D(X/\mathfrak{r})$ as follows: if $\varrho(x) = \varrho(y)$, then $a(x) = a(y)$ for all $a \in A$, so defining a'': $X/\mathfrak{r} \to \gamma R$ by $a''(\varrho(x)) = a(x)$ yields an isomorphism $a \to a''$ of A into $D(X/\mathfrak{r})$.

The homeomorphism $\tau^{\leftarrow}$: $\mathcal{M}(A) \to X/\mathfrak{r}$ induces an isomorphism $(\tau^{\leftarrow})^*$: $D(X/\mathfrak{r}) \to D(\mathcal{M}(A))$ by $(\tau^{\leftarrow})^*(f) = f \circ \tau^{\leftarrow}$. Then $a \to (\tau^{\leftarrow})^*(a'')$ is an embedding of A into $D(\mathcal{M}(A))$. It is natural to ask whether this embedding agrees with the Henriksen–Johnson embedding, $a \to a'$, of A into $D(\mathcal{M}(A))$ (see 1.4).

THEOREM. *The homeomorphism of theorem 3.2 induces the Henriksen–Johnson embedding of A into $D(\mathcal{M}(A))$; i.e., in the notation of the above paragraphs, $a' = (\tau^{\leftarrow})^*(a'')$.*

Proof. Let $0 \leqslant a \in A$, $M_x \in \mathcal{M}(A)$. Recall that $a'(M_x) = \inf\{r \in \mathbf{R}: M_x(a) \leqslant r\}$. If $M_x(a) \leqslant r$ for some $r \in \mathbf{R}$, then there is $m \in M_x$ such that $a \leqslant \mathbf{r} + m$, so $r = r + m(x) = (\mathbf{r} + m)(x) \geqslant a(x) = a''(\tau^{\leftarrow}(M_x)) = (\tau^{\leftarrow})^*(a'')(M_x)$. Hence $(\tau^{\leftarrow})^*(a'') \leqslant a'$. If $r \in \mathbf{R}$ with $r < a'(M_x)$, then $r < M_x(a)$, so there is $m \in M_x$ with $\mathbf{r} + m \leqslant a$; then $r = (\mathbf{r} + m)(x) \leqslant a(x) = (\tau^{\leftarrow})^*(a'')(M_x)$. Hence $a' \leqslant (\tau^{\leftarrow})^*(a'')$.

We will usually not distinguish notationally between elements of A and their images in $D(X/\mathfrak{r})$.

3.4. It is clear that if A is a maximal Φ-subalgebra of $D(X)$, then A is a maximal Φ-subalgebra of $D(\mathcal{M}(A))$. In the latter case, A is said to be a *maximal Φ-algebra.*

4. Stationary sets for maximal Φ-subalgebras.

4.1. Connectedness in the next theorem depends only on the fact that A is local and contains the constant functions.

THEOREM. *Let A be a maximal Φ-subalgebra of $D(X)$. Every stationary set of A with more than one point is closed, nowhere dense, and connected.*

Proof. Let S be a stationary set of A with more than one point, and let $x \in S$. Then $S = \bigcap \{f^{\leftarrow} \circ f(x) \colon f \in A\}$, so S is closed.

If $y \in \operatorname{int} S$ and $y \neq x$, then there is an open neighborhood U of y such that $x \notin U \subseteq S$. There is a closed neighborhood V of y such that $V \subseteq U$. Let $g \in C(X)$ with $g(y) = 0$ and $g[X \sim V] = \{1\}$. Every element of A is constant on the open set U, and g is constant on the open set $X \sim V$, so $\sum f_i g^i$ is defined whenever $f_i \in A$ $(0 \leqslant i \leqslant n)$. Hence $g \in A$. But g separates x and y.

Suppose $S = B \cup C$ where B and C are closed and disjoint, and $C \neq \emptyset$. Let $g \in C(X)$ with $\mathbf{0} \leqslant g \leqslant \mathbf{1}$, $g[U] = \{0\}$, and $g[V] = \{1\}$ for some open neighborhoods U and V of C and B, respectively. Let $Z = \{x \in X \colon g \text{ is not in } A \text{ at } x\}$. Note that Z is closed and Z is disjoint from $U \cup V$. Now, A separates each point of Z from S, so by compactness of X, A separates Z from S; hence there is $f \in A$ such that f vanishes on a neighborhood of Z, is 1 on a neighborhood of S, and is bounded. Then fg is in A at every point of X, so $fg \in A$. Since fg separates B and C, we must have $B = \emptyset$. Hence S is connected.

It has been pointed out to the author that connectedness of stationary sets also arises in a different context. The following statement was proved by M. Katětov [10], Lemma 18, see [3], 16.31: If A is a subring of $C^*(X)$ containing the constant functions and if f belongs to A whenever $f^2 \in A$, then every stationary set of A is connected.

4.2. COROLLARY. *If X is a totally-ordered space or is zero-dimensional, then every maximal Φ-subalgebra of $D(X)$ separates points of X.*

4.3. LEMMA. *Let $h \in D(X)$ and let Y be a connected subset of $\mathcal{N}(h) = \{x \in X \colon |h(x)| = \infty\}$ satisfying*

(*) *If $f \colon Y \to [-1, 1]$ is continuous and onto, then there exists $r \in (-1, 1)$ such that $f^{\leftarrow}(r)$ is nowhere dense in Y.*

Then if $g \in D(X)$ is not constant on Y, g does not belong to any Φ-subalgebra containing h.

Proof. Since Y is connected, either $Y \subseteq \operatorname{pos} h$ or $Y \subseteq \operatorname{neg} h$; we can assume $h \geqslant \mathbf{0}$. Suppose $x, y \in Y$ with $g(x) = -1$ and $g(y) = 1$. Let $r \in (-1, 1)$ such that $g^{\leftarrow}(r)$ is nowhere dense in Y—we can assume that $r = 0$. Finally, we suppose that gh is defined and arrive at a contradiction.

Suppose that there is $z \in Y$ such that every Y-neighborhood U of z contains points $x(U)$ and $y(U)$ at which g is, resp., strictly negative and strictly positive. Let U be a Y-neighborhood of z. Clearly $gh\big(x(U)\big) = -\infty$ and $gh\big(y(U)\big) = \infty$. Hence every neighborhood, V, of z contains points $x(V \cap Y)$ and $y(V \cap Y)$ at which gh is $-\infty$, $+\infty$, resp., contradicting continuity of gh.

Hence for each $z \in Y$ there exists a Y-neighborhood, U_z, of z such that either $U_z \subseteq \operatorname{pos} g \cup Z(g)$ or $U_z \subseteq \operatorname{neg} g \cup Z(g)$. Let B be the set of $z \in Y$ satisfying the former and C, the set of z satisfying the latter. Since $Z(g)$ is nowhere dense in Y, $B \cap C = \emptyset$. If $z \in B$, then $\operatorname{int}_Y U_z \subseteq B$, so B is open in Y; similarly, C is open in Y. Now, $Y = B \cup C$, $x \in C$, and $y \in B$. But this contradicts connectedness of Y.

Note that if Y is a space in which every disjoint family of open sets is countable—e.g., if Y is 2nd-countable—then Y satisfies condition (*) of the lemma.

4.4. Let A be a maximal Φ-subalgebra of $D(X)$. The sets $\mathcal{N}(g)$ for $g \in A$ are called *infinity sets* of A. Stationary sets of A with more than one point arise as a result of certain functions' being "kept out of" A. Since functions are kept out of A by infinity sets of A, it is reasonable to except a close relationship between stationary sets of A and infinity sets of A, and this is indeed the case (see 4.5). However, there may be a stationary set on which no function of A is infinite, and which is not even a zero set.

EXAMPLE. *A stationary set for a maximal Φ-subalgebra A of $D(X)$ which is not a zero set, which has more than one point, and which is disjoint from every infinity set of A.*

Let L be the 1-point compactification of the long line (see, for example, [3], p. 262). Let $X = L \times [0, 1]$. For convenience, if $x \in L$, we denote by $v(x)$ the "vertical line segment" $\{x\} \times [0, 1]$. It will be shown that $v(\infty)$ is a stationary set for some maximal Φ-subalgebra A.

Recall that the long line is the lexicographic product $W \times [0, 1)$ with the first coordinate dominating, where W is the space of countable ordinals, and the product is given the order topology. For $s \in W$, define $f_s \in D(X)$ by $f_s[v(\infty)] = \{0\}$, $f_s\big((w, x), y\big) = 0$ $(w \neq s)$, and $f_s\big((s, x), y\big) = (|x - \tfrac{1}{2}|^{-1} - 4) \vee 0$. Since $s \neq t$ implies that f_s vanishes on a neighborhood of $\mathcal{N}(f_t)$, $\{f_s \colon s \in W\}$ is contained in a subalgebra of $D(X)$. Let A be a maximal Φ-subalgebra containing all of the f_s.

Since $\mathcal{N}(f_s) = v(s, \tfrac{1}{2})$ is homeomorphic to $[0, 1]$, lemma 4.3 implies that each $\mathcal{N}(f_s)$ is a stationary set of A. Since ∞ is a limit point of $\{(s, \tfrac{1}{2}) \colon s \in W\}$, $v(\infty)$ is also a stationary set of A. Since ∞ is not a G_δ set, it is not a zero set.

If $f \in A$ is infinite at any point of $v(\infty)$, it is infinite on all of $v(\infty)$; $\mathcal{N}(f)$ is a zero set. But every G_δ set containing $v(\infty)$ has non-empty interior.

4.5. PROPOSITION. *If S is a stationary set of a maximal Φ-subalgebra A of $D(X)$ with more than one point, and if V is any open set intersecting S, then V contains points at which some element of A is infinite.*

Proof. Suppose $V \subseteq \mathfrak{R}(A)$. Since S is connected, $S \cap V$ has more than one point; let x and y be distinct points of $S \cap V$. Let $g \in C(X)$ with $g(x) = 1$ and $g[X \sim W] = \{0\}$ for some closed neighborhood W of x contained in $V \sim \{y\}$. Since $V \subseteq \mathfrak{R}(A)$, $g \in A$ (see 2.2). But g is not constant on S.

4.6. THEOREM. *If Y is a nowhere dense connected zero set of X and if Y satisfies $(*)$ of lemma 4.3, then Y is a stationary set for some maximal Φ-subalgebra of $D(X)$.*

Proof. If $Y = Z(f)$, let $h = |f|^{-1}$ and let A be a maximal Φ-subalgebra of $D(X)$ containing h.

This theorem is a partial converse to theorem 4.1. As example 4.4 shows, theorem 4.1 cannot be strengthened to include zero set in its conclusion.

4.7. As the next two examples show, stationary sets for maximal Φ-subalgebras may be very large.

EXAMPLE. *A 1-dimensional space X with maximal Φ-subalgebra which has a 1-dimensional stationary set.*

Let $X = \{(x, \sin(1/x)):\ x \in (0,1]\} \cup (\{0\} \times [-1, 1])$. The set $Y = \{0\} \times [0, 1]$ is a nowhere dense connected zero set satisfying $(*)$ of lemma 4.3. By theorem 4.6, Y is a stationary set for some maximal Φ-subalgebra A of $D(X)$.

4.8. EXAMPLE. *A maximal Φ-subalgebra of $D([0,1]^{\alpha})$ for any cardinal $\alpha > 1$ with structure space $[0,1]$.*

Index the rationals in $[0,1]$: $Q \cap [0,1] = \{q_i:\ i \in N\}$ so that $q_1 = 0$. Define $f_1 \in D([0,1])$ by $f_1(x) = 1/x$ ($f_1(0) = \infty$). Let U_n be a closed interval containing q_n in its interior but not containing q_i for $i < n$. Let I_n be a closed interval contained in the interior of U_n and containing q_n in its interior. Let $h_n \in C([0,1])$ such that $h_n[[0,1] \sim U_n] = \{0\}$, $h_n[I_n] = \{1\}$, and $0 \leqslant h_n \leqslant 1$.

For $1 < n \in N$, define f_n inductively by

$$f_n(x) = h_n(x)\big[\ln\big(1 \vee \inf\{|f_i(x) - f_i(q_n)|^{-1}:\ 1 \leqslant i < n\}\big)\big],$$
$$f_n(q_n) = \infty.$$

Then $f_n \in D([0,1])$.

(a) Since h_n vanishes on a neighborhood of $\{q_i:\ i < n\}$, so does f_n.

(b) Since $f_1(x) = f_1(q_n)$ iff $x = q_n$, f_n is infinite only at q_n.

(c) If $j < n$, then

$$\begin{aligned} 0 &\leqslant \lim_{x\to q_n} \big[|f_j(x) - f_j(q_n)|\, f_n^{m}(x)\big] \\ &= \lim_{x\to q_n} \big[|f_j(x) - f_j(q_n)|\big(\ln\inf\{|f_i(x) - f_i(q_n)|^{-1}:\ i < n\}\big)^m\big] \\ &\leqslant \lim_{x\to q_n} \big[|f_j(x) - f_j(q_n)|\big(\ln|f_j(x) - f_j(q_n)|^{-1}\big)^m\big] \\ &= 0 \quad \text{for all } m \in N. \end{aligned}$$

(d) If P is a polynomial in $f_1, \ldots, f_{n-1}$ which vanishes at q_n, then $\lim_{x\to q_n} f_n^{m}(x)P(x) = 0$.

Proof. For fixed $n > 1$, the proof is by induction on the degree of P. The statement is clear if degree of P is 0.

Suppose that degree of P is $s > 0$, and the statement holds for every polynomial in $f_1, \ldots, f_{n-1}$ of degree $< s$. For notational reasons, we assume that there are only two terms of P of highest degree: $rf_1^{i_1} \ldots f_{n-1}^{i_{n-1}}$ and $tf_1^{j_1} \ldots f_{n-1}^{j_{n-1}}$. We suppose that $i_1 \neq 0$, $j_2 \neq 0$. Write P as $\big(f_1 - f_1(q_n)\big)Q + \big(f_2 - f_2(q_n)\big)R + S$ where $Q = rf_i^{i_1-1} \ldots f_{n-1}^{i_{n-1}}$, $R = tf_1^{j_1} f_2^{j_2-1} \ldots f_{n-1}^{j_{n-1}}$, and S has degree $< s$. Since P, $\big(f_1 - f_1(q_n)\big)$, and $\big(f_2 - f_2(q_n)\big)$ vanish at q_n, so does S. By the induction hypothesis and (c) above, $\lim_{x\to q_n} f_n^{m}P = 0$.

(e) $\{f_i:\ i \in N\}$ belongs to some subalgebra of $D([0,1])$.

Proof. Let S be a polynomial in $f_1, \ldots, f_n$. We can write S as $\sum_0^m f_n^{\,i} P_i(f_1, \ldots, f_{n-1})$ where (by induction) each P_i is defined as a continuous function on $[0,1]$ to γR and which is finite except (possibly) at $q_1, \ldots, q_{n-1}$. $\sum_0^m f_n^{\,i} P_i$ is defined and continuous on $[0,1] \sim \{q_n\}$ since f_n vanishes on a neighborhood of $\{q_i:\ i < n\}$.

Write $P_i'(x)$ for $P_i(x) - P_i(q_n)$. Then

$$S = \sum_1^m f_n^{\,i} P_i' + \sum_1^m f_n^{\,i} P_i(q_n) + P_0.$$

By (d), the first term vanishes at q_n; P_0 is finite at q_n; the middle term is $+\infty$, $-\infty$, or 0 according as $P_j(q_n) > 0$, $P_j(q_n) < 0$ where j is the highest index for which $P_j(q_n) \neq 0$ and $j > 0$, or all $P_i(q_n)$ $(i > 0)$ are zero.

(f) Now, write $[0,1]^{\alpha}$ as $[0,1] \times [0,1]^{\beta}$ and define $g_n \in D([0,1]^{\alpha})$ by $g_n(x, y) = f_n(x)$. Let A be a maximal Φ-subalgebra of $D([0,1]^{\alpha})$ containing the g_n $(n \in N)$.

K. A. Ross and A. H. Stone prove ([14], theorem 2) that any disjoint family of open subsets of a product of separable spaces is countable.

Hence $[0,1]^\beta$ satisfies $(*)$ of lemma 4.3. By that lemma, each $\{q_i\}\times[0,1]^\beta$ is a stationary set of A. By denseness of Q, each $\{r\}\times[0,1]^\beta$ is a stationary set of A $(r\in[0,1])$. Hence $\mathcal{M}(A)$ is homeomorphic to $[0,1]$.

5. Intersection of maximal Φ-subalgebras.

5.1. Lemma. *If $x\in X$ has a countable base of neighborhoods and if f belongs to all maximal Φ-subalgebras of $D(X)$, then f is constant on some neighborhood of x.*

Proof. Suppose that $f\in D(X)$ is not constant on any neighborhood of x, and that x has a countable base of neighborhoods. Then f takes infinitely many values on each neighborhood of x. Let $\{U_n\colon n\in N\}$ be a base of neighborhoods of x such that $\operatorname{cl}U_{n+1}\subseteq U_n$ for all $n\in N$. Let $(a_n)_{n\in N}$ be a sequence in X such that $f(a_i)\neq\pm\infty$, $a_i\in U_i$, $a_i\neq x$, and $i\neq j$ implies that $f(a_i)\neq f(a_j)$. We can assume that $a_i\in U_i\sim\operatorname{cl}U_{i+1}$ (by choosing subsequences of $(U_i)_{i\in N}$ and $(a_i)_{i\in N}$). Now, $\big(f(a_n)\big)_{n\in N}$ is a sequence in $\boldsymbol{R}$ converging to $f(x)$, so $\big(f(a_n)\big)_{n\in N}$ has either an increasing or a decreasing subsequence; suppose the former, and—by a change of notation—suppose that $\big(f(a_n)\big)_{n\in N}$ is increasing.

Case 1. If $f(x)=\infty$. We can assume that $f(a_n)>0$ for all $n\in N$. For each $n\in N$, there exists $g_n\in C(X)$ satisfying

$$g_n(a_n)=(-1)^n\big(f(a_n)\big)^{-1/2},\qquad g_n[(X\sim U_n)\cup\operatorname{cl}U_{n+1}]=\{0\},$$

the g_n are alternately positive and negative, and g_n is bounded by $g_n(a_n)$. Since $\{g_n|_{X\sim\{x\}}\colon n\in N\}$ is locally finite and $\big(f(a_n)\big)_{n\in N}$ converges to ∞, we may define $g=\sum g_n$, continuous. However, $fg(a_n)=(-1)^n\big(f(a_n)\big)^{1/2}$, so fg is not defined at x.

Case 2. If $f(x)<\infty$. Since f belongs to a maximal Φ-subalgebra A iff $f-\boldsymbol{f(x)}\in A$, we may assume $f(x)=0$; also, $f\in A$ iff $-f\in A$, so we treat the case $f(x)=0$, $1>f(a_n)>0$ and $\big(f(a_n)\big)_{n\in N}$ is decreasing. There exist $g_n\in C(X)$ satisfying:

$$g_n[\operatorname{cl}(U_n)\sim U_{n+1}]=\left\{\frac{2+(-1)^n}{f(a_n)}\right\},$$

$$0\leqslant g_n\leqslant\frac{2+(-1)^n}{f(a_n)},$$

$$g_n[(\operatorname{cl}U_{n+2})\cup(X\sim U_{n-1})\cup\{a_{n-1},a_{n+1}\}]=\{0\}.$$

Define g by

$$g(y)=\sup\{g_n(y)\colon n\in N\}\quad(y\neq x),\qquad g(x)=\infty.$$

As before, $g\in D(X)$. For n odd, $fg(a_n)=f(a_n)g_n(a_n)=1$; for n even, $fg(a_n)=3$. Hence $fg(x)$ cannot be defined so that fg is continuous.

In either case, g belongs to some maximal Φ-subalgebra A and $f\notin A$.

5.2. Since maximal Φ-subalgebras of $D(X)$ are local and contain the constant functions, we have the following corollary.

Corollary. *Let X be a 1st-countable space. Then an element f of $D(X)$ belongs to all maximal Φ-subalgebras of $D(X)$ iff f is locally constant.*

The restriction that X be 1st-countable cannot be dropped completely. For $D(\beta N)$ contains non-locally constant functions; e.g., any unbounded function of $D(\beta N)$ is not locally constant. But βN is extremally disconnected, so $D(\beta N)$ is a Φ-algebra.

5.3. Theorem. *Let Y be a 1st-countable space and let $X=\beta Y$. Then $f\in D(X)$ belongs to all maximal Φ-subalgebras of $D(X)$ iff $f|_Y$ is locally constant.*

Proof. If f belongs to all maximal Φ-subalgebras, lemma 5.1 implies that f is constant on some X-neighborhood of each point of Y—since each point of Y has a countable base of neighborhoods in X (see [3], section 9.7).

Conversely, suppose that $f|_Y$ is locally constant. Let A be a maximal Φ-subalgebra of $D(X)$. Let $a_i\in A$ $(0\leqslant i\leqslant n)$; then since $f|_Y$ is locally constant, $g=\sum(a_i|_Y)(f|_Y)^i$ is defined and continuous. Since Y is C^*-embedded in X, the extension, $\sum a_if^i$, of g is defined. By corollary 2.2, f belongs to A.

5.4. As the next theorem shows, not all Φ-subalgebras of $D(X)$ are intersections of maximal Φ-subalgebras.

Theorem. *If X is zero-dimensional, then the intersection of all maximal Φ-subalgebras of $D(X)$ separates points of X.*

Proof. $D(X)$ contains a family of locally constant functions which separates points. The intersection contains all locally constant functions.

For 1st-countable spaces, the converse of this theorem is also true.

5.5. Theorem. *If X is 1st-countable, then X is zero-dimensional iff the intersection of all maximal Φ-subalgebras of $D(X)$ separates points of X.*

Proof. By corollary 5.2, $D(X)$ contains a family of locally constant functions which separates points of X. Noting that any member of this family takes on only finitely many values, one easily obtains a family $\mathcal{F}$ of coverings of X each consisting of two disjoint open sets such that for x, $y\in X$, there is $\{U,V\}\in\mathcal{F}$ with $x\in U$ and $y\in V$. Hence X is totally disconnected.

6. Characterization of uniformly closed maximal Φ-algebras. Let A be a Φ-algebra. If

$$U(r)=\{(a,b)\in A\times A\colon |a-b|<r\},$$

then the set of $U(r)$ for strictly positive real numbers r is a base for a uniformity on A. The uniform topology is 1st-countable and convergence in this topology is just uniform convergence as extended real-valued functions on $\mathcal{M}(A)$. If A is a complete uniform space with this uniformity, then A is said to be *uniformly closed*.

The major result of this section is theorem 6.3: There is a unique maximal Φ-subalgebra, $U(X)$, of $D(X)$ containing $C(X)$. $U(X)$ is the set of all functions in $D(X)$ whose coinfinity sets are C^*-embedded. Since every maximal uniformly closed Φ-algebra A contains $C(\mathcal{M}(A))$, this theorem yields a very useful characterization of uniformly closed maximal Φ-algebras.

6.1. If A is a Φ-subalgebra of $D(X)$ containing the constant functions and separating points of X, then by the Stone–Weierstrass theorem, A is uniformly closed iff $C(X) \subseteq A$ iff $A^* = C(X)$ (by [6], 3.7, A is uniformly closed iff A^* is). In view of the first paragraph of 3.3, the following proposition is clear.

PROPOSITION. *Let A be a Φ-subalgebra of $D(X)$ containing the constant functions. Then A is uniformly closed iff A contains $\{f \in C(X)$: f is constant on all stationary sets of $A\}$.*

6.2. LEMMA. *Let Y be a completely regular space, and let $f \in D(Y)$. If fg is defined whenever $g \in (C^*(Y))^+$ with $Z(g) = \mathcal{N}(f)$, then $\mathcal{R}(f)$ is C^*-embedded in Y.*

Proof. First suppose $\mathbf{1} \leqslant f \in D(Y)$ such that fg is defined whenever $g \in (C^*(Y))^+$ with $Z(g) = \mathcal{N}(f)$. Let $\mathbf{1} \leqslant h \in C^*(\mathcal{R}(f))$. Define $g\colon Y \to \boldsymbol{R}$ by $g(x) = h(x)/f(x)$ for $x \in \mathcal{R}(f)$, and $g(x) = 0$ for $x \in \mathcal{N}(f)$. Then $g \in (C^*(Y))^+$ and $Z(g) = \mathcal{N}(f)$. Now, fg extends h over Y. If $k \in C^*(\mathcal{R}(f))$, then $k + \boldsymbol{n} \geqslant \mathbf{1}$ for some $n \in N$, so $k+\boldsymbol{n}$, and hence k, has an extension over Y. Hence, in this case, $\mathcal{R}(f)$ is C^*-embedded.

Let $f \in D(Y)$ such that fg is defined whenever g belongs to $(C^*(Y))^+$ with $Z(g) = \mathcal{N}(f)$. Let $f_1 = f^+ + \mathbf{1}$ and $f_2 = f^- + \mathbf{1}$. Let $h \in (C^*(Y))^+$ with $Z(h) = \mathcal{N}(f_1)$; then $Z(h) = \mathcal{N}(f^+)$. Let $k \in (C^*(Y))^+$ with $Z(k) = \mathcal{N}(f^-)$ and $k[Z(f^-)] = \{1\}$. Then $Z(hk) = \mathcal{N}(f)$ and hk agrees with h on pos f. Now, by hypothesis, hkf is defined and hkf equals hf^+ on pos f; f^+ is bounded on $\{y \in Y\colon f(y) < 1\}$, so hf^+ is defined. Then $hf_1 = hf^+ + h$ is defined since a bounded function can always be added to any element of $D(Y)$. By the first part, $\mathcal{R}(f_1)$ is C^*-embedded in Y. Similarly, $\mathcal{R}(f_2)$ is C^*-embedded in Y. Since the intersection of two open dense C^*-embedded sets is always C^*-embedded (see [3], 9 N), $\mathcal{R}(f) = \mathcal{R}(f_1) \cap \mathcal{R}(f_2)$ is C^*-embedded.

6.3. THEOREM. *Let Y be a completely regular space and let $U(Y) = \{f \in D(Y)\colon fg$ is defined for all $g \in C(Y)\}$. Then*

(i) *$f \in U(Y)$ iff $\mathcal{R}(f)$ is C^*-embedded in Y, and*

(ii) *$U(Y)$ is the unique maximal Φ-subalgebra of $D(Y)$ containing $C(Y)$.*

Proof. (i) follows from the lemma.

Since $C(Y)$ is an algebra, $C(Y) \subseteq U(Y)$; moreover, any subalgebra of $D(Y)$ containing $C(Y)$ lies in $U(Y)$. To show that $U(Y)$ is an algebra, it suffices to observe that if f and g belong to $U(Y)$, then $\mathcal{R}(f) \cap \mathcal{R}(g)$ is C^*-embedded.

While the last statement in the proof of 6.3 is known (see [3], 9 N), I cannot resist giving the following proof (a portent of things to come): If $h \in C^*(\mathcal{R}(f))$, then h has an extension $h' \in C^*(Y)$; since gh' is defined, $(g|_{\mathcal{R}(f)})h$ is defined. This implies, by the lemma, that $\mathcal{R}(f) \cap \mathcal{R}(g)$ is C^*-embedded in $\mathcal{R}(f)$, hence in Y.

Hence $D(X)$ contains a unique maximal uniformly closed Φ-subalgebra which separates points of X.

6.4. A space Y is said to be a CM-*space* iff $C(Y)$ is a maximal Φ-subalgebra of $D(Y)$, or equivalently, iff $C(Y) = U(Y)$.

COROLLARY. *A space Y is a* CM-*space iff Y has no proper dense C^*-embedded cozero subset.*

Proof. If $X \sim Z(f)$ is dense and C^*-embedded, then f^{-1} is defined and if, in addition $Z(f) \neq \emptyset$, then f^{-1} belongs to $U(Y) \sim C(Y)$.

6.5. The next corollary was pointed out to the author by J. R. Isbell.

COROLLARY. *If X is a compact space of power $< 2^c$, then X is a* CM-*space.*

Proof. If X had a proper dense C^*-embedded cozero subset, S, then, by [3], 9.5, $X \sim S$ would have power at least 2^c.

6.6. THEOREM. *$U(X)$ is the intersection of all of the maximal l-subspaces of $D(X)$.*

Proof. Suppose that f belongs to the intersection. Since every element of $D(X)$ belongs to some maximal l-subspace, $f+h$ is defined for all $h \in D(X)$. Let $g \in C^*(\mathcal{R}(f))$. Define $h\colon X \to \gamma R$ by $h(x) = g(x) - f(x)$ for $x \in \mathcal{R}(f)$ and $h(x) = -f(x)$ for $x \in \mathcal{N}(f)$. Since g is bounded, h is continuous; $\mathcal{N}(h) = \mathcal{N}(f)$ so $h \in D(X)$. Clearly $f+h$ extends g over X. By theorem 6.3, $f \in U(X)$.

Suppose $f \in U(X)$ and let $h \in D(X)$. Define $k\colon \mathcal{R}(f) \to \gamma R$ by $k(x) = f(x) + h(x)$ for $x \in \mathcal{R}(f) \cap \mathcal{R}(h)$ and $k(x) = h(x)$ for $x \in \mathcal{R}(f) \sim \mathcal{R}(h)$. Since every point of $\mathcal{R}(f)$ has a neighborhood on which f is bounded, k is continuous. Since $f \in U(X)$, $\mathcal{R}(f)$ is C^*-embedded, so k has a continuous extension k' over X. Since $\mathcal{N}(k') \subseteq \mathcal{N}(f) \cup \mathcal{N}(h)$, $k' \in D(X)$. Hence $f+h$ is defined for all $h \in D(X)$. This implies that f belongs to every maximal subspace of $D(X)$, hence to every maximal l-subspace of $D(X)$ (see 2.1).

6.7. COROLLARY. *The following are equivalent:*

(a) $D(X)$ *is a vector space.*

(b) $D(X)$ *is a Φ-algebra.*

(c) *Every dense cozero subset of X is C^*-embedded.*

Proof. (The equivalence of (b) and (c) is a result of [6]—see 1.3.) Each of the three statements is equivalent to $D(X) = U(X)$.

6.8. By a construction similar to the proof (case 2) of 5.1, one easily obtains the following lemma.

LEMMA. *Let Y be a completely regular space and let x be a non-isolated point of Y with countable base of neighborhoods. Then $Y\sim\{x\}$ is not C^*-embedded in Y.*

6.9. PROPOSITION. *If x is a 1st-countable point of Y and if $f \in D(Y)$ is infinite at x, then there is some $g \in C(Y)$ for which fg is not defined.*

Proof. By the lemma, $\mathcal{R}(f)$ is not C^*-embedded, since $Y\sim\{x\}$ is not and $\mathcal{R}(f)$ is dense in $Y\sim\{x\}$. By theorem 6.3, there is a g with the desired property.

6.10. COROLLARY. *Every 1st-countable space is a* CM-*space.*

6.11. PROPOSITION. *If $X = \beta Y$ for some non-pseudocompact space Y, then X is not a* CM-*space.*

Proof. In this case, $C(Y)$ properly contains $C(X)$ (in $D(X)$).

6.12. THEOREM. *$U(X)$ is isomorphic to $C(Z)$ for some Z iff $X = \beta Y$ for some* CM-*space Y, and in this case, $U(X)$ is isomorphic to $C(Y)$.*

Proof. If $U(X)$ is isomorphic to $C(Z)$, then 1.6 implies that $U(X) = C\big(\mathcal{R}(U(X))\big)$ and that $\mathcal{R}(U(X))$ is dense and C^*-embedded in X; i.e., $X = \beta\big(\mathcal{R}(U(X))\big)$. Let $Y = \mathcal{R}(U(X))$. If $S = Y\sim Z(f)$ is a dense C^*-embedded cozero set of Y, then S is C^*-embedded in X, so $|f|^{-1}$ has an extension $g \in D(X)$. Now, $Y \subseteq \mathcal{R}(g) \cap Y \subseteq S$, so by 6.4, Y is a CM-space.

For the converse, if $f \in U(X)$, then (by 6.3 and [3], 9 N), $\mathcal{R}(f) \cap Y$ is C^*-embedded in Y; since $\mathcal{R}(f)$ is open, $\mathcal{R}(f) \cap Y$ is dense in Y; clearly $\mathcal{R}(f) \cap Y$ is a cozero set of Y; hence, since Y is a CM-space, $Y \subseteq \mathcal{R}(f)$. Since Y is dense and C^*-embedded in X, 6.3 implies that $f \to f|_Y$ is an isomorphism of $U(X)$ onto $C(Y)$.

6.13. COROLLARY. *If X has a minimal C^*-embedded dense cozero subset, Y, then $C(Y) = U(X)$.*

For example, $U(\beta \boldsymbol{R}) = C(\boldsymbol{R})$.

6.14. PROPOSITION. *If $U(X)$ is isomorphic to $C(Y)$ for a pseudocompact space Y, then X is a* CM-*space.*

Proof. If $\varphi\colon C(Y) \to U(X)$ is an isomorphism, then $\varphi(\boldsymbol{n}) = n\varphi(\boldsymbol{1}) = n\boldsymbol{1} = \boldsymbol{n}$. Hence every element of $U(X)$ is bounded.

6.15. Suppose that $U(X)$ is $C(Z)$ for some space Z, and let B be a Φ-algebra with structure space X. Then $B = C(Y)$ for some Y iff B is uniformly closed and closed under inversion. For, in this case, since B is uniformly closed, separates points of X and contains the constant functions, $C(X) \subseteq B$. Since $U(X)$ is the unique maximal Φ-algebra containing $C(X)$, $B \subseteq U(X)$. Hence $\mathcal{R}(U(X)) \subseteq \mathcal{R}(B)$. By 1.6, $\mathcal{R}(U(X))$ is dense and C^*-embedded in X, so $\mathcal{R}(B)$ is dense and C^*-embedded. Hence, by 1.6, B is isomorphic to $C(\mathcal{R}(B))$. Necessity is immediate from 1.6.

7. Stationary sets for uniformly closed maximal Φ-subalgebras. It is still an open question whether uniformly closed maximal Φ-subalgebras of $D(X)$ must separate points of X. This section provides some partial answers and gives a condition which stationary sets for such a Φ-algebra must satisfy.

7.1. PROPOSITION. *Let A be a uniformly closed maximal Φ-subalgebra of $D(X)$. If $\mathcal{M}(A)$ is a* CM-*space, then A separates points of X, and X is a* CM-*space.*

Proof. Since $C(\mathcal{M}(A))$ is maximal and $C(\mathcal{M}(A)) \subseteq A$, we have $A = C(\mathcal{M}(A))$, so A contains only bounded functions. Hence $A \subseteq C(X)$, and, since A is maximal, $U(X) = A = C(X)$.

7.2. COROLLARY. *If A is a uniformly closed maximal Φ-subalgebra of $D(X)$ and if $\mathcal{M}(A)$ is 1st-countable, then $\mathcal{M}(A) = X$.*

Proof. This follows from 6.10 and 3.2.

7.3. THEOREM. *If X is a metric space, then every uniformly closed maximal Φ-subalgebra of $D(X)$ separates points of X. Hence $C(X)$ is the unique uniformly closed maximal Φ-subalgebra of $D(X)$.*

Proof. Let A be a uniformly closed maximal Φ-subalgebra of $D(X)$. Let $\varrho\colon X \to X/\mathrm{r}$ be the natural projection. By [11], 3.12, ϱ induces an upper semicontinuous decomposition of X; by [11], 5.20 ff, $\mathcal{M}(A)$ is a metric space. By corollary 7.2, $\mathcal{M}(A) = X$.

7.4. PROPOSITION. *If X is 1st-countable and if A is a uniformly closed maximal Φ-subalgebra of $D(X)$, then whenever $f \in A$ with $\mathcal{N}(f)$ non-empty, $\mathcal{N}(f)$ is a union of stationary sets each with more than one point.*

Proof. Clearly, if any stationary set meets $\mathcal{N}(f)$, then it is contained in $\mathcal{N}(f)$.

Suppose that there is some $x \in \mathcal{N}(f)$ such that $\{x\}$ is a stationary set of A. Let $\varrho\colon X \to \mathcal{M}(A)$ be the projection; then $\varrho(x)$ is a 1st-countable point of $\mathcal{M}(A)$ ([11], 3.12). By proposition 6.9, A cannot contain $C(\mathcal{M}(A))$. But A is uniformly closed and separates points of $\mathcal{M}(A)$.

7.5. PROPOSITION. *Let A be a uniformly closed maximal Φ-subalgebra of $D(X)$. If S is a non-trivial stationary set of A and if U is any open set containing S, then $U\sim S$ contains points at which some $f \in A$ is infinite.*

Proof. Suppose that U is an open set containing S for which $U\sim S\subseteq\mathcal{R}(A)$. Since U contains a point of $X\sim\mathcal{R}(A)$ (by proposition 4.5), $S=\mathcal{N}(g)$ for some $g\in A$. Let $\varrho\colon X\to X/\mathfrak{r}$ be the projection. Now A is the set of all $f\in D(X/\mathfrak{r})$ for which fh is defined whenever $h\in C(X/\mathfrak{r})$, so if $f\in A$, then $\varrho[\mathcal{R}(f)]$ is C^*-embedded in $X/\mathfrak{r}$. Let $x_0=\varrho[S]$. Then $X/\mathfrak{r}\sim\{x_0\}$ is C^*-embedded in $X/\mathfrak{r}$. Let $x\neq y$ in S; let $f\in C(X)$ with $f(x)=1$ and, for some closed neighborhood W of x contained in $U\sim\{y\}$, $f[X\sim W]=\{0\}$. By proposition 4.5 (with $V=U\sim S$), no other non-trivial stationary set meets U, so every $k\in C^*(U\sim S)$ can be dropped to $k'\in C^*(\varrho[U]\sim\{x_0\})$. It follows from the fact that $X/\mathfrak{r}\sim\{x_0\}$ is C^*-embedded in $X/\mathfrak{r}$ that $\varrho[U]\sim\{x_0\}$ is C^*-embedded in $\varrho[U]$, so k' has an extension k'' over $\varrho[U]$; $k''\circ\varrho$ extends k over U. Hence $U\sim S$ is C^*-embedded in U. Any $h\in A$ taking on an infinite value on U is infinite on S and nowhere else on U. Hence $\sum h_i(f|_U)^i$ is defined whenever $h_i\in A$ $(0\leqslant i\leqslant n)$; $\sum h_i(f|_{X\sim W})^i=\{0\}$. Thus $f\in A$. But f is not constant on S.

8. Non-uniformly closed maximal Φ-subalgebras. The existence of uniformly closed maximal Φ-algebras is immediate from section 6. If $D(X)$ is a Φ-algebra, then $D(X)$ contains no non-uniformly closed maximal Φ-subalgebras. Otherwise, $D(X)$ contains many such Φ-subalgebras.

8.1. THEOREM. *If $f\in D(X)\sim U(X)$, then f belongs to a maximal Φ-subalgebra M_f of $D(X)$ which is not uniformly closed and whose only non-trivial stationary sets are contained in $\mathcal{N}(f)$.*

Proof. Since $\mathcal{R}(f)$ is not C^*-embedded, lemma 6.2 implies that there is $g\in\big(C^*(X)\big)^+$ with $Z(g)=\mathcal{N}(f)$ for which fg is not defined. Let $S=\{h\in C(X)\colon h$ vanishes on a neighborhood of $\mathcal{N}(f)\}$ and let M_f be a maximal subalgebra of $D(X)$ containing $S\cup\{f\}$. Clearly the only non-trivial stationary sets of M_f lie in $\mathcal{N}(f)$. Now, if $g_n=(g-\mathbf{1}/\boldsymbol{n})^+$, then $g_n\in S\subseteq M_f$; but $(g_n)_{n\in N}$ converges (uniformly) to $g\notin M_f$. Hence M_f is not uniformly closed.

8.2. COROLLARY. *If $D(X)$ is not a Φ-algebra, then $D(X)$ contains a non-uniformly closed maximal Φ-subalgebra.*

8.3. EXAMPLE. *$D([0,1])$ contains at least c distinct (but perhaps isomorphic) non-uniformly closed maximal Φ-subalgebras.* (By starting with one of the M_f of this example and a family of c homeomorphisms of $[0,1]$ onto itself, one obtains a family of c *isomorphic* but distinct maximal Φ-subalgebras.)

For each $x\in[0,1]$, let $f_x\in D([0,1])$ with $\mathcal{N}(f_x)=\{x\}$. Then if $x\neq y$, there is $g\in C([0,1])$ vanishing on a neighborhood of x and having the property that gf_y is not defined (see 6.9). Then (by construction of M_f), $g\in M_{f_x}\sim M_{f_y}$.

8.4. A Φ-subalgebra A of $D(X)$ is *closed under composition* iff whenever $f\in A$ and $g\in C(\mathbf{R})$, then $g\circ(f|_{\mathcal{R}(f)})$ has an extension belonging to A. The extension is denoted $g(f)$.

PROPOSITION. *Let A be a non-uniformly closed maximal Φ-subalgebra of $D(X)$. If X is 1st-countable or locally connected, then A is not closed under composition.*

Proof. First, suppose that X is locally connected. Let $\mathbf{1}\leqslant f\in A$ with $f(x_0)=\infty$ and let $g\in C(\mathbf{R})$ be given by $g(x)=\sin x$. Let U be a connected neighborhood of x_0; $f[U]$ is connected and contains ∞, so it contains $n\pi+\pi/2$ for two successive integers n. Hence every neighborhood of x_0 contains points at which $g\circ(f|_{\mathcal{R}(f)})$ is 1 and -1.

Suppose that X is 1st-countable. Let $f\in A\sim U(X)$ and let $x_0\in\mathcal{N}(f^+)$. Let $(a_i)_{i\in N}$ be a sequence in $\mathcal{R}(f)$ converging to x_0; then $\big(f(a_i)\big)_{i\in N}$ converges to ∞. We suppose that all $f(a_i)$ are distinct. Let $g\in C(\mathbf{R})$ with $g\big(f(a_i)\big)=(-1)^i$. Then $g\circ(f|_{\mathcal{R}(f)})$ has no continuous extension over X.

9. Quotient fields. A totally ordered set S is an *η_1-set* iff whenever Q and R are (perhaps empty) countable subsets of S with every element of Q less than every element of R (denoted $Q<R$), then there exists $s\in S$ greater than every element of Q and less than every element of R. A totally-ordered field which is an η_1-set is called an *η_1-field*.

It is proved in [7], 1.5 that every prime ideal in a uniformly closed Φ-algebra is an l-ideal.

Let A be a uniformly closed Φ-algebra. A maximal ideal I of A is said to be *real* iff A/I is isomorphic to the real field; otherwise, it is *hyper-real*. In either case, A/I is a totally-ordered field containing a cannonical copy of $\mathbf{R}$—i.e., the set of $I(r\cdot 1)$ for $r\in\mathbf{R}$.

It is known ([3], 13.8, 13.4) that if I is a hyper-real maximal ideal of $C(Y)$, then $C(Y)/I$ is a real closed η_1-field.

This statement is not true for arbitrary uniformly closed Φ-algebras. M. Henriksen, J. R. Isbell, and D. G. Johnson give an example ([7], 1.9) of a uniformly closed Φ-algebra A, closed under composition, with a hyper-real maximal ideal M such that A/M has a countable cofinal subset.

It is the purpose of this section to show that the above-mentioned theorem does hold for uniformly closed *maximal* Φ-algebras.

9.1. If A is any uniformly closed maximal Φ-algebra, then $A=U\big(\mathcal{M}(A)\big)$. The next proposition is clear from 6.3.

PROPOSITION. *Every uniformly closed maximal Φ-algebra is closed under composition.*

9.2. The next proposition is proved in the same way as the special case for $C(Y)$. See [3], 7.15.

PROPOSITION. *Let A be a uniformly closed Φ-algebra. Every prime ideal P of A contains*

$$O_x = \{f \in A\colon Z(f) \text{ is a neighborhood of } x\}$$

for a unique $x \in \mathcal{M}(A)$. Hence M_x is the unique maximal ideal of A containing P.

9.3. THEOREM. ([7], 1.7.) *Let P be a prime ideal of the uniformly closed Φ-algebra A. If S and T are non-empty countable subsets in A/P such that $S < T$, then there is an $a \in A/P$ such that $S \leqslant a \leqslant T$.*

9.4. A Φ-algebra A is *closed under countable composition* iff whenever $(f_n)_{n\in N}$ is a sequence in A and $g \in C(\mathbf{R}^\infty)$, then there exists $h \in A$ such that $h(x) = g(f_1(x), f_2(x), \ldots)$ for all $x \in \bigcap \mathcal{R}(f_n)$. If it were known that every maximal uniformly closed Φ-algebra is closed under countable composition, the next theorem would be an easy consequence of the following result ([7], 2.6): *Every Φ-algebra closed under countable composition is a homomorphic image of $C(Y)$ for some space Y.* However, it is not known whether this is true, so a direct approach must be taken.

THEOREM. *Let M_y be a hyper-real maximal ideal of $U(X)$ containing a prime ideal P. Then $U(X)/P$ has no countable cofinal subset.*

Proof. (This proof is a modification of the proof [12], 2.6, for $C(Y)$.) Let $a_1 \leqslant a_2 \leqslant \ldots$ be a sequence of elements of $U(X)/P$. We can suppose that $a_1 \geqslant 0$. By [3], 13.5, there exist $f_1 \leqslant f_2 \leqslant \ldots$ in $U(X)^+$ with $P(f_i) = a_i$. Since M_y is hyper-real, $U(X)/M_y$ is non-Archimedean—let $\mathbf{1} \leqslant g \in U(X)$ such that $M_y(g) \geqslant n$ in $U(X)/M_y$ for all $n \in N$. Note that $g(y) = \infty$. We can assume that $\mathcal{N}(f_i)$ contains $\mathcal{N}(g)$ (since we can replace f_i with $f_i \vee g$). For $i \in N$, define $\Phi_i \in C(\mathbf{R})$ by $\Phi_i(x) = 0$ if $x \leqslant i-1$ or $x \geqslant i+1$, $\Phi_i(i) = 1$, and Φ_i is linear on $[i-1, i]$ and $[i, i+1]$.

Since $U(X)$ is closed under composition, $\Phi_i(g) \in U(X)$ for all $i \in N$. Let $x \in \mathcal{R}(g)$; let $V_x = \{z \in X\colon g(x)-1 < g(z) < g(x)+1\}$. For $z \in V_x$,

$$\sum_1^\infty \Phi_i\big(g(z)\big)f_i(z) = \sum_{n-2}^{n+2} \Phi_i\big(g(z)\big)f_i(z)$$

where n is the greatest integer in $g(x)$. Hence the function h' given by

$$h'(z) = \sum_1^\infty \Phi_i\big(g(z)\big)f_i(z) \qquad (z \in \mathcal{R}(g))$$

is defined and continuous on $\mathcal{R}(g)$. Since $\mathcal{R}(g)$ is C^*-embedded in X, h' has a continuous extension h; by the Baire category theorem, $\bigcap \mathcal{R}(f_i)$ is dense; hence $\bigcap \mathcal{R}(f_i) \subseteq \mathcal{R}(h)$ implies $h \in D(X)$.

Let $k \in C^*(X)$. Since $\Phi_i(g)f_ik$ is defined for all $i \in N$, $\sum_1^\infty\big(\Phi_i(g)f_ik\big)|_{\mathcal{R}(g)}$ is defined and continuous. But $\mathcal{R}(g)$ is C^*-embedded in X, so this function has a continuous extension which must be hk. Hence hk is defined for all $k \in C^*(X)$, so $h \in U(X)$.

Let $U_n = \{x \in X\colon g(x) > n\}$. For $x \in U_n \cap \mathcal{R}(g)$,

$$h(x) = \sum_n^\infty \Phi_i\big(g(x)\big)f_i(x) \geqslant f_n(x),$$

so $h \geqslant f_n$ on the neighborhood U_n of y. Hence $Z\big((h-f_n)^-\big)$ is a neighborhood of y, so $(h-f_n)^- \in O_y \subseteq P$. This implies $P(h) \geqslant P(f_n) \geqslant a_n$ for all $n \in N$.

It is interesting to note that among all uniformly closed Φ-algebras with structure space X, the largest, $U(X)$ and the smallest, $C(X)$ share an algebraic property which need not be enjoyed by Φ-algebras between them. It seems that in going from $C(X)$ to a larger Φ-algebra $A \neq U(X)$, one can add enough functions to create problems without adding enough to solve them.

9.5. This theorem together with the Henriksen, Isbell, Johnson result (9.3) yields the following, just as in the $C(Y)$ case (see [3], 13.8).

THEOREM. *If M is a hyper-real maximal ideal of a uniformly closed maximal Φ-algebra A, then A/M is an η_1-field.*

10. Completion.

10.1. Let C be a Φ-subalgebra of B. Then C is said to be *order-dense* in B iff $b = \sup\{c \in C\colon c \leqslant b\}$ for all $b \in B$.

PROPOSITION. *Let B be a Φ-algebra and let C be a Φ-subalgebra of B containing the identity element of B. Then there is a continuous onto function $\varrho\colon \mathcal{M}(B)\to\mathcal{M}(C)$ which induces the embedding of C in B. Moreover, if C is order-dense in B, then ϱ is tight.*

Proof. We have $C \subseteq D\big(\mathcal{M}(B)\big)$, C contains the constant functions, and $\mathcal{M}(B)$ is compact. Identify $\mathcal{M}(C)$ with $\mathcal{M}(B)/\mathrm{r}$ by theorem 3.2. Theorem 3.3 implies that ϱ, the projection of $\mathcal{M}(B)$ onto $\mathcal{M}(C)$ induces the embedding of C into B (*via* $f \to f \circ \varrho$).

We will consider B and C as Φ-subalgebras of $D\big(\mathcal{M}(B)\big)$. For $x \in \mathcal{M}(B)$, M_x^B denotes the maximal l-ideal $\{f \in B\colon fg(x) = 0 \text{ for all } g \in B\}$, and M_x^C denotes $\{f \in C\colon fg(x) = 0 \text{ for all } g \in C\}$. We next prove that if C is dense in B, then $M_x^C = C \cap M_x^B$. Suppose $|fg|(x) > 0$ for some $f \in C$, $g \in B$. Denseness of C in B implies that there exists $h \in C$ with $h \geqslant |g|$, so $|fh|(x) > 0$.

Let $\{M \in \mathcal{M}(B)\colon a \notin M\}$ be a non-empty basic open subset of $\mathcal{M}(B)$.

Let $b \in C$ such that $0 < b \leqslant |a|$. If $M \in \varrho^{\leftarrow}(\{I \in \mathcal{M}(C): b \notin I\})$, then $b \notin M$, so $a \notin M$. Since $b \neq 0$, $\{I \in \mathcal{M}(C): b \notin I\}$ is non-empty. Hence φ is tight.

10.2. D. G. Johnson has proved [8] the following theorem:

If A is a Φ-algebra, there is a complete isomorphism θ of A onto an order-dense Φ-subalgebra of a complete Φ-algebra A'. Moreover, A' is unique in the following sense: if Φ is an isomorphism of A onto a dense Φ-subalgebra of a complete Φ-algebra B, then there is an isomorphism Ψ of B onto A' such that $\Psi \circ \Phi = \theta$.

The next proposition characterizes the structure space of A'.

If B is a complete Φ-algebra, then B is uniformly closed ([6], page 94), so B^*, the set of bounded elements of B, is isomorphic to $C(\mathcal{M}(B))$. Since B^* is complete, $\mathcal{M}(B)$ is extremally disconnected ([15], 12).

PROPOSITION. *Let A be a Φ-algebra. Then the structure space of the completion of A is the minimal projective extension of the structure space of A: $\mathcal{M}(A') = (\mathcal{M}(A))_\infty$.*

Proof. As remarked above, $\mathcal{M}(A')$ is extremally disconnected. By proposition 10.1 there is a tight map of $\mathcal{M}(A')$ onto $\mathcal{M}(A)$. By the uniqueness statement in 1.1, the proof is complete.

It is remarked in [13] that the above proposition holds when $A = C(Y)$ for some Y.

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