STUDIA MATHEMATICA, T. XXX. (1968)

The sets of convergence of power series in B_0-algebras

by

Z. SAWOŃ (Warszawa)

A B_0-*algebra* is a completely metrizable, locally convex topological algebra over real or complex scalars.

We shall also assume that the algebras have the unit element.

The topology in a B_0-algebra X may be introduced by means of a denumerable sequence of pseudonorms satisfying

(1) $$\|x\|_i \leqslant \|x\|_{i+1}, \quad i = 1, 2, \ldots,$$

and

(2) $$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1}$$

(see [3]).

A sequence $x_n \to 0$ if and only if $\lim\limits_n \|x_n\|_i = 0$ for $i = 1, 2, \ldots$

A B_0-algebra X is called *m-convex* if there exists an equivalent system of pseudonorms satisfying

(3) $$\|xy\|_i \leqslant \|x\|_i\|y\|_i, \quad i = 1, 2, \ldots$$

Let X be a B_0-algebra and let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers. We write

(4) $$V(a_n) = \Big\{x \in X\colon \sum_{n=0}^{\infty} a_n x^n \text{ is convergent}\Big\}.$$

This paper contains some theorems on topological properties of sets $V_{(a_n)}$; some of those theorems are generalizations of the theorems proved by W. Żelazko in [3].

For instance, in [3] it is proved that if X is an m-convex B_0-algebra which is not a Q-algebra, then every function $\sum\limits_{n=0}^{\infty} a_n x^n$ convergent for every x belonging to a non-void open subset of X converges for every $x \in X$.

In this paper we replace an open set by a set A such that $\operatorname{int}\bar{A} \neq \emptyset$.

It is easy to see that

1. THEOREM. *If X is a B_0-algebra with a unit and $V_{(a_n)} = X$, then*

$$\overline{\lim_n} \sqrt[n]{|a_n|} = 0.$$

2. DEFINITION. Let X be a B_0-algebra and let $\mathfrak{M}(X)$ be the set of all continuous, linear and multiplicative functionals on X. If there exists an $M > 0$ and a positive integer i such that for every $x \in X$ and $x^* \in \mathfrak{M}(X)$

$$|x^*(x)| \leqslant M\|x\|_i, \tag{5}$$

then X is said to have property (α).

It follows from Michael's theorem [2] that if X is an m-convex B_0-algebra, then the following conditions are equivalent:

(i) X is a Q-algebra;

(ii) X has the (α)-property.

3. THEOREM. *If X is an m-convex B_0-algebra and not a Q-algebra, then $V_{(a_n)} = X$ or $V_{(a_n)}$ is a nowhere-dense set.*

Proof. Write

$$\varrho = \overline{\lim_n} \sqrt[n]{|a_n|}.$$

If $\varrho = 0$, then $V_{(a_n)} = X$. If $\varrho = +\infty$, then

$$V_{(a_n)} \subset H_{x^*} = \{x \in X: x^*(x) = 0\},$$

where $x^* \in \mathfrak{M}(X)$. Hence $V_{(a_n)}$ is a nowhere-dense set. Now suppose that $0 < \varrho < +\infty$ and $V_{(a_n)}$ is not a nowhere-dense set. Thus there exist an $\varepsilon > 0$, a positive integer i and an $x_0 \in X$ such that $V_{(a_n)}$ is dense in the set

$$K(\varepsilon, x_0, i) = \{x \in X: \|x - x_0\|_i < \varepsilon\}.$$

Let $x^* \in \mathfrak{M}(X)$. Then $|x^*(x)| \leqslant 1/\varrho$ for every $x \in \overline{K(\varepsilon, x_0, i)}$ and $|x^*(x)| \leqslant 2\|x\|_i/\varrho\varepsilon$ for every $x \in X$, $x^* \in \mathfrak{M}(X)$. Hence X has the (α)-property and X is a Q-algebra, which contradicts our assumptions and the proof is completed.

4. REMARK. If X is a B_0-algebra which does not have the (α)-property and $0 < \overline{\lim_n} \sqrt[n]{|a_n|} < +\infty$, then $V_{(a_n)}$ is a nowheredense set.

In the next part of this paper we shall use the following Chebyshev's theorem of the approximation theory:

Let

$$\Big|\sum_{i=0}^{k} a_i \varrho^i\Big| \leqslant M \quad \text{for} \quad -1 \leqslant \varrho \leqslant 1,$$

where a_i are complex numbers for $i = 1, 2, \ldots, k$. Then (see [1])

$$|a_k| \leqslant M 2^k.$$

5. LEMMA. *Let X be a linear complex space with pseudonorm $\|\cdot\|$. We put*

$$w(\varrho) = \sum_{i=0}^{k} x_i \varrho^i, \quad \text{where} \quad -1 \leqslant \varrho \leqslant 1, \ x_i \in X, \ i = 0, 1, \ldots, k.$$

If for every ϱ such that $-1 \leqslant \varrho \leqslant 1$ the inequality $\|w(\varrho)\| \leqslant M$ holds, then $\|x_k\| \leqslant 2^k M$.

Proof. Let us remark that in the space X^* there exists a functional x^* such that: 1° $|x^*(x)| \leqslant \|x\|$, 2° $|x^*(x_k)| = \|x_k\|$. We write

$$v(\varrho) = \sum_{i=0}^{k} x^*(x_i) \varrho^i.$$

It is easy to see that for $-1 \leqslant \varrho \leqslant 1$

$$|v(\varrho)| = |x^*(w(\varrho))| \leqslant \|w(\varrho)\| \leqslant M$$

and from Chebyshev's theorem we obtain our lemma.

6. THEOREM. *If X is a B_0-algebra with unit e and*

$$\overline{\lim_n} \sqrt[n]{|a_n|} = +\infty,$$

then $V_{(a_n)}$ is a set of the first category.

Proof. Suppose that $\overline{\lim_n} \sqrt[n]{|a_n|} = +\infty$ and that $V_{(a_n)}$ is a set of the second category. Let $x^* \in X^*$ be a linear functional on X such that $x^*(e) = 1$. We write

$$A_k = \{x \in X: |a_n|\,|x^*(x^n)| \leqslant k \text{ for } n = 0, 1, \ldots\}, \qquad k = 1, 2, \ldots$$

It is trivial that $V_{(a_n)} \subset \bigcup_k A_k$. Thus there exists an $\varepsilon > 0$, an $i_0 = 0, 1, \ldots$, a $k_0 = 1, 2, \ldots$, an $x_0 \in X$ such that $K(\varepsilon, x_0, i_0) \subset A_{k_0}$. Besides there exists a $\varrho_0 > 0$ such that if $-\varrho_0 \leqslant \varrho \leqslant \varrho_0$, then $x_0 + \varrho e \in K(\varepsilon, x_0, i_0)$. Hence

$$|a_n| \Big|\sum_{i=0}^{n} \binom{n}{i} x^*(x_0^{n-i}) \varrho^i\Big| \leqslant k_0$$

for every $-\varrho_0 \leqslant \varrho \leqslant \varrho_0$, $n = 0, 1, \ldots$

By Chebyshev's theorem we obtain $|a_n| \leqslant k_0 \cdot 2^n \varrho_0^{-n}$ $(n = 0, 1, \ldots)$ and $\overline{\lim_n} \sqrt[n]{|a_n|} \leqslant M < +\infty$, which contradicts our assumptions and the proof is completed.

7. DEFINITION. Let X be a B_0-algebra. A sequence $(\varrho_n)_{n=0}^{\infty}$ such that $\varrho_n \geqslant 0$, $\varrho_n \to 0$, is called the *rate of growth* of X if

$$O = \{x \in X : \lim_n \varrho_n \sqrt[n]{\|x^n\|_i} = 0,\ i = 0, 1, \ldots\}$$

is a set of the second category.

It is easy to observe that if $\overline{\lim_n} \sqrt[n]{|a_n|} = 0$ and $V_{(a_n)} = X$, then the sequence $\varrho_n = \sqrt[n]{|a_n|}$ is the rate of growth of the algebra X. The converse is true. We have namely

8. THEOREM. *If the sequence (a_n) is the rate of growth of X, then $V_{(\varrho_n^n)} = X$.*

Proof. Let i be a non-negative integer and

$$A_k = \{x \in X : \varrho_n \sqrt[n]{\|x^n\|_i} \leqslant k,\ n = 0, 1, \ldots\}.$$

The sets A_k are closed and $O \subset \bigcup_k A_k$. Thus there exist an $\varepsilon_i > 0$, a j_i, a k_i and an $x_{0,i} \in X$ such that

$$K(\varepsilon_i, x_{0,i}, j_i) \subset A_{k_i}.$$

We may suppose without loss of generality that $j_i \geqslant i$. If $\|x\|_{j_i} < \varepsilon_i$, then for every ϱ such that $-1 \leqslant \varrho \leqslant 1$ we have $x_{0,i} + \varrho x \in K$, whence

$$\varrho_n \left\| \sum_{s=0}^{n} \binom{n}{s} (x_{0,i}^{n-s} x^s) \varrho^s \right\|_i \leqslant k_i^n.$$

From Lemma 5 it follows that $\varrho_n \|x^n\|_i \leqslant 2^{n+1} k_i^n$. Because $j_i \geqslant i$, we obtain for every $x \in X$

$$\varrho_n \|x^n\|_i \leqslant 2 \left(\frac{2k_i}{\varepsilon}\right)^n \|x\|_{j_i}^n.$$

This implies that for $i = 0, 1, \ldots$ there exist an $M_i > 0$ and a $j_i \geqslant i$ such that

$$\varrho_n \|x^n\|_i \leqslant 2 M_i^n \|x\|_{j_i}^n \quad \text{for} \quad x \in X.$$

Because

$$\varphi(z) = \sum_{n=0}^{\infty} \varrho_n^{n-1} z^n$$

is an entire function, we infer that

$$\sum_{n=0}^{\infty} \varrho_n^n \|x^n\|_i < +\infty$$

for every $i = 0, 1, \ldots$, $x \in X$ and $V_{(a_n)} = X$.

Theorem 8 implies the following corollaries.

9. COROLLARY. *If $\lim_n \sqrt[n]{|a_n|} = 0$, then $V_{(a_n)} = X$ or $V_{(a_n)}$ is a set of the first category.*

Proof. Suppose that $V_{(a_n)}$ is a set of the second category and $\lim_n \sqrt[n]{|a_n|} = 0$. Then $\varrho_n = \sqrt[n]{|a_n|^2}$ $(n = 0, 1, \ldots)$ is the rate of growth of X. From Theorem 8 it follows that $V_{(|a_n|^2)} = X$ and $V_{(a_n)} = X$.

10. COROLLARY. *X is an m-convex B_0-algebra if and only if every sequence (ϱ_n) such that $\varrho_n \geqslant 0$, $\varrho_n \to 0$, is the rate of growth of X.*

The proof is an immediate consequence of the following theorem of Mitiagin, Rolewicz and Żelazko (see [3]): X is an m-convex B_0-algebra if and only if $V_{(a_n)} = X$ for every $(a_n)_{n=0}^{\infty}$ such that $\lim_n \sqrt[n]{|a_n|} = 0$.

11. COROLLARY. *If there exists a sequence (a_n) such that $V_{(a_n)}$ is a set of the second category, then X has the rate of growth.*

Proof. Suppose that $V_{(a_n)}$ is a set of the second category. Repeating the argument of the proof of Theorem 8, we infer that for every i there exists a $j_i \geqslant i$ and an $M_i > 0$ such that

$$|a_n| \, \|x^n\|_i \leqslant 2 M_i^n \|x\|_{j_i}^n$$

for every $x \in X$ and $n = 0, 1, \ldots$ Putting

$$\varrho_n = \frac{1}{n+1} |a_n|^{1/n}, \quad n = 0, 1, \ldots,$$

we find that

$$\lim_n \varrho_n \sqrt[n]{\|x^n\|_i} = 0$$

for every $x \in X$ and $i = 0, 1, \ldots$, whence (ϱ_n) is the rate of growth of X, q.e.d.

Finally we give the following

12. THEOREM. *If $\underline{\lim}_n \sqrt[n]{|a_n|} = \delta > 0$ and $V_{(a_n)}$ is a set of the second category, then X is a Q-algebra.*

Proof. Reasoning as before we find that for every i there exist a $j_i \geqslant i$ and $M_i > 0$ such that

$$|a_n| \, \|x^n\|_i \leqslant 2 M_i^n \| \, \|x\|_{j_i}^n \quad \text{for every } x \in X,\ i = 0, 1, \ldots$$

Considering the new system of pseudonorms $\|\cdot\|_m^*$

$$\|x\|_0^* = \|x\|_0, \quad \|x\|_1^* = \|x\|_{j_0}, \quad \|x\|_2^* = \|x\|_{j_{j_0}}, \quad \ldots,$$

we observe that $|a_n|\,\|x^n\|_i \leqslant 2M_i^n(\|x\|_{i+1}^*)^n$ for every $x \in X$, $n = 0, 1, \ldots$

From [3] it follows that X is an m-convex B_0-algebra. By remark 4 we infer that X is a Q-algebra, q.e.d.

References

[1] Н. И. Ахиезер, *Лекции по теории аппроксимации*, Москва 1965.

[2] E. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[3] W. Żelazko, *Metric generalization of Banach algebras*, Rozprawy Matematyczne 47, Warszawa 1965.

Reçu par la Rédaction le 4. 7. 1967

STUDIA MATHEMATICA, T. XXX. (1968)

On the generation of tight measures

by

J. KISYŃSKI (Warszawa)

A non-negative measure μ defined on a σ-algebra $\mathscr{A}$ of subsets of a topological space is called *tight* if

$$\mu(A) = \sup\{\mu(C) \colon A \supset C \in \mathscr{A},\ C\text{—compact}\}$$

for every $A \in \mathscr{A}$. The main result of this paper is theorem 2.1 concerning extensions to tight measures of some set functions in arbitrary Hausdorff spaces. This theorem generalizes a theorem given by Bourbaki ([1], Chap. IV, § 4, N° 10, theorem 5) for locally compact spaces. The proof of theorem 2.1 is based on the idea of Halmos ([3], § 53 and 54) of extending to a measure a certain "semi-regular content" obtained from a given set function. However, the method of such extension presented here is different from that of Halmos.

Throughout this paper the Borel subsets of any topological space X are defined as elements of the smallest σ-algebra of subsets of X, containing all the closed subsets of X.

1. Extension of a content to a tight measure. We call a *content* any non-negative, finite, non-decreasing set function λ defined on the class of all compact subsets of topological space X, such that for every pair A, B of compact subsets of X we have

$$\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$$

and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B) \quad \text{if} \quad A \cap B = \emptyset.$$

We say that a content λ is *tight* if

$$\lambda(A) - \lambda(B) = \sup\{\lambda(C) \colon C \subset A \setminus B,\ C\text{—compact}\}$$

for every pair A, B of compacts such that $B \subset A$.

We say that a content λ is *semi-regular*, if for every compact A and every $\varepsilon > 0$ there is an open set U such that $A \subset U$ and $\lambda(B) < \lambda(A) + \varepsilon$ for every compact $B \subset U$.