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On differentiability in an important class of locally convex spaces

by

J. KIJOWSKI and W. SZCZYRBA (Warszawa)

The most interesting facts in the theory of differentiation in Banach spaces are based on the fact that in a Banach space there exist bounded neighbourhoods of zero. In more general cases these two properties (of being open or bounded) exclude one another. So the generalizations of this theory which are known to us have followed two different ways: defining differentiability "with respect to bounded sets" or "with respect to open sets". The first way was chosen by Sebastião e Silva [12]. A very disagreeable defect of this theory is that differentiability does not imply continuity. However, this implication is valid for Fréchet spaces but it requires a non-trivial proof. Besides, the lack of the mean value theorem in Silva's theory makes it impossible to estimate the remainder by the derivative.

The second way was chosen by several authors, e. g. Marinescu [11], Bastiani [1], Binz [2], Keller [8], Frölicher and Bucher [6]. As is well known (cf. an excellent review of Keller [9]), in the case of a general locally convex space E there does not exist any locally convex topology in $\mathscr{L}(E,F)$ in which the mapping

$$E\times\mathscr{L}(E,F)\ni(h,L)\to L(h)\in F$$

is continuous. No wonder that nobody succeeded in obtaining in the general case the mean value theorem or an equivalent theorem stating that the continuously Gâteaux differentiable mapping is also Fréchet differentiable. Replacing the continuity of a Gâteaux derivative by a much stronger non-topological condition, Marinescu [11], Bastiani [1] and other authors mentioned above obtain the Fréchet differentiability.

We prove in the present paper that in an important for applications class of Fréchet spaces one can develop a theory of differentiation "with respect to open sets" without assuming this condition. Many theorems known in the classical theory of differentiation in Banach spaces are proved here. We give also (in section 3) a natural criterion for the existence of a Fréchet derivative in the Schwartz spaces. As is well known, this class embraces all nuclear spaces, hence all spaces of the theory of distributions[1].

1. Definitions and formal laws of differentiation. Let $E, F, G, \ldots$ be locally convex topological Hausdorff vector spaces over the field $\boldsymbol{R}$ or $\boldsymbol{C}$. By $\mathfrak{N}(E)$ we denote the set of all absolutely convex closed neighbourhoods of zero in E. By $\| \|_U$, $\| \|_V$ etc. we shall denote the seminorms generated by elements $U, V \in \mathfrak{N}(E)$. By $\mathscr{L}_s(E, F)$ we denote the space of all continuous linear mappings from E to F endowed with weak topology (the topology of simple convergence).

Let T be a mapping from E to F (E and F are both topological vector spaces). By a *weak derivative* (*Gâteaux derivative*) of T at a point $x \in E$ we call a mapping $\nabla T(x) \in \mathscr{L}(E, F)$ such that for every vector $h \in E$ the mapping

$$E \ni h \to r_x(h) := T(x+h) - T(x) - \nabla T(x) h \in F$$

has the following property:

$$\frac{r_x(th)}{t} \underset{t\to 0}{\to} 0 \qquad (t \in \boldsymbol{R} \text{ or } t \in \boldsymbol{C}).$$

Definition. Let T be a mapping of an open set $\Omega \subset E$ to F. We say that T is *Fréchet differentiable at a point* $x \in \Omega$ if there exists a mapping $L \in \mathscr{L}(E, F)$ such that the mapping

$$E \ni h \to r_x(h) := T(x+h) - T(x) - Lh \in F$$

has the following property: for every $V \in \mathfrak{N}(F)$ there exists $U \in \mathfrak{N}(E)$ such that for every Moore-Smith sequence $\{h_\lambda\}_{\lambda \in \Lambda}$ convergent to zero in E we have

$$\lim_{\lambda \in \Lambda} \frac{\|r_x(h_\lambda)\|_V}{\|h_\lambda\|_U} = 0.$$

We collect some elementary facts.

PROPOSITION 1. *If the mapping T is Fréchet differentiable at $x \in E$, then the mapping L is uniquely determined.*

The mapping L is then called the *Fréchet derivative* of T at x and denoted by $T'(x)$.

PROPOSITION 2. 1° *Every mapping which is Fréchet differentiable at $x \in E$ is also continuous at x.*

(1) The theory developed in this paper could be applied to some problems in the calculus of variations and the classical field theory. A corresponding paper will be published in Studia Math.

2° *A linear combination of differentiable mappings at x is differentiable at x and its derivative is a linear combination of derivatives.*

3° *A superposition of a mapping T_1 differentiable at $x \in E$ with a mapping T_2 differentiable at $T_1(x) \in F$ is differentiable at x and*

$$(T_2 \circ T_1)'(x) = T_2'(T_1(x)) \circ T_1'(x).$$

2. Some properties of partially continuous mappings in Fréchet spaces. In this section we shall prove some facts which will be necessary in developing the theory of continuously differentiable mappings.

LEMMA 1. *Let E be a Fréchet space and F an arbitrary locally convex space. Let $E \times E \ni (x, h) \to g(x, h) \in F$ be partially continuous and linear in the second variable and partially continuous in the first variable at the point $x_0 \in E$. Then the mapping g is continuous at the point $(x_0, 0) \in E \times E$.*

This lemma is a generalization of the Mazur-Orlicz theorem (see [3]).

Let E be a locally convex space and $U \in \mathfrak{N}(E)$. By E_U we shall denote the normed space $E/N(U)$ (where $N(U) = \{x \in E : \|x\|_U = 0\}$) with a norm $\| \|_U$ defined in a standard way.

Definition. By a *Schwartz space* (S-space) we mean a locally convex space E which satisfies the following condition: for every $U \in \mathfrak{N}(E)$ there exists $W \in \mathfrak{N}(E)$, $W \subset U$, such that the natural injection $E_W \to E_U$ is a precompact mapping (cf. [7]).

THEOREM 1. *Let E be an S-space and F an arbitrary locally convex space. Suppose that we are given a mapping $E \times E \ni (x, h) \to g(x, h) \in F$ continuous in a neighbourhood of the point $(x_0, 0) \in E \times E$ and linear in the second variable. Then for every $V \in \mathfrak{N}(F)$ there exist $U \in \mathfrak{N}(E)$ and $W \in \mathfrak{N}(E)$ such that for every $\tilde{x} \in x_0 + U$*

$$\mathscr{L}(E, F_V) \ni g(x, \cdot) \underset{x\to\tilde{x}}{\to} g(\tilde{x}, \cdot) \in \mathscr{L}(E, F_V)$$

uniformly on W (which means for $h \in W$).

Proof. For every $V \in \mathfrak{N}(F)$ there exists $U_1 \in \mathfrak{N}(E)$ such that g is continuous on $\{x_0 + U_1\} \times U_1$ and $\|g(x, h)\|_V \leqslant 1$ for $h \in U_1$, $x \in x_0 + U_1$. Let us take $U \in \mathfrak{N}(E)$ such that $U + U \subset U_1$ and $\tilde{x} \in x_0 + U$. Then

$$\{g(x, \cdot) : x \in \tilde{x} + U\} \subset \mathscr{L}(E_U, F_V)$$

is an equicontinuous set in $\mathscr{L}(E_U, F_V)$. But the topologies of simple convergence and precompact convergence are the same for equicontinuous sets (see [3]). Since E is an S-space, if one takes $W \in \mathfrak{N}(E)$ which is precompact in E_U, then from the simple convergence

$$\mathscr{L}(E_U, F_V) \ni g(x, \cdot) \underset{x\to\tilde{x}}{\to} g(\tilde{x}, \cdot) \in \mathscr{L}(E_U, E_V)$$

one obtains uniform convergence on W.

In the same way one can prove

THEOREM 1′. *Let F and E be as in theorem* 1. *Let $g(\cdot,\cdot)$ be linear in the second variable and continuous at $(x_0, 0)$ (not necessarily in the neighbourhood of $(x_0, 0)$) Then for every $V \in \mathfrak{N}(F)$ there exists $W \in \mathfrak{N}(E)$ such that $\mathscr{L}(E, F_V) \ni g(x,\cdot) \underset{x\to x_0}{\to} g(x_0,\cdot) \in \mathscr{L}(E, F_V)$ uniformly on W.*

3. The mean value theorem.

Definition. The mapping T from an open set $\Omega \subset E$ to F (E and F are both arbitrary topological vector spaces) is called *continuously Gâteaux differentiable* if the mapping

$$E \supset \Omega \ni x \to \nabla T(x) \in \mathscr{L}_s(E, F)$$

is continuous.

Plainly a linear combination of continuously differentiable mappings is a continuously differentiable mapping.

PROPOSITION 3. *Let E and F be Fréchet spaces ($\mathscr{F}$-spaces) and G an arbitrary locally convex space. If $T: E \to F$ is continuously Gâteaux differentiable at $x_0 \in E$ and $S: F \to G$ is continuously Gâteaux differentiable at $T(x_0) \in F$, then $S \circ T: E \to G$ is continuously Gâteaux differentiable at x_0.*

Proof. One must prove only the continuity of the derivative. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a sequence convergent to x_0

$$(S\circ T)'(x_\lambda)h = S'\big(T(x_\lambda)\big)\big(T'(x_\lambda)h\big).$$

Let us write $h_\lambda := T'(x_\lambda)h$. Of course, $h_\lambda \underset{\lambda\in\Lambda}{\to} T'(x_0)h$. But the mapping $(x_\lambda, h_\lambda) \to S'\big(T(x_\lambda)\big)h_\lambda \in G$ satisfies the assumptions of lemma 1 and then $S'\big(T(x_\lambda)\big)h_\lambda \to S'\big(T(x_0)\big)\big(T'(x_0)h\big)$.

So we have removed the difficulty sygnalized by Keller [9], namely that in more general spaces and other topologies proposition 3 is not necessarily valid.

PROPOSITION 4. *Let E be a Fréchet space and also a Schwartz space ($\mathscr{F}$-$\mathscr{S}$-space) and F an arbitrary locally convex space. Let a mapping T: $E \supset \Omega \to F$ be continuously Gâteaux differentiable in a neighbourhood of the point $x_0 \in \Omega$. Then for every $V \in \mathfrak{N}(F)$ there exist such $U \in \mathfrak{N}(E)$ and $W \in \mathfrak{N}(E)$ that*

$$\sup_{h\in W}\|\nabla T(x)h - \nabla T(\tilde{x})h\|_V \underset{x\to\tilde{x}}{\to} 0$$

for every $\tilde{x} \in x_0 + U$.

The proof immediately follows from lemma 1 and theorem 1.

THEOREM 2 (mean value theorem). *Let the assumptions of proposition 4 be satisfied. Then for every $V \in \mathfrak{N}(F)$ there exist such $U \in \mathfrak{N}(E)$ and $W \in \mathfrak{N}(E)$ that for every $\tilde{x} \in x_0 + U$ and $h \in W$ we have*

$$\|T(\tilde{x}+h) - T(\tilde{x})\|_V \leqslant C\|h\|_W,$$

where

$$C := \sup_{k,s\in W}\|\nabla T(\tilde{x}+k)s\|_V.$$

Proof. Let us take the mapping $\boldsymbol{R} \supset [0,1] \ni t \to T(\tilde{x}+th) \in F_V$. From the ordinary mean value theorem (see [4]) one has

$$\|T(\tilde{x}+h) - T(\tilde{x})\|_V \leqslant \sup_{0\leqslant\Theta\leqslant1}\|\nabla T(\tilde{x}+\Theta h)h\|_V \leqslant \sup_{k,s\in W}\|\nabla T(\tilde{x}+k)s\|_V\|h\|_W$$

for every $h \in W$ and an arbitrary $W \in \mathfrak{N}(E)$.

From lemma 1 we know that there exist such $M < \infty$, $U_1 \in \mathfrak{N}(E)$ and $W_1 \in \mathfrak{N}(E)$ that $\|\nabla T(x)s\|_V \leqslant M\|s\|_{W_1}$ for every $x \in x_0 + U_1$. Let U be such that $U + U \subset U_1$ and $W \subset W_1 \cap U$. Then for every $\tilde{x} \in x_0 + U$ we have

$$\sup_{k,s\in W}\|\nabla T(\tilde{x}+k)s\|_V \leqslant M < \infty.$$

From the above considerations we get the following

THEOREM 3 (fundamental theorem). *Let the assumptions of proposition 4 be satisfied. Then for every $V \in \mathfrak{N}(F)$ there exist such U, $W \in \mathfrak{N}(E)$ that for every $\tilde{x} \in x_0 + U$ and for arbitrary $U_1 \in \mathfrak{N}(E)$ we have*

$$\|r_{\tilde{x}}(h)\|_V \leqslant C_{\tilde{x}}(U_1)\|h\|_W$$

for every $h \in U_1$. Besides,

$$C_{\tilde{x}}(U_1) := \sup_{\substack{k\in U_1\\ s\in W}}\|\nabla T(\tilde{x}+k)s - \nabla T(\tilde{x})s\|_V$$

and

$$\lim_{U_1\in\mathfrak{N}(E)} C_{\tilde{x}}(U_1) = 0.$$

Proof. Let us consider the following mapping:

$$E \ni y \to \psi(y) := T(y) - \nabla T(\tilde{x})y, \qquad \nabla\psi(y) = \nabla T(y) - \nabla T(\tilde{x}).$$

In the same way as in the proof of theorem 2 we obtain:

$$\begin{aligned}\|T(\tilde{x}+h) - T(\tilde{x}) - \nabla T'(\tilde{x})h\|_V &= \|\psi(\tilde{x}+h) - \psi(\tilde{x})\|_V\\ &\leqslant \|h\|_W \sup_{\substack{k\in U_1\\ s\in W}}\|\nabla T(\tilde{x}+k)s - \nabla T(\tilde{x})s\|_V.\end{aligned}$$

The last point of the theorem immediately follows from proposition 4.

If one uses theorem 1′ instead of theorem 1, on can prove the following

THEOREM 3′. *Let E be an $\mathscr{F}$-$\mathscr{S}$-space and F an arbitrary locally convex space. Let a mapping $T: E \supset \Omega \to F$ be continuously Gâteaux differentiable at the point $x_0 \in \Omega$ (which means that it is differentiable in some neighbourhood* *of x_0 and $\nabla T(x)$ is continuous at x_0). Then for every $V \in \mathfrak{N}(F)$ there exists such $W \in \mathfrak{N}(E)$ that for every $U \in \mathfrak{N}(E)$ and an arbitrary $h \in U$ we have*

$$\|r_{x_0}(h)\|_V \leqslant \|h\|_W C(U),$$

where

$$C(U) := \sup_{\substack{k \in U \\ s \in W}} \|\nabla T(x_0 + k)s - \nabla T(x_0)s\|_V.$$

Besides, $\lim_{U \in \mathfrak{N}(E)} C(U) = 0$.

THEOREM 4. *Let E be an $\mathscr{F}$-S-space and F an arbitrary locally convex space. Let $T: E \supset \Omega \to F$ be a mapping continuously Gâteaux differentiable at a point $x \in \Omega$. Then T is Fréchet differentiable at x and $\nabla T(x) = T'(x)$.*

The proof immediately follows from theorem 3'.

The fact that E is an $\mathscr{F}$-space enters in the above discussions only through lemma 1. So all the conclusions of the present section can be transferred onto an arbitrary S-space in the following way:

THEOREM 5. *Let E be an S-space and F an arbitrary locally convex space. Let $T: E \supset \Omega \to F$ be a mapping continuously Gâteaux differentiable at the point $x_0 \in \Omega$. If the mapping*

$$E \times E \ni (x, h) \to g(x, h) := \nabla T(x)h \in F$$

is continuous at the point $(x_0, 0) \in E \times E$, then the assertions of propositions 3 and 4 and theorems 2, 3, 3' and 4 are valid.

Remark. The above assumption means that for every $V \in \mathfrak{N}(F)$ there exists such $U \in \mathfrak{N}(E)$ that the set

$$\{\nabla T(x): x \in x_0 + U\} \subset \mathscr{L}(E, F_V)$$

is equicontinuous.

Marinescu's assumption about continuous differentiability in the sense of "réunion pseudotopologique" [11] means that the mapping $E \times E \ni (x, h) \to g(x, h) := \nabla T(x)h \in F$ satisfies the condition given by the assertion of proposition 4. So we have proved theorem 4 — which we recognize as the fundamental fact of every reasonable theory of differentiation — without Marinescu's assumption in the case of $\mathscr{F}$-S-spaces and with a weaken assumptions in the case of S-spaces.

4. Higher order derivatives. If one wants to construct a theory of higher order derivatives, one must choose a suitable topology in the spaces $\mathscr{L}(E, \mathscr{L}(E, \ldots, \mathscr{L}(E, F)\ldots))$ and spaces $\mathscr{L}(E, E, \ldots, E; F)$ (the space of all multilinear continuous mappings from $E \times \ldots \times E$ to F). Consequently, as in the theory of the first order derivative, we choose weak topology (simple convergence topology) in all those spaces. This choice

of topology permits us to avoid difficulties in the definition of higher order derivatives (cf. Keller [9]).

LEMMA 2. *If E is an $\mathscr{F}$-space and F an arbitrary locally convex space, then the spaces $\mathscr{L}_s(E, \mathscr{L}_s(E, F))$ and $\mathscr{L}_s(E, E; F)$ (weak topologies) are canonically isomorphic.*

This lemma follows from the Mazur-Orlicz theorem. Of course, if $u \in \mathscr{L}_s(E, \mathscr{L}_s(E, F))$, then $u \to \tilde{u} \in \mathscr{L}_s(E, E; F)$ is defined by

$$\tilde{u}(h, s) := u(h)s, \quad h, s \in E.$$

Similarly, $\mathscr{L}_s(E, \mathscr{L}_s(\ldots \mathscr{L}_s(E, F))) \cong \mathscr{L}_s(E, \ldots, E; F)$.

Now we shall quote the generalizations of lemma 1 and theorem 1 for the case of several variables.

LEMMA 3. *Let E be an $\mathscr{F}$-space and F an arbitrary locally convex space. Let the mapping*

$$E \times E \times \ldots \times E \ni (x, h_1, \ldots, h_n) \to g(x, h_1, \ldots, h_n) \in F$$

be n-linear in the variables $(h_1, \ldots, h_n)$ and partially continuous in all variables at the point $(x_0, h_1, \ldots, h_n) \in E \times E \times \ldots \times E$ for every set $(h_1, \ldots, h_n)$. Then g is continuous at the point $(x_0, 0, \ldots, 0) \in E \times E \times \ldots \times E$.

Proof by induction.

THEOREM 6. *Let E be an S-space and F an arbitrary locally convex space. Suppose we are given a mapping*

$$E \times E \times \ldots \times E \ni (x, h_1, \ldots, h_n) \to g(x, h_1, \ldots, h_n) \in F$$

continuous in a neighbourhood of the point $(x_0, 0, \ldots, 0)$ and n-linear in the variables $(h_1, \ldots, h_n)$. Then for every $V \in \mathfrak{N}(F)$ there exist $U \in \mathfrak{N}(E)$ and $W \in \mathfrak{N}(E)$ such that for every $\tilde{x} \in x_0 + U$

$$\mathscr{L}(E, \ldots, E; F_V) \ni g(x, \cdot, \ldots, \cdot) \underset{x \to \tilde{x}}{\to} g(\tilde{x}, \cdot, \ldots, \cdot) \in \mathscr{L}(E, \ldots, E; F_V)$$

uniformly on $W \times \ldots \times W$.

Proof. The equicontinuous family of n-linear mappings on the precompact set is a uniformly equicontinuous family. Then the rest of the proof follows as in theorem 1.

Definition. Let E be an $\mathscr{F}$-space and F an arbitrary locally convex space. Let $T: E \supset \Omega \to F$ be Fréchet differentiable in a neighbourhood of the point $x_0 \in \Omega$. We say that T is *twice Fréchet differentiable at x_0* if the mapping $E \supset \Omega \ni x \to T'(x) \in \mathscr{L}_s(E, F)$ is Fréchet differentiable at x_0.

The derivative of mapping $x \to T'(x)$ at the point x_0 will be called the *second derivative of T at x_0* and denoted by $T''(x_0)$. Of course, $T''(x_0) \in \mathscr{L}_s(E, \mathscr{L}_s(E, F)) = \mathscr{L}_s(E, E; F)$.

PROPOSITION 5. *Let E be an $\mathcal{F}$-space and F an arbitrary locally convex space. If the mapping $T\colon E \supset \Omega \to F$ is twice differentiable at x_0, then $T''(x_0)$ is a symmetrical bilinear mapping.*

Proof. For every $V \in \mathfrak{N}(F)$ and every $h, s \in E$ the mapping

$$\boldsymbol{R}^2 \ni (a, b) \to \psi(a, b) := T(x_0 + ah + bs) \in F_V$$

is twice differentiable at the point $(0, 0)$ in the sense of the theory of differentiation in normed spaces (it is, with strong topology in $\mathscr{L}(\boldsymbol{R}^2, F_V)$). Because

$$\psi''(0,0)(\xi, \eta)(\zeta, \varrho) = T''(x_0)(\xi h + \eta s)(\zeta h + \varrho s),$$

from the symmetry of second derivatives in normed spaces we have

$$\|T''(x_0)(h, s) - T''(x_0)(s, h)\|_V = 0.$$

Since F is a Hausdorff space, $T''(x_0)(h, s) = T''(x_0)(s, h)$.

In the similar way one can define higher order derivatives.

Definition. Let E be a Fréchet space and F an arbitrary locally convex space. We say that the *mapping $T\colon E \supset \Omega \to F$ is n times Fréchet differentiable at the point $x_0 \in \Omega$ if it is* $n-1$ times differentiable in some neighbourhood of x_0 and if the mapping $E \ni x \to T^{(n-1)}(x) \in \mathscr{L}_s(E, \ldots, E; F)$ is differentiable at x_0. The derivative of this mapping at the point x_0 is called the *n-th order derivative of T at x_0* and denoted by $T^{(n)}(x_0)$. Of course,

$$T^{(n)}(x_0) \in \mathscr{L}_s\big(E, \mathscr{L}_s(E, \ldots, E; F)\big) \cong \mathscr{L}_s(E, \ldots, E; F).$$

As in the case of second order derivatives one can prove

PROPOSITION 6. *The n-th order derivative is an n-linear symmetrical continuous mapping from $E \times \ldots \times E$ to F.*

5. Taylor's formula. Let F be a complete locally convex Hausdorff space. In the standard way the Riemann integral can be constructed for continuous vector fields on intervals $[a, b] \subset \boldsymbol{R}$ with the values in F. If F is a Banach space, this fact is well known (see e. g. [10]). The standard properties of the integral, given in [10], can also be verified in this case. Also the following lemma can be proved (see [4]):

LEMMA 4. *Let F be a locally convex complete Hausdorff space. If a function $f\colon]a, b[\to F$ is $p+1$ times continuously differentiable in $]a, b[$, then for every $t, t_0 \in]a, b[$ we have*

$$f(t) = \sum_{k=0}^{p} f^{(k)}(t_0)\frac{(t-t_0)^k}{k!} + \int_{t_0}^{t} f^{(p+1)}(s)\frac{(t-s)^p}{p!}\,ds.$$

THEOREM 7 (Taylor's formula). *Let T be a mapping from the Fréchet space E to the locally convex complete Hausdorff space F. If T is $p+1$ times continuously differentiable on $\Omega \subset E$, then for every $x_0 \in \Omega$ there exists $U \in \mathfrak{N}(E)$ that for every $h \in U$ we have*

$$T(x_0+h) - T(x_0) = \sum_{n=1}^{p} \frac{1}{n!} T^{(n)}(x_0)(\underbrace{h, \ldots, h}_{n}) + \left(\int_0^1 \frac{(1-s)^p}{p!} T^{(p+1)}(x_0 + sh)\,ds\right)(\underbrace{h, \ldots, h}_{p+1}).$$

Proof. Let $U \in \mathfrak{N}(E)$ be such that $x_0 + U \subset \Omega$. Let us take the following vector function:

$$]-\delta, 1+\delta[\ni t \to f(t) := T(x_0 + th) \in F.$$

Of course, f is $p+1$ times continuously differentiable on $]-\delta, 1+\delta[$ and

$$f^{(n)}(t) = T^{(n)}(x_0 + th)(\underbrace{h, \ldots, h}_{n}), \quad n = 1, \ldots, p+1.$$

Using lemma 4 for $t = 1$, $t_0 = 0$, we obtain the theorem. We shall use the following notation:

$$r_x^{(p)}(h) := T(x+h) - \sum_{n=0}^{p} \frac{1}{n!} T^{(n)}(x)(\underbrace{h, \ldots, h}_{n}),$$

where $T^{(0)}(x) := T(x)$.

THEOREM 8. *Let E be an $\mathcal{F}$-S-space and F a locally convex complete Hausdorff space. If the mapping $T\colon E \supset \Omega \to F$ is p times continuously differentiable on Ω, then for every $x_0 \in \Omega$ and every $V \in \mathfrak{N}(F)$ there exist $U, W \in \mathfrak{N}(E)$ such that for every $U_1 \in \mathfrak{N}(E)$ and $\tilde{x} \in x_0 + U$ and $h \in U_1$ we have*

$$\|r_{\tilde{x}}^{(p)}(h)\|_V \leqslant C_{\tilde{x}}(U_1)\|h\|_W^p,$$

where

$$C_{\tilde{x}}(U_1) := \frac{1}{p!} \sup_{\substack{k \in U_1 \\ q_i \in W}} \|T^{(p)}(\tilde{x}+k)(q_1, \ldots, q_p) - T^{(p)}(\tilde{x})(q_1, \ldots, q_p)\|_V$$

and

$$\lim_{U_1 \in \mathfrak{N}(E)} C_{\tilde{x}}(U_1) = 0.$$

Proof. We use Taylor's formula:

$$\|r_{\tilde{x}}^{(p)}(h)\|_V = \left\|\int_0^1 \frac{(1-s)^{p-1}}{(p-1)!}\big(T^{(p)}(\tilde{x}+sh)(h, \ldots, h) - T^{(p)}(\tilde{x})(h, \ldots, h)\big)\,ds\right\|_V$$

$$\leqslant \int_0^1 \frac{(1-s)^{p-1}}{(p-1)!}\,ds \cdot \sup_{\substack{k \in U_1 \\ q_i \in W}} \|T^{(p)}(\tilde{x}+k)(q_1, \ldots, q_p) - T^{(p)}(\tilde{x})(q_1, \ldots, q_p)\|_V \|h\|_W^p$$

$$= C_{\tilde{x}}(U_1)\|h\|_W^p.$$

From theorem 6 it follows that for every $V \in \mathfrak{N}(F)$ the $W, U \in \mathfrak{N}(E)$ can suitably be chosen.

6. Partial differentiability.

Definition. Let E and G be Fréchet spaces and T a mapping from $E \times G$ to the locally convex space F. We say that *T is partially differentiable at the point $(x_0, y_0) \in E \times G$ in the direction of the space E* if the mapping $E \ni x \to T(x, y_0) \in F$ is differentiable at $x_0 \in E$. Its derivative at the point x_0 is called *partial derivative of T at the point (x_0, y_0) in the direction of the space E* and denoted by $T'_E(x_0, y_0)$. Of course, $T'_E(x_0, y_0) \in \mathscr{L}(E, F)$. Similarly the derivative in the direction of space G can be defined.

THEOREM 9. *Let E and G be $\mathscr{F}$-$\mathcal{S}$-spaces and F an arbitrary locally convex space. The mapping $T: E \times G \to F$ is continuously differentiable on $\Omega \subset E \times G$ if and only if it is continuously partially differentiable (in both variables) on Ω.*

Proof. Because of the existence of the mean value theorem in this theory, the proof proceeds almost in the same way as in normed spaces (see e. g. [4]).

Also in the same way one can prove

THEOREM 10. *Let E, F and G be as in theorem 9. If $T: E \times G \to F$ is differentiable in the first variable at the point $(x_0, y_0) \in E \times G$ and continuously differentiable in second variable in some neighbourhood of the point (x_0, y_0), then T is differentiable at the point (x_0, y_0).*

In the standard way one can define partial derivatives of higher order. In this case a theorem similar to theorem 9 can be proved. Also if all spaces of all variables are $\mathscr{F}$-spaces, the continuous partial derivatives do not depend on the order of iteration (for example $T''_{EG}(x_0, y_0) = T''_{GE}(x_0, y_0)$). This fact follows from the symmetry of the p-th derivative for $p \geqslant 2$.

*

Let us notice that the above construction does not follow strictly the construction for Banach spaces. In the theory of differentiation in Banach spaces the continuity of the derivative is understood in the sense of strong topology in $\mathscr{L}(E, F)$. In our paper we take weak topology in $\mathscr{L}(E, F)$, and so we must make an additional assumption about the space E: we assume that E is an $\mathscr{F}$-$\mathcal{S}$-space. Of course, the common domain of these two theories is the class of finite-dimensional spaces. For these spaces both theories are the same.

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INSTITUTE OF MATHEMATICAL METHODS IN PHYSICS
OF THE WARSAW UNIVERSITY

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