STUDIA MATHEMATICA, T. XXX. (1968)

Algebraic spectral problems*

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Contents

Introduction

The origin of the present paper lies in the development of general finite-dimensional perturbations of spectral problems, the investigation of which was started by the first author in 1956.

Finite-dimensional perturbations turned out to be a unifying tool for the consideration of almost all the existing variational approximation methods for eigenvalue problems. On the other hand, they are also instrumental in the spectral analysis of extensions of a symmetric operator in a Hilbert space of finite deficiency indices (such as occur in ordinary differential eigenvalue problems). In this connection it became apparent that the right setting for the spectral problems would be to consider a system $(V, W; C, D)$ formed by two Banach spaces V and W and two bounded linear operators C and D of V into W.

In an ordinary system which corresponds to usual bounded spectral problems we would have $V = W$, $C = I$ (identity) and D a bounded

* Research sponsored by NSF Grant G 8154 and ONR Contract Nonr 583 (13).

operator of V into itself. However, in case of a problem concerning an unbounded closed operator D, instead of putting it in the preceding frame we will write it as a system $(\mathfrak{D}_D, V; I, D)$ where $\mathfrak{D}_D$ is the domain of D provided with its graph norm.

We may consider a system $(V, W; C_1, D_1)$ as another representation of $(V, W; C, D)$ with the change of "basis" C, D into the "basis" C_1, D_1 if $C_1 = \alpha C + \beta D$, $D_1 = \gamma C + \delta D$, where the complex matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is non-singular.

If there exists a linear combination $\alpha C + \beta D$ giving an operator C_1 which is a continuous linear isomorphism of V onto W, then the system is called *non-singular*. Via a change of basis to C_1, D_1 and the isomorphism C_1, the non-singular system can be transformed into an isomorphic ordinary system $(V, V; I, C_1^{-1}D_1)$. Hence for non-singular systems it would not be necessary to consider the more general setting for spectral problems. However, a general finite-dimensional perturbation may transform a non-singular system into a singular system, which is not isomorphic to an ordinary system. This is the reason why such general systems have to be considered, their spectral decompositions into subsystems investigated (1), the classification of their spectral subsystems undertaken and so forth.

The first author presented these developments in a number of talks (2). It became apparent that especially for consideration of finite-dimensional spectral subsystems, a preliminary analysis of algebraic systems was needed, i.e. systems where V and W are just algebraic vector spaces (with complex scalars) and C and D are just linear operators of V into W (3).

An investigation of this aspect of the problem was started in 1959 in collaboration with the second author. A preliminary draft of the results was prepared in 1960 but it was only in 1966 that the authors were able to renew their collaboration and to finish the present paper.

In the meantime many results of this paper were stated (without proofs) and referred to by the two authors in other papers (see [1] and [2]) and reference was made to the preliminary draft "to appear shortly". Since 1960 the authors have developed the theory further, but it was decided to include in this paper only those results which were part of the original draft. Some of the proofs have been simplified and in many instances the terminology and notations have been changed.

(1) In the present paper we speak of a direct decomposition into subsystems which is in line with algebraic terminology.

(2) E. g., 1956 Northwestern University—lecture; 1957 London Mathematical Society—lecture; 1960 International Symposium on Linear Spaces, Jerusalem—lecture.

(3) Such a system is referred to as a "quadruple" in the paper itself, the term "system" being reserved for a technically convenient substitute.

Since a non-singular algebraic system is isomorphic to an ordinary algebraic system, its algebraic study reduces to that of a module over the polynomial ring in one variable. It is therefore evident that the results for general algebraic systems have their analogs in the theory of modules. We point out these connections in the text. However, the theory in the general case is often much more complicated and requires the introduction of special tools. Among the tools introduced and used in the present paper are: the notion of chains of different types and the corresponding vector spaces of chains (4) (Sections 2 and 6), the correcting transformation (Section 3), quasi-spectral subsystems (analogous to pure submodules), the eigenvalue part of a system (analogous to the torsion part of a module), etc.

The main results of the paper are in Sections 5, 6, 8 and 9.

In Section 5 the theorem is given that a quasi-spectral finite-dimensional subsystem is spectral (analog of the corresponding theorem for pure submodules).

In Section 6, Theorem 6.6 gives a complete description of all the quasi-spectral subsystems of a given system which are direct sums of finite-dimensional indecomposable subsystems. A maximal subsystem of this kind corresponds to bases in quotient spaces of spaces of chains. The multiplicity of appearance of a finite-dimensional indecomposable type in such a maximal subsystem is given by the dimension of the corresponding quotient space which is an isomorphism invariant of the given system. Theorem 6.7 says that the quotient of the whole system modulo such a maximal subsystem does not have any finite-dimensional spectral subsystems.

In Section 7 divisible systems are defined. Certain isomorphism types of divisible systems (which we denote by II_0^∞) are introduced in Section 8, and Theorem 6.6 is then extended to include these types (5) (Theorem 8.8). The structure of divisible systems is determined in Section 9, and it is shown that every system is a direct sum of a divisible system and one which does not have divisible subsystems (although divisible subsystems are not necessarily spectral). This enables us to reduce the study of eigenvalue systems (defined in the same section) to the well known theory of reduced primary modules over a ring of polynomials in one variable.

We should add that in Section 4 we give a proof of the classical theorem due essentially to Kronecker (see [5] and also [10] and [4])

(4) Although chains have been considered before for finite-dimensional systems (see [9], [6] and [4]) the corresponding spaces do not seem to have been used.

(5) However Theorem 6.7 does not extend in general. The existence of such an extension was stated by mistake in an abstract by N. Aronszajn and U. Fixman (see Notices A. M. S. 6 (1959), p. 429).

concerning the direct decomposition of a finite-dimensional system into indecomposable subsystems; we believe that this proof is simpler than other available.

Before ending the introduction we would like to stress the fact mentioned before that the original notation and terminology has been changed in many instances. For example, the type III^{m-1} of the original draft is III^m in the present paper. The reader who would like to compare the results of this paper with those stated in [1] and [2] without proof, would do well by checking the notation and terminology used in these papers.

1. Preliminary definitions

Let V and W be vector spaces over the field $\boldsymbol{C}$ of complex numbers and let C and D be linear transformations of V into W.

The study of the quadruples $(V, W; C, D)$ involves consideration of all the complex linear combinations of C and D. In case (α_1, α_2) and (β_1, β_2) are distinct (ordered) pairs of complex numbers, it is desirable to regard $\alpha_1 C+\alpha_2 D$ and $\beta_1 C+\beta_2 D$ as formally distinct linear combinations of C and D even if they coincide as linear transformations of V into W. It will not do to confine attention to the interesting case in which C and D are linearly independent, because one would like to consider also their restrictions to subspaces of V (or transformations induced by C, D on quotient spaces of V) and these may well be linearly dependent. It is therefore technically convenient to regard the pairs of coefficients (α_1, α_2) themselves as acting on V. We are thus led to the following notion analogous to the notion of a module. Let $\boldsymbol{C}^2$ denote the two-dimensional complex vector space of pairs of complex numbers. A *system* (V, W) is a pair of complex vector spaces V, W together with a *system operation* which assigns to every pair of elements $a \in \boldsymbol{C}^2$, $v \in V$ an element av of W so that:

(a) *for every* $a \in \boldsymbol{C}^2$ *the map* $v \to av$ *is a linear transformation of* V *into* W;

(b) $(\alpha a+\beta b)v = \alpha(av)+\beta(bv)$ *for all* $v \in V$, $a, b \in \boldsymbol{C}^2$ *and* $\alpha, \beta \in \boldsymbol{C}$.

Otherwise expressed, the system operation $(a, v) \to av$ is a bilinear transformation of $\boldsymbol{C}^2 \times V$ into W.

When the system (V, W) is assumed to be given, we shall denote by $T(a)$ the linear transformation $v \to av$ corresponding to the element a of $\boldsymbol{C}^2$. The range and null space of $T(a)$ will then be denoted by aV and $\mathcal{N}(a|V)$ respectively.

To a given quadruple $(V, W; C, D)$ we attach the system (V, W) the system operation of which is defined by $(\alpha_1, \alpha_2)v = (\alpha_1 C+\alpha_2 D)v$. Then the elements $(1, 0)$ and $(0, 1)$ of $\boldsymbol{C}^2$ act as C and D, i.e., $T\big((1, 0)\big) = C$ and $T\big((0, 1)\big) = D$. Obviously, this yields a one-to-one correspondence between the class of all the above quadruples and the class of all the systems. Some of the concepts defined in the sequel correspond to those introduced above for quadruples, and all the results obtained for systems can be reinterpreted in terms of the original quadruples.

If (V, W) is a system, the spaces V and W are its *domain* and *range-space* respectively. If V^1 is a subspace of V, then the subspace of W spanned by $\{cv\colon c \in \boldsymbol{C}^2, v \in V^1\}$ is denoted by $\boldsymbol{C}^2V^1$ (the system in question being supposedly known). The subspace $\boldsymbol{C}^2V$ of W is called the *range* of (V, W). Evidently, if a, b are linearly independent elements of $\boldsymbol{C}^2$, then $\boldsymbol{C}^2V = aV+bV$ and

$$\bigcap_{c\in\boldsymbol{C}^2} \mathcal{N}(c|V) = \mathcal{N}(a|V) \cap \mathcal{N}(b|V).$$

A system (V^1, W^1) is said to be a *subsystem* of the system (V, W) if and only if V^1 is a subspace of V, W^1 is a subspace of W and the system operation of (V^1, W^1) is the restriction of the system operation of (V, W) to $\boldsymbol{C}^2\times V^1$. This implies of course that $\boldsymbol{C}^2V^1$ is contained in W^1. The subsystems of a given system (V, W) form a complete modular lattice under the partial order: $(V^1, W^1) \subset (V^2, W^2)$ if and only if $V^1 \subset V^2$ and $W^1 \subset W^2$. The join and the meet of a family $\big((V^i, W^i)\big)_{i\in I}$ of subsystems are their *sum* $\sum(V^i, W^i) = (\sum V^i, \sum W^i)$ and *intersection* $\bigcap (V^i, W^i) = (\bigcap V^i, \bigcap W^i)$ (with the induced system operations). Here the sum of a family of subspaces of a vector space is of course taken in the algebraic sense, $\sum V^i$ consisting of all the sums of *finite* subsets of $\bigcup_{i\in I} V^i$. The "null" and "all" elements of the lattice of subsystems of (V, W) are the *trivial* subsystems $(0, 0)$ and (V, W). A *proper* subsystem of (V, W) is one which does not coincide with (V, W). The usual convention that an empty sum of subsystems (subspaces) is the zero subsystem $(0, 0)$ (zero subspace) will be followed. The sum of a family $\big((V^i, W^i)\big)_{i\in I}$ of subsystems of (V, W) is said to be a *direct sum* if and only if

$$(V^i, W^i) \cap \sum_{j\neq i} (V^j, W^j) = (0, 0) \quad \text{for every } i \in I.$$

The notations $\sum\cdot(V^i, W^i)$ or, in case of a finite number of summands, $(V^1, W^1)\dot{+}\ldots\dot{+}(V^n, W^n)$ will be used to denote direct sums of subsystems, with similar notations for direct sums of vector subspaces. Note that $(U, Z) = \sum\cdot(V^i, W^i)$ means not only that $U = \sum\cdot V^i$, $Z = \sum\cdot W^i$ as vector spaces, but also that $\boldsymbol{C}^2V^i \subset W^i$ for every i. A subsystem (V^1, W^1) of (V, W) is said to be *spectral in* (V, W) if and only if there exists a subsystem (V^2, W^2) of (V, W) such that $(V, W) = (V^1, W^1)\dot{+}(V^2, W^2)$. An *indecomposable* system is a non-zero system which has no spectral subsystems except the trivial ones. A system (V, W) is said to be *of finite dimension* if and only if V and W are finite-dimensional vector spaces. This is the case if and only if the lattice of subsystems of (V, W) is of finite dimension, and we then assign to (V, W) the dimension of this lattice, namely: $\dim(V, W) = \dim V + \dim W$.

A *homomorphism* of a system (V, W) into a system (V^1, W^1) is a pair (P, Q) where P and Q are linear transformations of V into V^1 and of W into W^1 respectively such that $aPv = Qav$ for all $a \in \boldsymbol{C}^2$, $v \in V$. The terms *isomorphism*, *isomorphic systems*, *endomorphism* and *homomorphic image* are now self explanatory. The *isomorphism type* of a system (V, W) is the class of all systems isomorphic to (V, W). If (P, Q) is a homomorphism of (V, W) into (V^1, W^1), its *image* $(P, Q)(V, W) = (PV, QW)$ is a subsystem of (V^1, W^1), and its *kernel* $\mathrm{Ker}(P, Q) = (\mathrm{Ker}P, \mathrm{Ker}Q)$ is a subsystem of (V, W) (the kernel $\mathrm{Ker}P$ of a linear transformation P is its null space). If (V^1, W^1) is a subsystem of (V, W), then the pair of spaces $(V/V^1, W/W^1)$ with the (well defined) system operation

$$(a, v+V^1) \to a(v+V^1) = av + W^1$$

is a system called the *quotient system* of (V, W) modulo (V^1, W^1). Both notations $(V/V^1, W/W^1)$ and $(V, W)/(V^1, W^1)$ will be used for a quotient system. The usual homomorphism and isomorphism theorems hold for systems and will be used freely. An endomorphism (P, Q) of (V, W) which is an idempotent (i.e. satisfies $P^2 = P, Q^2 = Q$) will be called a *projection* of (V, W). Projections of systems are related to their decompositions into direct sums of subsystems in the same manner that projections of vector spaces are related to vector space decompositions.

Some of our proofs will use duality arguments. If V is a vector space over $\boldsymbol{C}$, we denote by V' its anti-dual, namely, the space of all the anti-linear functionals defined on V. Note that the multiplication of a functional $v' \in V'$ by a scalar $a \in \boldsymbol{C}$ is defined by $(av')(v) = av'(v)$ $(= v'(\bar{a}v))$ for all $v \in V$, where $\bar{a}$ is the complex conjugate of a. If $a = (a_1, a_2) \in \boldsymbol{C}^2$, we write $\bar{a} = (\bar{a}_1, \bar{a}_2)$. Let (V, W) be a system. Then the pair (W', V'), consisting of the algebraic anti-duals of V, W in reverse order, together with the system operation $(a, w') \to aw'$, where aw' is defined by $(aw')(v) = w'(\bar{a}v)$ for all $v \in V$, constitute a system. The last system will be called the *dual system* of (V, W). It will be denoted by (W', V') or $(V, W)'$ according to convenience. The dual system was defined in such a manner that the linear transformation corresponding to an element $\bar{a} = (\bar{a}_1, \bar{a}_2)$ of $\boldsymbol{C}^2$ in the dual is the adjoint of the linear transformation $a_1C + a_2D$ corresponding to a in the original system. We denote by $\perp$ the operation of taking polars: if L is a subset of V, then

$$L^\perp = \{v' \in V' : v'(v) = 0 \text{ for all } v \in L\}.$$

If M is a subset of V' and P is the canonical isomorphism of V into its second anti-dual V'' (defined by $(Pv)(v') = \overline{v'(v)}$ for all $v' \in V'$), we have

$$\{v \in V : v'(v) = 0 \text{ for all } v' \in M\} = P^{-1}(PV \cap M^\perp).$$

However, no confusion will arise if we identify here V with its image under P and hence simplify the notation of the last set to $V \cap M^\perp$. With this notation, we have the following relations between ranges and null spaces in the system (V, W) and its dual:

$$\bar{a}W' = \mathcal{N}(a \mid V)^\perp, \qquad aV = W \cap \mathcal{N}(\bar{a} \mid W')^\perp.$$

If (V, W) is of finite dimension, then it is reflexive in the sense that it is isomorphic to its second dual $(V, W)''$ under the *canonical isomorphism* (P, Q), where P and Q are the canonical isomorphisms of V onto V'' and of W onto W'' respectively. Let $(V, W) = \sum\cdot(V^i, W^i)$ be a decomposition of (V, W), and let (P_i, Q_i) be the associated projections. Then the pairs (Q'_i, P'_i) of the adjoint transformations in reverse order are projections of $(V, W)'$ onto subsystems isomorphic to the systems $(V^i, W^i)'$. The system $(V, W)'$ is canonically isomorphic to the *complete* direct sum of its subsystems $(Q'_i, P'_i)(W', V')$. (The domain of the *complete direct sum* of a family of systems $\big((X^i, Y^i)\big)_{i \in I}$ consists of all the families $(x_i)_{i \in I}$ such that $x_i \in X^i$ for every i. The range space is defined similarly; and the operations are defined componentwise.) If the given decomposition has only a finite number of summands, we have the *dual decomposition*

$$(W', V') = \sum\cdot(Q'_i, P'_i)(W', V').$$

A system (V, W) is said to be *exact* if and only if $\boldsymbol{C}^2 V = W$. It is said to be *co-exact* if and only if $\bigcap_{a \in \boldsymbol{C}^2} \mathcal{N}(a \mid V) = 0$. It is easily seen that (V, W) is exact if and only if $(V, W)'$ is co-exact, and vice versa. Let (V, W) be an arbitrary system. Let U be a direct complement of $\bigcap_{a \in \boldsymbol{C}^2} \mathcal{N}(a \mid V)$ in V, $(x_j)_{j \in J}$ a (Hamel) basis of $\bigcap_{a \in \boldsymbol{C}^2} \mathcal{N}(a \mid V)$ and $(y_k)_{k \in K}$ a basis of a direct complement of $\boldsymbol{C}^2 V$ in W. Then we have the decomposition

$$(V, W) = (U, \boldsymbol{C}^2 U) \dotplus \sum_j \cdot([x_j], 0) \dotplus \sum_k \cdot(0, [y_k])$$

(brackets denote the vector subspace spanned by the enclosed set or family). The subsystem $(U, \boldsymbol{C}^2 U)$ being both exact and co-exact, this decomposition shows that the only isomorphism types of indecomposable systems which are either not exact or not co-exact are given respectively by $(0, [x])$ and $([x], 0)$, where $x \neq 0$. Since $(U, \boldsymbol{C}^2 U)$ is determined up to isomorphism by (V, W), only systems which are both exact and co-exact play any significant role in the isomorphism problem. However, the class of exact systems is not closed under intersections of subsystems, the class of co-exact systems is not closed with respect to taking quotients, and the class of systems which are exact and co-exact is not closed under either of these operations. Therefore, it is not convenient to restrict our considerations to any of these classes.

Exact systems are determined up to isomorphism by linear relations. A *linear relation* in a complex vector space X is a binary relation Φ defined in X which has the following properties:

(a) there exists at least one pair (x_1, x_2) of elements of X satisfying $x_1\Phi x_2$;

(b) if $x_1\Phi x_2$, $x_3\Phi x_4$ and $\alpha, \beta \epsilon C$, then $(\alpha x_1+\beta x_3)\Phi(\alpha x_2+\beta x_4)$.

Thus, from the extensional point of view which we shall adopt here, a linear relation in X is just a subspace of the external direct sum $X\dotplus X$ of X with itself (denoting the external direct sum here by the symbol used for direct sums of subspaces will not give rise to any confusion).

If (V, W) is a system and a, b is a basis of C^2, then the set

$$\Phi = \{(v_1, v_2)\,\epsilon\, V\dotplus V\colon bv_1 = av_2\}$$

is a linear relation in V which will be called the *domain relation* of (V, W) relative to the basis a, b. Conversely, if Φ is a linear relation in V and a, b is a basis of C^2, then the pair $\big(V, (V\dotplus V)/\Phi\big)$ together with the system operation $\circ$ defined by

$$(\alpha a+\beta b)\circ v = (\beta v, -\alpha v)+\Phi$$

constitute an exact system. The domain relation of the last system relative to a, b is Φ. Conversely, if Φ is the domain relation of an *exact* system (V, W) relative to a, b, then the pair (I, Q), where I is the identity mapping of V and Q is the (well defined) linear transformation $(v_1, v_2)+\Phi \to bv_1-av_2$, is an isomorphism of the above system $\big(V, (V\dotplus V)/\Phi\big)$ onto (V, W).

If Φ is the domain relation of a system (V, W) relative to a, b and $c = \alpha a+\beta b$, $d = \gamma a+\delta b$ is another basis of C^2, then the domain relation of (V, W) relative to c, d is

$$\{(\alpha v_1+\beta v_2, \gamma v_1+\delta v_2)\colon (v_1, v_2)\,\epsilon\,\Phi\}.$$

Remark. Similarly, given a basis a, b of C^2 one establishes a one-to-one correspondence up to isomorphism between co-exact systems and their *range relations* $\Psi = \{(av, bv)\colon v\,\epsilon\, V\}$. The co-exact system corresponding to a linear relation Ψ in W is the system (Ψ, W) having the system operation $\circ$ defined by

$$(\alpha a+\beta b)\circ(w_1, w_2) = \alpha w_1+\beta w_2.$$

The range relation of the last system (relative to a, b) is Ψ. Conversely, if Ψ is the range relation of a *co-exact* system (V, W), then the pair (P, I), where P is defined by $Pv = (av, bv)$ and I is the identity mapping of W is an isomorphism of (V, W) onto the above system (Ψ, W). The transformation of range relations under a change of basis of C^2 is given by the same formula as for domain relations. Some concepts and results involving domain relations have analogs for range relations. However, the latter will be omitted as they will not be used in the present work.

We now define a proper subclass of the class of exact and co-exact systems. A system (V, W) is *non-singular* if and only if there exists an element a in C^2 such that the linear transformation $A = T(a)$ is an isomorphism of V onto W. In this case (I, A^{-1}), where I denotes the dentity operator of V, is an isomorphism of (V, W) onto the system (V, V) with the system operation $\circ$ defined by

$$b\circ v = A^{-1}bv, \quad b\,\epsilon\, C^2,\ v\,\epsilon\, V.$$

In this new system the elements of C^2 act like linear operators of V into itself; in particular, a acts like the identity. A system of this kind, in which the domain and the range space coincide and there exists an element of C^2 which acts like the identity operator of the domain, will be called an *ordinary* system. The data V, a, an element $b\,\epsilon\, C^2$ linearly independent of a, and the linear operator $B\colon v\to b\circ v$ determine the above ordinary system (V, V). Thus, since a non-singular system is isomorphic to an ordinary one, it may be regarded as a representation of a single linear operator B of a vector space V. The latter is often studied by means of the module over the ring of complex polynomials defined by $p(\lambda)v = p(B)v$, $p(\lambda)$ a polynomial, $v\,\epsilon\, V$. Hence, arbitrary systems may be viewed as a generalization of a special case of modules over principal ideal domains. Some concepts and results for such modules carry over to systems, but in many instances in a non-trivial manner, since C^2 is just a vector space, not a ring. Some of the differences between systems and modules over principal ideal domains will be pointed out later in the appropriate places.

We shall have to refer below to the ranges and the null spaces of linear transformations corresponding to elements of C^2. If $c\,\epsilon\, C^2$ and $c \neq (0, 0)$, then the range cV and the null space $\mathcal{N}(c\,|\,V)$ of $T(c)$ in a given system (V, W) evidently depend only on the point of the complex projective line which c represents. We shall regard the complex projective line $P^1(C)$ as the quotient set of $C^2-\{(0, 0)\}$ modulo the equivalence relation of proportionality and denote the point $\{\gamma c\colon 0\neq\gamma\,\epsilon\, C\}$ of $P^1(C)$ corresponding to an element $c\,\epsilon\, C^2-\{(0, 0)\}$ by $\dot{c}$. If a, b is an (ordered) basis of C^2, then

$$(\alpha a+\beta b)\to -\alpha/\beta \quad ((0, 0)\neq(\alpha, \beta)\,\epsilon\, C^2,\ -\alpha/0 = \infty)$$

induces a one-to-one map of $P^1(\boldsymbol{C})$ onto the extended complex plane $\tilde{\boldsymbol{C}}$. Thus the map

$$\theta \to b_\theta = \begin{cases} b-\theta a & \text{for} \quad \theta \in \boldsymbol{C}, \\ a & \text{for} \quad \theta = \infty \end{cases}$$

is a parametrization of a complete set of representatives of the points of $P^1(\boldsymbol{C})$. Note that for $\theta_1 \neq \theta_2$, $b_{\theta_1}, b_{\theta_2}$ is a basis of $\boldsymbol{C}^2$. Generalizing from the special case $T(a) = I$, we say that a vector v is an *eigenvector* of the system (V, W) corresponding to the *eigenvalue* θ if and only if it is a *null vector* of $T(b_\theta)$; i.e. $b_\theta v = 0$, $v \neq 0$. It should be noted that the parametrization, and therefore the eigenvalues, depend on the chosen basis a, b (which is not necessarily such that $T(a)$, $T(b)$ are the originally given transformations C, D). If $c = \alpha a + \beta b$, $d = \gamma a + \delta b$ is another basis, then d_η and b_θ are proportional (i.e. $\dot{d}_\eta = \dot{b}_\theta$) if and only if η is the Moebius transform $\eta = (\delta\theta+\gamma)/(\beta\theta+\alpha)$ of θ. *Whenever the above kind of parametrization is used it is tacitly if not explicitly assumed that a basis of $\boldsymbol{C}^2$, usually denoted by a, b, has been chosen.*

2. Finite-dimensional indecomposable systems. Chains

We now exhibit certain types of systems which will turn out to be all the isomorphism types of indecomposable systems of finite dimension. *In the definition below and henceforth the letters l, m, n with or without subscripts will be reserved for positive rational integers.*

Definition 2.1. A system (V, W) is said to be of type:

I^m if and only if $\dim V = m$, $\dim W = m-1$ and $\dim \mathcal{N}(c \mid V) = 1$ for every $c \in \boldsymbol{C}^2 - \{(0, 0)\}$.

$\mathrm{II}^m_{\dot{c}}$ if and only if $\dim V = \dim W = m$, $\dim \mathcal{N}(d \mid V) = 1$ for every $d \in \dot{c}$ and $\mathcal{N}(d \mid V) = 0$ for every $d \in \boldsymbol{C}^2 - \{(0, 0)\}$, $d \notin \dot{c}$.

III^m if and only if $\dim V = m-1$, $\dim W = m$ and $\mathcal{N}(c \mid V) = 0$ for every $c \in \boldsymbol{C}^2 - \{(0, 0)\}$.

Obviously, systems of distinct types are not isomorphic. Our characterization of these types by means of chains (to be defined later) will make it evident that systems of the same type are isomorphic, so that $\mathrm{I}^m, \mathrm{II}^m_{\dot{c}}, \mathrm{III}^m$ may be regarded as notations for isomorphism types. It is clear that systems of type III^1 (respectively I^1) are of the single non-exact (respectively non-co-exact) indecomposable isomorphism type mentioned in Section 1.

Using the parametrization $\theta \to b_\theta$ defined in Section 1, we may denote the types $\mathrm{II}^m_{\dot{c}}$, $\dot{c} \in P^1(\boldsymbol{C})$ by $\mathrm{II}^m_{\dot{b}_\theta}$, $\theta \in \tilde{\boldsymbol{C}}$. We shall abbreviate this to II^m_θ, keeping in mind that the designation of a system as being of type II^m_θ depends on the choice of a basis a, b for $\boldsymbol{C}^2$ (see the convention made at the end of Section 1). Thus, a system (V, W) is of type II^m_θ if and only if $\dim V = \dim W = m$, $\dim \mathcal{N}(b_\theta \mid V) = 1$ and $\mathcal{N}(b_\eta \mid V) = 0$ for $\theta \neq \eta \in \tilde{\boldsymbol{C}}$.

PROPOSITION 2.2. *The systems of Definition 2.1 are indecomposable (and therefore except for types III^1, I^1 exact and co-exact).*

Proof. Let (V, W) be of type II^m_θ and let $(V, W) = (V^1, W^1) \dotplus (V^2, W^2)$ be a decomposition. For $\eta \neq \theta$ we have $\dim W^i \geqslant \dim b_\eta V^i = \dim V^i$. Since $\dim W^1 + \dim W^2 = m = \dim V^1 + \dim V^2$, it follows that $\dim W^i = \dim V^i$, $i = 1, 2$. Hence the two component subsystems are non-singular. Thus, if both components were non-zero, both would have an eigenvector. As θ is the only possible eigenvalue, this would imply that $\dim \mathcal{N}(b_\theta \mid V) \geqslant 2$. Therefore the decomposition must be a trivial one. If (V, W) is of type III^m, we have again $\dim W^i \geqslant \dim V^i$. Since $\dim W^1 + \dim W^2 = m = \dim V^1 + \dim V^2 + 1$, we have $\dim W^i = \dim V^i$ for $i = 1$ or $i = 2$. For this value of i it follows that $(V^i, W^i) = (0, 0)$ since otherwise the system (V, W) would have an eigenvector. A similar proof can be given for type I^m. However, by the next proposition the indecomposability of systems of type I^m follows from that of systems of type III^m. One has only to apply the remarks on duality made in Section 1.

PROPOSITION 2.3. *The dual of a system of type III^m is of type I^m (and conversely by reflexivity). The dual of a system of type $\mathrm{II}^m_{\dot{c}}$ is of type $\mathrm{II}^m_{\dot{\bar{c}}}$. Thus the dual of a system of type II^m_θ relative to the basis a, b is of type $\mathrm{II}^m_{\bar{\theta}}$ relative to the basis $\bar{a}, \bar{b}$ (here $\overline{\infty} = \infty$).*

Proof. Let (V, W) be the given system. The domain and the range space of (W', V') obviously have the appropriate dimensions in both cases. For a subset L of V let $L^\perp$ be the polar $\{v' \in V' : v'(v) = 0 \text{ for all } v \in L\}$. We then have the relations

$$\dim V' = \dim \mathcal{N}(d \mid V) + \dim \mathcal{N}(d \mid V)^\perp,$$

$$\bar{d} W' = \mathcal{N}(d \mid V)^\perp \quad \text{and} \quad \dim W' = \dim \bar{d} W' + \dim \mathcal{N}(\bar{d} \mid W').$$

Using these one checks that $\dim \mathcal{N}(\bar{d} \mid W')$ matches too in all cases. The final statement of the proposition follows from the fact that $\overline{(\dot{b}_\theta)} = \dot{\bar{b}}_{\bar{\theta}}$.

Remark. Using the formula for change of parametrization given at the end of Section 1, we obtain that if $a = (\alpha_1, \alpha_2)$, $b = (\beta_1, \beta_2)$ then b_η is proportional to $\bar{b}_{\bar{\theta}}$ if and only if

$$\eta = \frac{(\bar{\alpha}_1\beta_2 - \beta_1\bar{\alpha}_2)\,\bar{\theta} + \beta_1\bar{\beta}_2 - \bar{\beta}_1\beta_2}{(\bar{\alpha}_1\alpha_2 - \alpha_1\bar{\alpha}_2)\,\bar{\theta} + \alpha_1\bar{\beta}_2 - \alpha_2\bar{\beta}_1}.$$

It is easy to verify that the map $\theta \to \eta$ is the inversion (or reflection) in the circle (or straight line) given parametrically by

$$z = \frac{(\bar{\alpha}_1\beta_2-\beta_1\bar{\alpha}_2)t+\beta_1\bar{\beta}_2-\bar{\beta}_1\beta_2}{(\bar{\alpha}_1\alpha_2-\alpha_1\bar{\alpha}_2)t+\alpha_1\bar{\beta}_2-\alpha_2\bar{\beta}_1}, \qquad t \text{ real}.$$

Thus a system of type II_θ^m (w.r.t. the basis a, b) is isomorphic to its own dual if and only if θ lies on this circle. In the particular case $a = (1, 0)$, $b = (0, 1)$ this condition means that θ is a real number.

As a clue to a more detailed description of systems of our types, we consider a system (V, W) of type II_θ^m, $\theta \neq \infty$. Since the operator $A = T(a) = T(b_\infty)$ is an isomorphism of V onto W, such a system is non-singular. As explained in Section 1, the isomorphism (I, A^{-1}) maps it onto an ordinary system (V, V) which is essentially determined by the single operator $B\colon v \to A^{-1}bv$ of V into itself. The space V is indecomposable into subspaces invariant under B, and B has the single eigenvalue θ (in the ordinary sense). We conclude from the classical canonical form that there exists a basis $(v_k)_{k=1}^m$ of V with respect to which B is described by the matrix corresponding to the elementary divisor $(\lambda-\theta)^m$. That is, we have the relations $Bv_{k-1} = \theta v_{k-1}+v_k$, $k = 2, \ldots, m$, $Bv_m = \theta v_m$, which imply $b_\theta v_{k-1} = av_k$, $k = 2, \ldots, m$, $b_\theta v_m = 0$ in the original system. The following concept of chain is a generalization of this type of basis.

Let a, b be a basis of C^2, (V, W) a system, p a rational integer or the symbol $-\infty$ and q a rational integer or the symbol ∞ such that $p-1 \leqslant q$ (we use the usual conventions on order and addition in the rational integers extended by the symbols $-\infty, \infty$). The symbol $(u_k)_p^q$ will denote a sequence $k \to u_k$ defined on the segment $[p, q] = \{k\colon k$ integer, $p \leqslant k$, $k \leqslant q\}$ of the rational integers. In particular, the segments of the form $[p, p-1]$, p integer, are empty, and $(u_k)_p^{p-1}$ denotes the empty sequence. The set of all the pairs of sequences $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ with $v_k \in V$, $w_k \in W$, which satisfy the conditions

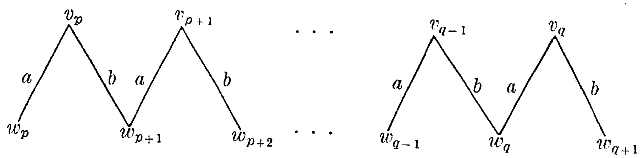
$$av_k = w_k, \; bv_k = w_{k+1} \quad \text{for} \quad k \in [p, q]$$

constitutes a complex vector space under the operations

$$\big((v_k^1)_p^q, (w_k^1)_p^{q+1}\big)+\big((v_k^2)_p^q, (w_k^2)_p^{q+1}\big) = \big((v_k^1+v_k^2)_p^q, (w_k^1+w_k^2)_p^{q+1}\big),$$
$$\alpha\big((v_k)_p^q, (w_k)_p^{q+1}\big) = \big((\alpha v_k)_p^q, (\alpha w_k)_p^{q+1}\big)$$

(here the sum of the empty sequence with itself and the product of the empty sequence by a complex number are the empty sequences). This vector space will be denoted by $C^{p,q}(a, b; V, W)$. An element $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ of $C^{p,q}(a, b; V, W)$ will be called a *chain* (w.r.t. a, b) in (V, W) with *domain sequence* $(v_k)_p^q$ and *range sequence* $(w_k)_p^{q+1}$. If $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ is a chain in (V, W) and (X, Y) is a subsystem of (V, W), we shall say that Γ *is contained in* (X, Y) and write $\Gamma \subset (X, Y)$ if and only if $v_k \in X$ for every $k \in [p, q]$ and $w_k \in Y$ for every $k \in [p, q+1]$. The pair of spaces $\big([v_k\colon k \in [p, q]], [w_k\colon k \in [p, q+1]]\big)$ is the smallest subsystem of (V, W) which contains Γ. This subsystem will be called the *subsystem of* (V, W) *spanned by* Γ. If $p \leqslant q$, then the subsystem spanned by Γ is exact. If $q = p-1$, then $[v_q\colon k \in [p, q]] = 0$ and the subsystem spanned by Γ is either of type III^1 ($w_p \neq 0$) or the zero subsystem ($w_p = 0$).

It is helpful to visualize a chain of $C^{p,q}(a, b; V, W)$, $-\infty < p \leqslant q < \infty$, by means of a diagram



with corresponding infinite diagrams for infinite segments.

If $p \leqslant q$ and $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ is a chain of $C^{p,q}(a, b; V, W)$, then the domain sequence $(v_k)_p^q$ satisfies the conditions

$$av_k = bv_{k-1}, \qquad k-1, k \in [p, q].$$

Conversely, if the basis a, b and the system (V, W) are given, such a sequence uniquely determines a chain in $C^{p,q}(a, b; V, W)$. Similarly, for $q = p-1$ a chain $\big((v_k)_p^{p-1}, (w_k)_p^p\big)$ is determined by the index p and the element w_p. *When the integer p may be assumed to be known, we shall denote such a chain by* $(\varnothing, w_p)$.

Let (P, Q) be a homomorphism of (V, W) into a system (U, Z). It is immediate that if $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ is a chain of $C^{p,q}(a, b; V, W)$, then its *homomorphic image* $(P, Q)\Gamma = \big((Pv_k)_p^q, (Qw_k)_p^{q+1}\big)$ is a chain of $C^{p,q}(a, b; U, Z)$.

We now define certain subspaces of $C^{p,q}(a, b; V, W)$ and operations on these subspaces, keeping the former notations and restrictions on p, q. If $q < \infty$, we denote by $C_{\to}^{p,q}(a, b; V, W)$ the subspace of $C^{p,\infty}(a, b; V, W)$ consisting of all the chains $(v_k)_p^\infty$ which satisfy $v_k = 0$ for $k > q$. In particular, if $-\infty < p$, then $C_{\to}^{p,p-1}(a, b; V, W)$ is the zero subspace of $C^{p,\infty}(a, b; V, W)$. Similarly, if $-\infty < p$ and $q < \infty$, we denote by $C_{\leftrightarrow}^{p,q}(a, b; V, W)$ the subspace of $C^{-\infty,\infty}(a, b; V, W)$ consisting of all the chains $\big((v_k)_{-\infty}^\infty, (w_k)_{-\infty}^\infty\big)$ with domain sequences which vanish outside the segment $[p, q]$, i.e., satisfy $v_k = 0$ for $k < p$ and $k > q$. In particular, if $-\infty < p$, then $C_{\leftrightarrow}^{p,p-1}(a, b; V, W)$ is the zero subspace of $C^{-\infty,\infty}(a, b; V, W)$. In an obvious way we could have introduced also the spaces $C_{\leftarrow}^{p,q}(a, b; V, W)$, however, the chains can always be oriented in such a way that these spaces are superfluous in the present work.

We make the convention that the pair a, b_∞ in a symbol for a space of chains stands for the pair b, a.

Let r, s satisfy similar restrictions to those imposed on p and q, namely, r is a rational integer or the symbol $-\infty$, s is a rational integer or the symbol ∞ and $r-1 \leqslant s$. Suppose in addition that $p \leqslant r$ and $s \leqslant q$. Then $R^{r,s}$ will denote the restriction map

$$\big((v_k)_p^q, (w_k)_p^{q+1}\big) \to \big((v_k)_r^s, (w_k)_r^{s+1}\big)$$

defined on any of the spaces $C^{p,q}(a, b; V, W)$. In particular,

$$R^{r,r-1}\big((v_k)_p^q, (w_k)_p^{q+1}\big) = (\emptyset, w_r).$$

The maps $R^{r,s}$ are obviously linear transformations of their domains $C^{p,q}(a, b; V, W)$ into the corresponding spaces $C^{r,s}(a, b; V, W)$. Thus $R^{r,s}\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ and $R^{r,s}\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$ denote certain subspaces of $C^{r,s}(a, b; V, W)$.

In case $r = p$ and $s = q$ the notations $R^{p,q}\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ and $R^{p,q}\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$ will be abbreviated to $R\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ and $R\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$ respectively.

Note that if Γ is a chain of $R^{r,s}\big(C^{p,q}(a, b; V, W)\big)$, $R^{r,s}\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ or $R^{r,s}\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$ and (P, Q) is a homomorphism of (V, W) into a system (U, Z), then $(P, Q)\Gamma$ belongs to

$$R^{r,s}\big(C^{p,q}(a, b; U, Z)\big),\quad R^{r,s}\big(C_{\rightarrow}^{p,q}(a, b; U, Z)\big)\quad \text{or}\quad R^{r,s}\big(C_{\leftrightarrow}^{p,q}(a, b; U, Z)\big)$$

respectively.

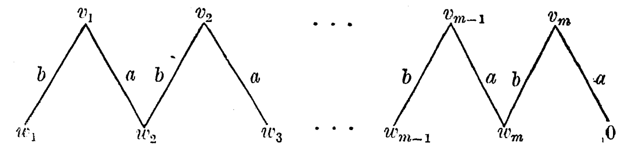
A chain $\big((v_k)_p^q, (w_k)_p^{q+1}\big)$ of $C^{p,q}(a, b; V, W)$, $R\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ or $R\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$ is said to be *proper* (relative to the respective space of chains) if and only if both its domain sequence $(v_k)_p^q$ and the sequence $(w_k)_p^{q+1}$, $(w_k)_p^q$ or $(w_k)_{p+1}^q$ respectively are linearly independent (note that in the second case $w_{q+1} = 0$, while in the third case $w_p = w_{q+1} = 0$). In particular, if $-\infty < p$, then a chain $(\emptyset, w_p)$ of $C^{p,p-1}(a, b; V, W)$ is proper if and only if $w_p \neq 0$ (according to the above definition the zero elements of $R\big(C_{\rightarrow}^{p,p-1}(a, b; V, W)\big)$ and $R\big(C_{\leftrightarrow}^{p,p-1}(a, b; V, W)\big)$ are to be considered as proper chains; however these spaces will not occur in our later work).

Finally we introduce the shift operators S. If $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ is a chain in $C^{p,q}(a, b; V, W)$, then

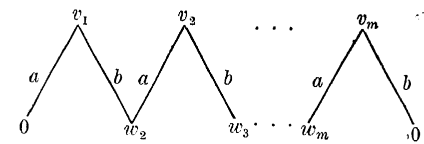
$$S\Gamma = \big((u_k)_{p-1}^{q-1}, (z_k)_{p-1}^q\big)$$

is defined by $u_k = v_{k+1}$, $z_k = w_{k+1}$. Thus, for every integer l the maps S^l are linear isomorphisms of their domains $C^{p,q}(a, b; V, W)$ onto the corresponding spaces $C^{p-l,q-l}(a, b; V, W)$.

The following examples will illustrate the above definitions. A chain $\big((v_k)_1^m, (w_k)_1^{m+1}\big)$ belongs to $R^{1,m}\big(C_{\rightarrow}^{0,m}(a, b_\infty; V, W)\big)$ if and only if it can be represented by the diagram



and there exists an element v_0 in V such that $av_0 = w_1$. A chain $\big((v_k)_1^m, (w_k)_1^{m+1}\big)$ belongs to $R\big(C_{\leftrightarrow}^{1,m}(a, b; V, W)\big)$ if and only if it can be represented by the diagram



It is a proper chain of $R\big(C_{\leftrightarrow}^{1,m}(a, b; V, W)\big)$ if and only if both the sequences $(v_k)_1^m$ and $(w_k)_2^m$ are linearly independent.

Proposition 2.4. *Let (V, W) be a system spanned by a proper chain $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ of $C^{p,q}(a, b; V, W)$, $R\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$ or $R\big(C_{\leftrightarrow}^{p,q}(a, b; V, W)\big)$, where $p \leqslant q$. Then the set*

$$\{(v_{k-1}, v_k)\colon k-1, k \in [p, q]\},\qquad \{(v_{k-1}, v_k)\colon k-1, k \in [p, q]\} \cup \{(v_q, 0)\},$$

or

$$\{(0, v_p)\} \cup \{(v_{k-1}, v_k)\colon k-1, k \in [p, q]\} \cup \{(v_q, 0)\},$$

respectively is a basis of the domain relation Φ of (V, W).

Proof. We prove the statement in the third case, the argument in the other cases being similar. Let

$$(u_1, u_2) \in \Phi \quad \text{with} \quad u_1 = \sum_{k=p}^q \beta_k v_k,\ u_2 = \sum_{k=p}^q \alpha_k v_k.$$

Then

$$\sum_{k=p+1}^q \beta_{k-1} w_k = bu_1 = au_2 = \sum_{k=p+1}^q \alpha_k w_k.$$

Since $(w_k)_{p+1}^q$ is linearly independent, this implies that $\beta_{k-1} = \alpha_k$ for $p+1 \leqslant k \leqslant q$. Hence

$$(u_1, u_2) = \alpha_p(0, v_p) + \sum_{k=p+1}^q \alpha_k(v_{k-1}, v_k) + \beta_q(v_q, 0).$$

The exhibited set is obviously contained in Φ, and the fact that $(v_k)_p^q$ is linearly independent clearly implies that the exhibited set is linearly independent.

Remark. It is evident that a sequence $(v_k)_p^q$ of elements of V is the domain sequence of a chain in $C^{p,q}(a, b; V, W)$, $R(C^{p,q}_{\rightarrow}(a, b; V, W))$ or $R(C^{p,q}_{\leftrightarrow}(a, b; V, W))$ if and only if the corresponding set mentioned in the proposition is contained in the domain relation of (V, W).

In proving that the indecomposable systems of our types are spanned by chains we shall use the following lemma:

Lemma 2.5. *Let a, b and c, d be bases of C^2. Let (V, W) be a system spanned by a chain Γ in $C^{1,s}(a, b; V, W)$, $s \geqslant 0$. Then there exists a chain Δ in $C^{1,s}(c, d; V, W)$ which spans (V, W). If Γ is proper, so is Δ.*

Proof. Let X, Y be the complex vector spaces of polynomials in an indeterminate λ of degree not exceeding $s-1$, s respectively (if $s = 0$, then $X = 0$). The pair (X, Y) with the map

$$\big(\varepsilon a+\zeta b, p(\lambda)\big) \to (\varepsilon a+\zeta b)p(\lambda) = (\varepsilon+\zeta\lambda)p(\lambda)$$

constitutes a system which is spanned by the chain $\big((\lambda^{k-1})_1^s, (\lambda^{k-1})_1^{s+1}\big)$. If $\Gamma = \big((v_k)_1^s, (w_k)_1^{s+1}\big)$, then the pair of linear transformations (P, Q) defined by the requirements $P\lambda^{k-1} = v_k$, $Q\lambda^{k-1} = w_k$ is a homomorphism of (X, Y) onto (V, W). If $c = \alpha a+\beta b$, $d = \gamma a+\delta b$, it is easy to verify that

$$\big((x_k)_1^s, (y_k)_1^{s+1}\big) = \Big(\big((\alpha+\beta\lambda)^{s-k}(\gamma+\delta\lambda)^{k-1}\big)_1^s, \big((\alpha+\beta\lambda)^{s-k+1}(\gamma+\delta\lambda)^{k-1}\big)_1^{s+1}\Big)$$

is a chain in $C^{1,s}(c, d; X, Y)$ which spans (X, Y). Therefore its homomorphic image $\Delta = \big((Px_k)_1^s, (Qy_k)_1^{s+1}\big)$ is the required chain. The chain Γ is proper if and only if $\dim V = s$, $\dim W = s+1$. Since Δ spans (V, W) it must be proper if Γ is.

Proposition 2.6. *A system (V, W) is of type I^m, II_0^m or III^m if and only if it is spanned by a proper chain of $R(C^{1,m}_{\leftrightarrow}(a, b; V, W))$, $R(C^{1,m}_{\rightarrow}(a, b_\theta; V, W))$ or $C^{1,m-1}(a, b; V, W)$ respectively.*

Proof. We omit the straightforward sufficiency proof. A necessity proof for the types II_θ^m was given in the remarks leading to the definition of chains. (Whenever a statement concerning all the values of the parameter $\theta \in \tilde{C}$ is proved in this work only for $\theta \neq \infty$, a proof for $\theta = \infty$ is obtainable on interchanging the roles of a and b.) A more direct proof might also be given.

Let (V, W) be of type III^m. If w_1 is any non-zero vector of W, then $(\emptyset, w_1)$ is a proper chain of $C^{1,0}(a, b; V, W)$. Let $\big((v_k)_1^s, (w_k)_1^{s+1}\big)$ be a proper chain in $C^{1,s}(a, b; V, W)$ with the maximal possible s (s is bounded by $m-1$). It suffices to show that the subsystem (V^1, W^1) spanned by it is the whole system (V, W). Assuming the contrary, we have $\dim V/V^1$

$= \dim W/W^1 = m-s-1 > 0$. Therefore some element $d \neq (0, 0)$ of C^2 has a null vector $x+V^1 \neq V^1$ in $(V, W)/(V^1, W^1)$. Let $c \in C^2$ be linearly independent of d. By Lemma 2.5 there exists a chain

$$\Gamma^1 = \big((v_k^1)_1^s, (w_k^1)_1^{s+1}\big) \in C^{1,s}(c, d; V^1, W^1)$$

which spans (V^1, W^1). We shall show that it is extendible to a proper chain Γ of $C^{0,s}(c, d; V, W)$. Applying the lemma again, this time to $S^{-1}\Gamma$, we shall obtain a proper chain in $C^{1,s+1}(a, b; V, W)$, which will contradict the maximality of s. We have $dx \in W^1$, therefore

$$dx = \sum_{k=1}^{s+1} a_k w_k^1 = a_1 w_1^1 + d\sum_{k=2}^{s+1} a_k v_{k-1}^1$$

(where the last sum is zero if $s = 0$). Here $a_1 \neq 0$ since otherwise, d having no null vector in V, we would have $x \in V^1$. Therefore w_1^1 is of the form $w_1^1 = dv_0^1$, with

$$v_0^1 = a^{-1}\Big(x - \sum_{k=2}^{s+1} a_k v_{k-1}^1\Big).$$

The chain $\big((v_k^1)_0^s, (w_k^1)_0^{s+1}\big)$ determined by the domain sequence $(v_k)_0^s$ is the required extension. Its domain sequence is linearly independent since

$$\sum_{k=0}^{s} \beta_k v_k^1 = 0$$

implies

$$\sum_{k=1}^{s+1} \beta_{k-1} w_k^1 = 0$$

on operating with d, and the elements in the last linear combination are linearly independent because the chain $\big((v_k)_1^s, (w_k)_1^{s+1}\big)$ is proper. If $(w_k^1)_{k=0}^{s+1}$ were linearly dependent, $([v_0^1, \ldots, v_s^1], [w_0^1, \ldots, w_{s+1}^1])$ would be a non-zero subsystem of (V, W) with the dimension of its range space not exceeding the dimension of its domain, which would imply the existence of an eigenvector in (V, W). Thus the chain $\big((v_k^1)_0^s, (w_k^1)_0^{s+1}\big)$ is proper as required. Finally, if (V, W) is of type I^m, its dual (W', V') is of type III^m and therefore is spanned by a proper chain $\big((w_k')_1^{m-1}, (v_k')_1^m\big)$ belonging to $C^{1,m-1}(\bar{a}, \bar{b}; W', V')$. Let $(w_k)_2^m$, $(v_k)_1^m$ denote the dual bases in reverse order (the first basis is empty if $m = 1$) defined by the relations $w_i'(w_j) = \delta_{i,m+1-j}$, $1 \leqslant i \leqslant m-1$, $2 \leqslant j \leqslant m$, and $v_i'(v_j) = \delta_{i,m+1-j}$, $i, j \in [1, m]$. Then it is easy to verify that $(v_k)_1^m$ is the domain sequence of a proper chain in $R(C^{1,m}_{\leftrightarrow}(a, b; V, W))$ (with range sequence $(0, w_2, \ldots \ldots, w_m, 0)$), and thus it spans (V, W).

Note that a system (V, W) is of type I^m or III^m if and only if it is spanned by a proper chain in $R(C^{1,m}_{\leftrightarrow}(a, b; V, W))$ or $C^{1,m-1}(a, b; V, W)$ respectively, where a, b is an *arbitrary* basis of C^2, while it is of type $\mathrm{II}^m_{\dot{c}}$ if and only if it is spanned by a proper chain in $R(C^{1,m}_{\leftrightarrow}(a, b; V, W))$ where a, b is any basis of C^2 *such that* $b \in \dot{c}$.

In view of Proposition 2.6 we introduce the following notations.

Definition 2.7. Let (V, W) be a system and a, b a basis of C^2. We write:

$$C\mathrm{I}^m(a, b; V, W) = R(C^{1,m}_{\leftrightarrow}(a, b; V, W)),$$

$$C\mathrm{II}^m_\theta(a, b; V, W) = R(C^{1,m}_{\rightarrow}(a, b_\theta; V, W)),$$

$$C\mathrm{III}^m(a, b; V, W) = C^{1,m-1}(a, b; V, W).$$

If Π is one of the types I^m, II^m_θ or III^m, a chain (not necessarily proper) in $C\Pi(a, b; V, W)$ will be said to be *of type* Π.

Clearly, if $\Gamma \in C\Pi(a, b; V, W)$ and (P, Q) is a homomorphism of (V, W) into a system (U, Z), then $(P, Q)\Gamma \in C\Pi(a, b; U, Z)$.

The following proposition is an immediate consequence of Definition 2.7:

Proposition 2.8. *Let (V, W) be a system and a, b a basis of C^2. A sequence $(v_k)_1^m$ of elements of V is the domain sequence of a chain in $C\mathrm{III}^{m+1}(a, b; V, W)$ if and only if there exist elements w_0, w_{m+1} in W such that the relation*

$$(b-\lambda a)\sum_{k=1}^{m}\lambda^{m-k}v_k = -\lambda^m w_0 + w_{m+1}$$

holds identically in the complex variable λ. In particular, $(v_k)_1^m$ is the domain sequence of a chain in $C\mathrm{II}^m_\theta(a, b; V, W)$, $\theta \neq \infty$, if and only if there exists an element w_0 in W such that

$$(b_\theta-\lambda a)\sum_{k=1}^{m}\lambda^{m-k}v_k = -\lambda^m w_0$$

holds identically in λ (if $\theta = \infty$, replace a, b_0 by b, a). It is the domain sequence of a chain in $C\mathrm{I}^m(a, b; V, W)$ if and only if it satisfies the identity

$$(b-\lambda a)\sum_{k=1}^{m}\lambda^{m-k}v_k = 0.$$

Remarks. The duality for indecomposable finite-dimensional systems (Proposition 2.3) is reflected in the following duality for chains (already used above). If $\Gamma = ((v_k), (w_k))$ is a proper chain in one of the spaces $C\mathrm{I}^m(a, b; V, W)$, $C\mathrm{II}^m_\theta(a, b; V, W)$ or $C\mathrm{III}^m(a, b; V, W)$, we define the *dual chain* $\Gamma' = ((w'_k), (v'_k))$ by the relations

$$w'_i(w_j) = \delta_{i,m+1-j}, \qquad v'_i(v_j) = \delta_{i,m+1-j}$$

with i, j varying in the appropriate domains. It is then easy to verify that Γ' is a proper chain in $C\mathrm{III}^m(\bar{a}, \bar{b}; W', V')$, $C\mathrm{II}^m_{\bar\theta}(\bar{a}, \bar{b}; W', V')$ or $C\mathrm{I}^m(\bar{a}, \bar{b}; W', V')$ respectively.

The following formulas allow transforming chains from one basis to another. Let a, b and c, d be bases of C^2 related by $c = \alpha a + \beta b$, $d = \gamma a + \delta b$.

If $\Gamma \in C\mathrm{III}^m(a, b; V, W)$, then, as explained in the proof of Lemma 2.5, its domain sequence may be represented by the polynomial sequence $(\lambda^{k-1})_1^{m-1}$ and then the sequence $((\alpha+\beta\lambda)^{m-k-1}(\gamma+\delta\lambda)^{k-1})_1^{m-1}$ represents the domain sequence of a chain Δ in $C\mathrm{III}^m(c, d; V, W)$.

The same formula allows to transform a chain $\Gamma \in C\mathrm{II}^m_\theta(a, b; V, W)$. If the domain sequence of Γ is represented by $(\lambda^{k-1})_1^m$, η is the unique element of $\tilde{C}$ such that d_η is proportional to b_θ and $c = \alpha_1 a + \beta_1 b_\theta$, $d_\eta = \delta_1 b_\theta$ (usual convention if θ or η equals ∞), then $((\alpha_1+\beta_1\lambda)^{m-k}(\delta_1\lambda)^{k-1})_1^m$ represents the domain sequence of a chain in $C\mathrm{II}^m_\eta(c, d; V, W)$.

Finally, if $(v_k)_1^m$ is the domain sequence of a chain in $C\mathrm{I}^m(a, b; V, W)$, then by Proposition 2.8, the relation

$$b_\theta\sum_{k=1}^{m}\theta^{m-k}v_k = 0$$

holds identically in the variable θ. If d_η is the element of $\tilde{C}$ proportional to b_θ, it must annihilate $\sum_{k=1}^{m}\theta^{m-k}v_k$. Therefore if we substitute $\theta = (\alpha\eta-\gamma)/(-\beta\eta+\delta)$ in this expression and multiply by $(-\beta\eta+\delta)^{m-1}$, the coefficient of η^{m-k} will be the k-th element in the domain sequence of a chain $\Delta \in C\mathrm{I}^m(c, d; V, W)$.

In all three cases, if Γ is proper relative to the respective space and spans (V, W), then Δ has the same properties. Under these conditions, if Π is one of the types I^m or III^m, then every chain in $C\Pi(c, d; V, W)$ is a multiple of Δ. This kind of uniqueness is not true for the type $C\mathrm{II}^m_\eta(c, d; V, W)$ (details will be given later).

3. Spectral subsystems. Correcting transformations

In this section we collect some of the tools for proving spectrality of subsystems.

The following proposition enumerates elementary properties of spectral subsystems. Their straightforward proofs will be omitted.

PROPOSITION 3.1. *Let* (V^1, W^1) *and* (V^2, W^2) *denote subsystems of a system* (V, W) *such that* $(V^2, W^2) \subset (V^1, W^1) \subset (V, W)$.

(a) *If* (V^2, W^2) *is spectral in* (V^1, W^1) *and* (V^1, W^1) *is spectral in* (V, W), *then* (V^2, W^2) *is spectral in* (V, W).

(b) *If* (V^2, W^2) *is spectral in* (V, W) *and* $(V^1, W^1)/(V^2, W^2)$ *is spectral in* $(V, W)/(V^2, W^2)$, *then* (V^1, W^1) *is spectral in* (V, W).

(c) *If* (V^2, W^2) *is spectral in* (V, W), *then* (V^2, W^2) *is spectral in* (V^1, W^1).

(d) *If* (V^1, W^1) *is spectral in* (V, W), *then* $(V^1, W^1)/(V^2, W^2)$ *is spectral in* $(V, W)/(V^2, W^2)$.

(e) *If* (V^1, W^1), $((U^i, Z^i))_{i\in I}$ *and* $((X^i, Y^i))_{i\in I}$ *(I an arbitrary index set) are subsystems of* (V, W) *such that* $(U^i, Z^i) = (V^1, W^1)\dot{+}(X^i, Y^i)$ *for every* i, *then we have* $(V, W) = (V^1, W^1)\dot{+}\sum\cdot(X^i, Y^i)$ *if and only if* $(V, W)/(V^1, W^1) = \sum\cdot(U^i, Z^i)/(V^1, W^1)$.

The following lemma is sometimes used in conjunction with Proposition 3.1(c) to prove spectrality:

LEMMA 3.2. *If* (V^1, W^1) *is a subsystem of* (V, W) *and* $(V, W)/(V^1, W^1)$ *is isomorphic to a subsystem of some system* (X, Y), *then there exists a system* (U, Z) *containing* (V, W) *as a subsystem such that* $(U, Z)/(V^1, W^1) \cong (X, Y)$.

Proof. By a standard application of Zorn's lemma it suffices to show that if (X^1, Y^1) is a proper subsystem of (X, Y) and (P_1, Q_1) is an isomorphism of (X^1, Y^1) onto a quotient system $(U^1, Z^1)/(V^1, W^1)$, then there exists a subsystem (X^2, Y^2) of (X, Y), properly containing (X^1, Y^1), and an extension (P_2, Q_2) of (P_1, Q_1) to an isomorphism of (X^2, Y^2) onto a quotient system $(U^2, Z^2)/(V^1, W^1)$ (this clearly implies that $(U^2, Z^2) \supset (U^1, Z^1)$). We can always choose (X^2, Y^2) so that $(X^2, Y^2)/(X^1, Y^1)$ is of type III^1 (if $Y^1 \neq Y$) or I^1 (if $Y^1 = Y$). We indicate how to extend (P_1, Q_1) to (P_2, Q_2) in both cases, omitting the simple verifications needed.

If $(X^2, Y^2)/(X^1, Y^1)$ is of type III^1, we have $X^2 = X^1$, $Y^2 = Y^1\dot{+}[y]$, $y \notin Y^1$. We then define: $U^2 = U^1$, $Z^2 = Z^1\dot{+}[z]$, where z is a new element, and we take the system operation in (U^2, Z^2) to be given by the same function as in (U^1, Z^1). We then take $P_2 = P_1$ and $Q_2(y^1 + ay) = Q_1y^1 + a(z + W^1)$ for all $y^1 \in Y^1$, $a \in C$.

If $(X^2, Y^2)/(X^1, Y^1)$ is of type I^1, we have $X^2 = X^1\dot{+}[x]$, $x \notin X^1$, $Y^2 = Y^1$. We define $U^2 = U^1\dot{+}[u]$, where u is a new element, $Z^2 = Z^1$. We make (U^2, Z^2) into an extension of (U^1, Z^1) by requiring for a basis a, b of C^2 that $au \in Q_1ax$ and $bu \in Q_1bx$. Finally, we take $P_2(x^1 + ax) = P_1x^1 + a(u + V^1)$ for all $x^1 \in X^1$, $a \in C$, and $Q_2 = Q_1$.

One of the main methods for showing that a subsystem (V^1, W^1) of (V, W) is spectral in (V, W) is the method of *correcting transformations*. We start by taking any direct complement V^2 in the domain, $V = V^1\dot{+}V^2$, and then change the decomposition to $V = V^1\dot{+}V^3$ so as to have $W^1 \cap W^3 = 0$, where $W^3 = C^2V^3$. Then taking any direct complement W^4 of $W^1 + W^3$ in W, we get a decomposition $(V, W) = (V^1, W^1)\dot{+}(V^3, W^3 + W^4)$ as desired. We can express the change of the decomposition of the domain as follows: Let E be the restriction to V^2 of the projection of V onto V^1 along V^3; then $V^3 = \{v - Ev : v \in V^2\}$. Conversely, if $V = V^1\dot{+}V^2$ and E is any linear transformation of V^2 into V^1, then $V^3 = \{v - Ev : v \in V^2\}$ satisfies $V = V^1\dot{+}V^3$. The condition that $W^1 \cap W^3 = 0$ is given in the following definition and proposition.

Definition 3.3. Let (V^1, W^1) be a subsystem of (V, W), V^2 a direct complement of V^1 in V and a, b a basis of C^2. A linear transformation E of V^2 into V^1 is said to be a *correcting transformation* of V^2 into V^1 (the systems and basis of C^2 involved being tacitly given) if and only if for every pair $(u_1, u_2) \in V^2\dot{+}V^2$ such that $(u_1 + V^1, u_2 + V^1)$ belongs to the domain relation of $(V, W)/(V^1, W^1)$, the pair $(u_1 - Eu_1, u_2 - Eu_2)$ belongs to the domain relation of (V, W). Explicitly, E is a *correcting transformation* of V^2 into V^1 if and only if $u_1, u_2 \in V^2$ and $bu_1 - au_2 \in W^1$ imply that $b(u_1 - Eu_1) - a(u_2 - Eu_2) = 0$.

PROPOSITION 3.4. *Let* (V^1, W^1) *be a subsystem of* (V, W). *Then the following four statements are equivalent*:

(1) (V^1, W^1) *is spectral in* (V, W).

(2) *Every system of equations*

$$\sum_{j\in J} a_{ij}x_j = w_i, \qquad i \in I$$

$(I, J$ possibly infinite index sets$)$, where $w_i \in W^1$ and a_{ij} are elements of C^2 such that for every i, $a_{ij} = 0$ except for a finite number of indices j, which is solvable for the unknowns x_j in V is solvable also in V^1.

(3) *For every basis* a, b *of* C^2 *and every direct complement* V^2 *of* V^1 *in* V, *there exists a correcting transformation of* V^2 *into* V^1.

(4) *There exists a correcting transformation* E *of a direct complement* V^2 *of* V^1 *in* V *into* V^1 *relative to a basis* a, b *of* C^2.

Proof. (1) $\Rightarrow$ (2). If $(v_j)_{j\in J}$ is a solution of the system in V and (P, Q) is a projection of (V, W) onto (V^1, W^1), then $(Pv_j)_{j\in J}$ is a solution of the system in V^1.

(2) $\Rightarrow$ (3). We construct a system of equations as follows. Let $(v_j)_{j\in J}$ be a basis of V^2. For every pair $(u_{i1}, u_{i2}) \in V^2\dot{+}V^2$ such that $bu_{i1} - au_{i2} = w_i \in W^1$ we include in the system the equation

$$b\sum \beta_{ij}x_j - a\sum \alpha_{ij}x_j = w_i,$$

where $u_{i1} = \sum \beta_{ij}v_j$ and $u_{i2} = \sum \alpha_{ij}v_j$. The system obtained is of the form considered in (2) since the i-th equation can be written

$$\sum(-\alpha_{ij}a + \beta_{ij}b)x_j = w_i$$

and for every i, $\alpha_{ij} = \beta_{ij} = 0$ for all but a finite number of j's. Since the system has the solution $(v_j)_{j\in J}$ in V, it has a solution $(v_j^1)_{j\in J}$ in V^1. The linear transformation E of V^2 into V^1 defined by the requirement that $Ev_j = v_j^1$, $j\in J$, is evidently a correcting transformation of V^2 into V^1.

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). By the remarks made before Definition 3.3 it suffices to verify that if $V^3 = \{v-Ev\colon v\in V^2\}$, then $W^1 \cap C^2 V^3 = 0$. Every element of $C^2 V^3$ is of the form $w = b(u_1-Eu_1)-a(u_2-Eu_2)$, where $u_1, u_2\in V^2$. If $w\in W^1$, then since $Eu_1, Eu_2\in V^1$, also $bu_1-au_2\in W^1$. Since E is a correcting transformation, this implies that $w = 0$.

Defining a correcting transformation (if such exists) is often facilitated by the use of chains, as the proof of the next proposition shows.

PROPOSITION 3.5. *Let* (V^1, W^1) *be a subsystem of* (V, W) *such that the quotient system* $(V, W)/(V^1, W^1)$ *is spanned by a proper chain* $\big((v_k^2+V^1)_p^q, (w_k^2+W^1)_p^{q+1}\big)$ *of one of the spaces* $C^{p,q}(a, b; V/V^1, W/W^1)$, $R\big(C^{p,q}_{\rightarrow}(a, b; V/V^1, W/W^1)\big)$ *or* $R\big(C^{p,q}_{\leftrightarrow}(a, b; V/V^1, W/W^1)\big)$, *where* $p\leqslant q$. *Then* (V^1, W^1) *is spectral in* (V, W) *if and only if there exists a sequence* $(x_k)_p^q$ *of elements of* V^1 *such that the sequence* $(v_k^3)_p^q = (v_k^2-x_k)_p^q$ *is the domain sequence of a chain* $\Gamma^3 = \big((v_k^3)_p^q, (w_k^3)_p^{q+1}\big)$ *in the corresponding space* $C^{p,q}(a, b; V, W)$, $C^{p,q}_{\rightarrow}(a, b; V, W)$ *or* $C^{p,q}_{\leftrightarrow}(a, b; V, W)$. *If this is the case, then* Γ^3 *is proper relative to the respective space of chains.*

Proof. Since the domain sequence $(v_k^2+V^1)_p^q$ is a basis of V/V^1, the sequence $(v_k)_p^q$ is a basis of a direct complement V^2 of V^1 in V. Hence a linear transformation E of V^2 into V^1 may be defined by assigning the values Ev_k^2, $k\in[p, q]$, arbitrarily in V^1. By Proposition 3.4, (V^1, W^1) is spectral in (V, W) if and only if these values can be chosen so that E be a correcting transformation of V^2 into V^1. Because of the linearity of E, this will be the case if and only if for every pair $(u_1, u_2)\in V^2+V^2$ such that (u_1+V^1, u_2+V^1) *belongs to a given system of generators* of the domain relation of $(V, W)/(V^1, W^1)$ the pair (u_1-Eu_1, u_2-Eu_2) belongs to the domain relation of (V, W). Applying Proposition 2.4 to $(V, W)/(V^1, W^1)$ and the remark following it to (V, W), we see that this condition is fulfilled if and only if $(v_k^2-Ev_k^2)_p^q$ is the domain sequence of a chain in the appropriate space. Hence the condition of the proposition with $x_k = Ev_k^2$ follows.

That Γ^3 is proper is a consequence of the fact that the elements $(v_k^3)_p^q$ (or $(w_k^3)_p^{q+1}$) are congruent to the corresponding elements of $(v_k^2)_p^q$ (or $(w_k^2)_p^{q+1}$) modulo V^1 (or W^1) and the chain $\big((v_k^2+V^1)_p^q, (w_k^2+W^1)_p^{q+1}\big)$ is proper.

As a first application of the method of correcting transformations we prove the following lemma:

LEMMA 3.6. *Let* (V^1, W^1) *be a subsystem of* (V, W) *spanned by a chain* $\Gamma^1 = \big((v_k^1)_1^m, (w_k^1)_1^{m+1}\big)$ *of* $CI^m(a, b; V, W)$. *Then* (V^1, W^1) *is spectral in* (V, W) *in each of the following cases:*

(a) $(V, W)/(V^1, W^1)$ *is of type* I^n *and either* $m\leqslant n$ *or* $m>1$ *and* $CI^{m-1}(a, b; V, W) = 0$.

(b) $(V, W)/(V^1, W^1)$ *is of one of the types* II_θ^n.

(c) $(V, W)/(V^1, W^1)$ *is of one of the types* III^n.

Proof. In case (a) let $(v_j^2+V^1)_1^n$ be the domain sequence of a proper chain spanning $(V, W)/(V^1, W^1)$. Write $v_0^2 = v_{n+1}^2 = 0$. Then we have the identity

$$(b-\lambda a)\sum_{j=1}^{n}\lambda^{n-j}v_j^2 = \sum_{j=0}^{n}\lambda^{n-j}(bv_j^2-av_{j+1}^2),$$

where each $bv_j^2-av_{j+1}^2$ belongs to W^1 and hence is of the form $\sum_{k=2}^{m}a_{jk}w_k^1$. Thus we get

$$(b-\lambda a)\sum_{j=1}^{n}\lambda^{n-j}v_j^2 = \sum_{j=0}^{n}\sum_{k=2}^{m}a_{jk}\lambda^{n-j}w_k^1. \tag{3.6.1}$$

According to Proposition 2.8 we have

$$\lambda^{n-j}w_k^1 = (b-\lambda a)\sum_{i=1}^{k-1}\lambda^{n-j+k-1-i}v_i^1 = (b-\lambda a)\cdot -\sum_{i=k}^{m}\lambda^{n-j+k-1-i}v_i^1. \tag{3.6.2}$$

If in (3.6.1) we replace the term $\lambda^{n-j}w_k^1$ by the first expression of (3.6.2) when $k-j\leqslant m-n$ and by the second expression when $k-j>m-n$, we obtain an identity of the form

$$(b-\lambda a)\Big(\sum_{j=1}^{n}\lambda^{n-j}v_j^2-\sum_{j=1}^{\max(n,m-1)}\lambda^{j-1}x_{n-j+1}\Big) = 0, \tag{3.6.3}$$

where $x_{n-j+1}\in V^1$. If $n<m$, then by Proposition 2.8 identity (3.6.3) yields a chain of $CI^{m-1}(a, b; V, W)$ which does not vanish because $v_1^2\notin V^1$. Thus the second assumption of (a) implies the first, that $m\leqslant n$. If $m\leqslant n$, then, by (3.6.3), $(v_j^2-x_j)_1^n$ is the domain sequence of a chain in $CI^n(a, b; V, W)$; and thus, by Proposition 3.5, (V^1, W^1) is spectral in (V, W).

From the characterization of the types by means of proper chains (Proposition 2.6) one sees that a system of type II_θ^n or III^n is isomorphic to a subsystem of a system of type I^l provided $l>n$. Thus in cases (b) and (c) it follows from Lemma 3.2 that there exists a system (U, Z) including (V, W) as a subsystem such that $(U, Z)/(V^1, W^1)$ is of type $\mathrm{I}^{\max(m,n+1)}$. By case (a) of the present lemma (V^1, W^1) is spectral in (U, Z). Hence by Proposition 3.1(c), (V^1, W^1) is spectral in (V, W).

4. Indecomposable systems of finite dimension

In the determination of the isomorphism types of finite-dimensional indecomposable systems we shall use the following lemmas:

LEMMA 4.1. *A finite-dimensional system (V, W) is non-singular if and only if it satisfies the following two conditions:*

(4.1.1) *Every set of eigenvectors in the system (V, W) corresponding to distinct eigenvalues is linearly independent.*

(4.1.2) *Every set of eigenvectors in the dual system (W', V') corresponding to distinct eigenvalues is linearly independent.*

Proof. By (4.1.1) and (4.1.2) the number of distinct eigenvalues of (V, W) and (W', V') is bounded by $\dim V$ and $\dim W$ respectively. Therefore, if (V, W) is of finite dimension, there exists a θ in C such that $T(b_\theta)$ has no null vector in V and its adjoint $T(\bar{b}_{\bar\theta})$ has no null vector in W'. This means that $T(b_\theta)$ is an isomorphism of V onto W and (V, W) is non-singular.

Conversely, if (V, W) is non-singular, then it is isomorphic to an ordinary system, for which condition (4.1.1) is well known for eigenvalues taken in the ordinary sense. Since a change of basis of C^2 involves only a Moebius transformation of the parameter giving the eigenvalues, (4.1.1) follows for the given system. Since the dual of a non-singular system is non-singular, (V, W) satisfies (4.1.2) also.

Remark. It can be verified that (4.1.2) is equivalent to the following condition which does not involve the dual system:

If $\theta_0, \ldots, \theta_m$, $m \geqslant 1$, are distinct elements of $\tilde{C}$ such that $b_{\theta_k}V \neq W$, $k = 0, \ldots, m$, then

$$b_{\theta_0}V + \bigcap_{k=1}^{m} b_{\theta_k}V = W.$$

LEMMA 4.2. *Suppose that a system (V, W) has a set $\{u_k\colon k = 1, \ldots \ldots, m+1\}$ of $m+1$ linearly dependent eigenvectors corresponding to distinct eigenvalues θ_k and that m is minimal with this property. Then (V, W) contains a subsystem of type I^m, and if $m > 1$, then $C\mathrm{I}^{m-1}(a, b; V, W) = 0$.*

Proof. Without loss of generality we may assume that the basis a, b has been chosen so that all the θ_k are finite and $\theta_{m+1} = 0$.

Let

$$(*)\qquad \sum_{k=1}^{m+1} a_k u_k = 0$$

be a non-trival dependence relation. Then the polynomial in λ

$$\prod_{j=1}^{m+1} (\theta_j - \lambda) \sum_{k=1}^{m+1} \frac{a_k}{\theta_k - \lambda} u_k$$

is identically annihilated by $b - \lambda a$ and has degree not exceeding $m-1$. By Proposition 2.8, the coefficients v_k of λ^{m-k} in this polynomial form a domain sequence of a chain $\Gamma = \big((v_k)_1^m, (w_k)_1^{m+1}\big) \in C\mathrm{I}^m(a, b; V, W)$.

Because of the minimality of m, the relation $(*)$ is the single dependence relation among the u_k up to proportionality. Therefore, since the θ_k's are distinct, the polynomial does not vanish identically. Hence there is at most a finite number of values of λ for which it vanishes. For every other value of λ the polynomial yields an eigenvector corresponding to the eigenvalue λ. Thus by the minimality of m, $(v_k)_1^m$ is linearly independent. The same argument shows that if $m > 1$, we have $C\mathrm{I}^{m-1}(a, b; V, W) = 0$.

Suppose that

$$\sum_{k=2}^{m} \beta_k w_k = 0$$

is a non-trivial dependence relation. Then $\sum_{k=2}^{m} \beta_k v_k$ and v_{m+1} are annihilated by a. By the minimality of m, they are both linear combinations of $u_1, \ldots, u_m$. Using the fact that $(v_k)_1^m$ is linearly independent, we can eliminate one u_k and find an eigenvector corresponding to the eigenvalue 0 which depends linearly on $m-1$ of the vectors $u_1, \ldots, u_m$. Therefore Γ is proper and spans a subsystem of type I^m.

The following version of the theorem of Kronecker mentioned in Introduction shows that the isomorphism types of finite-dimensional indecomposable systems listed in Definition 2.1 are all the possible ones.

THEOREM 4.3. *Every finite-dimensional system is a direct sum of subsystems of the types I^m, II_θ^m and III^m.*

Proof. The statement of the theorem is trivially valid for systems of dimension 0. We proceed by induction on the dimension of the system, assuming that every system of lower dimension than that of (V, W) is a direct sum of subsystems of the given types.

Suppose first that (V, W) is non-singular. Let a, b be the basis of C^2 with respect to which the parametrization $\theta \to b_\theta$ is defined, and let a^1, b^1 be a basis of C^2 such that $T(a^1)$ is an isomorphism of V onto W. Then (V, W) is isomorphic to an ordinary system (V, V) essentially determined by the operator $B^1\colon v \to T(a^1)^{-1} b^1 v$. Applying to the vector space V the classical decomposition corresponding to the elementary divisors of B^1, we obtain a direct decomposition of the system (V, V) into subsystems of the types $\mathrm{II}_{\theta_1}^m$, $\theta_1 \neq \infty$, with respect to the basis a^1, b^1 (cf. the remarks preceding the definition of chains in Section 2). Hence the isomorphic system (V, W) is a direct sum of subsystems of the types II_θ^m (the possibility $\theta = \infty$ included) with respect to the given basis a, b.

If (V, W) is singular, then by Proposition 4.1 it violates at least one of conditions (4.1.1), (4.1.2). Suppose that (V, W) violates condition (4.1.1). Then by Lemma 4.2 there exists a positive integer m such that (V, W) contains a subsystem (V^1, W^1) of type I^m and, in case $m > 1$, $C\mathrm{I}^{m-1}(a, b; V, W) = 0$. By the induction hypothesis we have a decomposition

$$(V, W)/(V^1, W^1) = \sum{}^{\cdot}\, (U^i, Z^i)/(V^1, W^1),$$

where each summand $(U^i, Z^i)/(V^1, W^1)$ is of one of our types. If $m > 1$, then $C\mathrm{I}^{m-1}(a, b; U^i, Z^i) \subset C\mathrm{I}^{m-1}(a, b; V, W) = 0$. Hence, by Lemma 3.6, (V^1, W^1) is spectral in each (U^i, Z^i). This implies, by Lemma 3.1 (e) that (V^1, W^1) is spectral in (V, W). Then (V^1, W^1) together with a decomposition of a direct complement in (V, W) into subsystems of our types yield the desired decomposition of (V, W).

Finally, if (V, W) violates condition (4.1.2), then the dual system (W', V'), which has the same dimension as (V, W), does not satisfy condition (4.1.1). Therefore by what we have proved, (W', V') is a direct sum of subsystems of our types. By Proposition 2.3 the dual decomposition of (V'', W''), which is isomorphic to (V, W), is of the same kind.

5. Quasi-spectral subsystems

The concept of a spectral subsystem is not sufficient for treating infinite-dimensional subsystems mainly because the sum of a set of spectral subsystems which is totally ordered by inclusion is not necessarily spectral. In particular, a direct sum of a sequence of subsystems, every finite subsum of which is spectral, need not be spectral or even contained in a proper spectral subsystem. For instance, it is well known that if N is the torsion submodule of the complete direct sum of the $\boldsymbol{C}[\lambda]$-modules $N_m = \boldsymbol{C}[\lambda]/(\lambda-\theta)^m$, where θ is a fixed complex number and m ranges over the positive integers, then the direct sum of the submodules N_m (embedded naturally in N) is not contained in a proper direct summand of N (although every finite subsum of the N_m is a direct summand). Using a correspondence between such modules and ordinary systems as in Section 1, we obtain such an example for systems with summands of the types II_θ^m (note that a spectral subsystem of an ordinary system is ordinary, hence must correspond to a submodule which is a direct summand). In the theory of modules the notion of a pure submodule is introduced as a remedy. In the present context it seems best to generalize likewise the concept of a spectral subsystem.

Definition 5.1. A subsystem (V^1, W^1) of (V, W) is said to be *quasi-spectral in* (V, W) if and only if it is spectral in every subsystem (V^2, W^2) of (V, W) such that $(V^1, W^1) \subset (V^2, W^2)$ and $(V^1, W^1)/(V^2, W^2)$ is of finite dimension.

A system (V, W) is said to be an *extension of finite type* (an *extension by type Π*) of its subsystem (V^1, W^1) if and only if $(V, W)/(V^1, W^1)$ is of finite dimension (has isomorphism type Π). It is immediate that (V, W) is an extension of finite type of (V^1, W^1) if and only if there exists a finite-dimensional subsystem (V^2, W^2) of (V, W) such that $(V, W) = (V^1, W^1) + (V^2, W^2)$ (here V^2 may be taken as a direct complement of V^1 in V). Using this terminology, a subsystem (V^1, W^1) of (V, W) is *quasi-spectral in* (V, W) if and only if it is spectral in all its extensions of finite type which are included in (V, W).

We have the following analog to Proposition 3.4:

Proposition 5.2. *Let (V^1, W^1) be a subsystem of (V, W). Then the following four statements are equivalent*:

1° *(V^1, W^1) is quasi-spectral in (V, W).*

2° *Every system of equations*

$$\sum_{j\in J} a_{ij}x_j = w_i, \quad i\in I \ (I \text{ a possibly infinite index set}),$$

with a finite number of unknowns x_j, $j\in J$, where $w_i \in W^1$ and $a_{ij}\in C^2$, which is solvable in V is solvable also in V^1.

3° *For a particular basis a, b of C^2 every system of equations of one of the forms*

(i) $ax_1 = w_1,\ bx_{i-1} - ax_i = w_i,\ i = 2, \ldots, m,\ bx_m = w_{m+1}$,

(ii) $b_\theta x_{i-1} - ax_i = w_i,\ i = 2, \ldots, m,\ b_\theta x_m = w_{m+1}\ (\theta \in \boldsymbol{C})$,

(iii) $bx_1 = w_1,\ ax_{i-1} - bx_i = w_i,\ i = 2, \ldots, m$, *(case* (ii) *for* $\theta = \infty$)

or

(iv) $bx_{i-1} - ax_i = w_i,\ i = 2, \ldots, m$,

with $w_i \in W^1$, which is solvable for the unknowns x_i in V, is solvable also in V^1.

4° *(V^1, W^1) is spectral in all its extensions by the types I^m, II_θ^m, or III^m which are contained in (V, W).*

Proof. 1° $\Rightarrow$ 2°. Let $(v_j)_{j\in J}$ be a solution of the system of equations in V. Since J is finite, $(V^1, W^1) + ([v_j : j\in J], \boldsymbol{C}^2[v_j : j\in J])$ is an extension of finite type of (V^1, W^1). Hence (V^1, W^1) is spectral in this extension and there exists a projection (P, Q) of the extension onto (V^1, W^1). Then $(Pv_j)_{j\in J}$ is a solution of the given system of equations in V^1.

2° $\Rightarrow$ 3°. Obvious.

3° $\Rightarrow$ 4°. This is an immediate corollary of Propositions 2.6 and 3.5.

4° $\Rightarrow$ 1°. This follows from Theorem 4.3 and Proposition 3.1(e).

We now enumerate elementary properties of the relation of quasi-spectrality. The first four are shared by spectrality (Proposition 3.1) while the sixth is the principal feature by which these relations differ.

Proposition 5.3. *Let (V^1, W^1) and (V^2, W^2) denote subsystems of a system (V, W) such that $(V^2, W^2) \subset (V^1, W^1) \subset (V, W)$. Then*:

(a) *If* (V^2, W^2) *is quasi-spectral in* (V^1, W^1) *and* (V^1, W^1) *is quasi-spectral in* (V, W), *then* (V^2, W^2) *is quasi-spectral in* (V, W).

(b) *If* (V^2, W^2) *is quasi-spectral in* (V, W) *and* $(V^1, W^1)/(V^2, W^2)$ *is quasi-spectral in* $(V, W)/(V^2, W^2)$, *then* (V^1, W^1) *is quasi-spectral in* (V, W).

(c) *If* (V^2, W^2) *is quasi-spectral in* (V, W), *then* (V^2, W^2) *is quasi-spectral in* (V^1, W^1).

(d) *If* (V^1, W^1) *is quasi-spectral in* (V, W), *then* $(V^1, W^1)/(V^2, W^2)$ *is quasi-spectral in* $(V, W)/(V^2, W^2)$.

(e) *If* (V^1, W^1) *is spectral in* (V, W), *then* (V^1, W^1) *is quasi-spectral in* (V, W).

(f) *If* $((V^k, W^k))_{k\in K}$ *is a family of quasi-spectral subsystems of* (V, W) *which is directed under the relation of inclusion* $\subset$, *then* $\sum_{k\in K}(V^k, W^k)$ *is quasi-spectral in* (V, W).

(g) *A sum* $\sum(V^k, W^k)$ *of subsystems of* (V, W) *is a direct sum which is quasi-spectral in* (V, W) *if and only if every finite subsum enjoys these properties.*

Proof. Statement (c) is obvious from the definition, (e) follows from Proposition 3.1 (c), while (g) is an immediate corollary of (f), (e), and (a).

The remaining statements can be easily derived from criterion 2° of Proposition 5.2. We shall indicate the proof only for statement (f). Let

$$\sum_{j\in J} a_{ij}x_j = w_i, \quad i\in I,$$

be a system of equations with a finite number of unknowns x_j and with $w_i\in\sum W^k$ which has a solution $(v_j)_{j\in J}$ in V. Since the number of v_j's is finite, the w_i's span a finite-dimensional subspace of $\sum W^k$. The family $(W^k)_{k\in K}$ being directed by $\subset$, this subspace must already be contained in some subspace W^{k_0}, $k_0\in K$. Since (V^{k_0}, W^{k_0}) is quasi-spectral in (V, W), the system of equations has a solution in V^{k_0} and a fortiori in $\sum V^k$.

Remarks. From the example mentioned in the beginning of the present section and Proposition 5.3 (g) it follows that a quasi-spectral subsystem is not necessarily spectral.

As concerns 5.3 (g), it should be remarked that the assumption that $\sum(V^k, W^k)$ is a direct sum and each summand (V^k, W^k) is quasi-spectral in (V, W) does not imply in general that $\sum(V^k, W^k)$ is quasi-spectral. This follows from the fact that a direct sum of two spectral subsystems of a finite-dimensional system is not necessarily spectral, as the following example shows. Let (V, W) be a direct sum of two subsystems (V^1, W^1) and (V^2, W^2) of the types II_0^1 and II_0^2 respectively. Let Γ^i be a proper chain (of type II_0^1 or II_0^2) which spans (V^i, W^i), $i = 1, 2$,

and let (V^3, W^3) be the subsystem spanned by $\Gamma^1+R^{11}(S\Gamma^2)$. Then it is easy to verify that (V^3, W^3) is a spectral subsystem of (V, W) which forms a direct sum with (V^1, W^1), but $(V^1, W^1)+(V^3, W^3)$ is not spectral in (V, W).

Definition 5.4. A system (V, W) is said to be *quasi-spectrally irreducible* if and only if it is not a zero system and it does not have non-trivial quasi-spectral subsystems.

From statements (a) and (c) of 5.3 it follows that (V^1, W^1) is minimal among the non-zero quasi-spectral subsystems of (V, W) if and only if it is a quasi-spectrally irreducible quasi-spectral subsystem of (V, W). Evidently, a quasi-spectrally irreducible system is indecomposable, a finite-dimensional indecomposable system is quasi-spectrally irreducible. Examples of quasi-spectrally irreducible systems which are not of finite dimension will be given in Sections 8 and 9.

Evidently, if $(V, W)/(V^1, W^1)$ is of finite dimension, then (V^1, W^1) is spectral in (V, W) if and only if it is quasi-spectral in (V, W). We shall show in the next theorem that the same holds under the assumption that (V^1, W^1) is of finite dimension. This means that for finite-dimensional subsystems spectrality is equivalent to quasi-spectrality.

Theorem 5.5. *If* (V^1, W^1) *is a finite-dimensional subsystem of* (V, W) *which is quasi-spectral in* (V, W), *then it is spectral in* (V, W).

Proof. Let V^2 be a direct complement of V^1 in V. If we show that there exists a correcting transformation of V^2 into V^1, the theorem will follow from Proposition 3.4.

A linearly topologized vector space is a vector space endowed with a topology which makes its additive group into a (Hausdorff) topological group and which has a basis of neighborhoods of zero consisting of subspaces. A subset of a vector space is said to be a *linear variety* if it is either empty or a coset modulo a subspace. A linearly topologized vector space is said to be *linearly compact* if and only if every family of closed linear varieties which has the finite intersection property has a non-empty intersection.

Since the space V^1 is of finite dimension, it is linearly compact when endowed with the discrete topology. According to an analog of Tychonoff's theorem, the space ${V^1}^{V^2}$ of all functions of V^2 into V^1 is linearly compact in the product topology [7]. If X is a subspace of V^2, we denote by $\mathfrak{M}(X)$ the set of all the functions E of V^2 into V^1 such that the restriction of E to X is a (linear) correcting transformation of X into V^1. It is easy to see that if X is of finite dimension, then $\mathfrak{M}(X)$ is a closed linear variety in ${V^1}^{V^2}$. If $X^1, \ldots, X^n$ is a finite family of finite-dimensional subspaces of V^2, then

$$\bigcap_{i=1}^{n}\mathfrak{M}(X^i) \supset \mathfrak{M}\Big(\sum_{i=1}^{n} X^i\Big).$$

By Proposition 3.4 the last set is not empty because (V^1, W^1) is spectral in

$$(V^1, W^1)+\Big(\sum_{i=1}^{n} X^i,\ C^2\sum_{i=1}^{n} X^i\Big).$$

Hence the set $\mathfrak{M} = \bigcap\{\mathfrak{M}(X): X$ a finite-dimensional subspace of $V^2\}$ is not empty. But, by definition, a function E of V^2 into V^1 is a linear transformation and a correcting transformation if and only if its restriction to every subspace X of V^2 of dimension at most 2 enjoys these properties. Hence $\mathfrak{M}(V^2) \supset \mathfrak{M}$ (actually $\mathfrak{M}(V^2) = \mathfrak{M}$) and $\mathfrak{M}(V^2)$ is not empty.

6. Bases for finite-dimensional spectral indecomposable subsystems

In the present section we describe in terms of chains all the quasi-spectral subsystems of a given system which are direct sums of finite-dimensional indecomposable subsystems. In particular, we obtain manageable criteria for finite-dimensional spectral subsystems. These will be stated in terms of the spaces of chains appearing in the following definition:

Definition 6.1. Let (V, W) be a system and a, b a basis of C^2. We write

$$\hat{C}\mathrm{I}^m(a, b;\ V, W) = R^{1,m}\big(C^{2,m}_{\leftrightarrow}(a, b;\ V, W)\big)+R^{1,m}\big(C^{1,m-1}_{\leftrightarrow}(a, b;\ V, W)\big),$$

$$\hat{C}\mathrm{II}^m_\theta(a, b;\ V, W) = R^{1,m}\big(C^{0,m}_{\rightarrow}(a, b_\theta;\ V, W)\big)+R^{1,m}\big(C^{1,m-1}_{\rightarrow}(a, b_\theta;\ V, W)\big)$$

and

$$\hat{C}\mathrm{III}^m(a, b;\ V, W) = R^{1,m-1}\big(C^{0,m-1}(a, b;\ V, W)\big)+R^{1,m-1}\big(C^{1,m}(a, b;\ V, W)\big).$$

Note that if Π is any isomorphism type of an indecomposable system of finite dimension, then $\hat{C}\Pi(a, b;\ V, W)$ is a subspace of $C\Pi(a, b;\ V, W)$. We denote the quotient space $C\Pi(a, b;\ V, W)/\hat{C}\Pi(a, b;\ V, W)$ by $Q\Pi(a, b;\ V, W)$.

The convention that for $\theta = \infty$ the pair a, b_∞ is to be replaced by b, a is still in force. According to the definitions in Section 2, for $m = 1$ the spaces $C^{2,m}_{\leftrightarrow}(a, b;\ V, W)$ and $C^{1,m-1}_{\leftrightarrow}(a, b;\ V, W)$ are the zero subspaces of $C^{-\infty,\infty}(a, b;\ V, W)$, so that $\hat{C}\mathrm{I}^1(a, b;\ V, W)$ is the zero subspace of $C\mathrm{I}^1(a, b;\ V, W)$. Similarly, for $m = 1$ the subspace $C^{1,m-1}(a, b_\theta;\ V, W)$, which figures in the definition of $\hat{C}\mathrm{II}^m_\theta(a, b;\ V, W)$, is the zero subspace of $C^{1,\infty}_{\rightarrow}(a, b_\theta;\ V, W)$ so that $\hat{C}\mathrm{II}^1_\theta(a, b;\ V, W) = R^{1,1}\big(C^{0,1}_{\rightarrow}(a, b_\theta;\ V, W)\big)$. Finally, a chain $(\emptyset, w_1)$ of $C\mathrm{III}^1(a, b;\ V, W)$ belongs to $\hat{C}\mathrm{III}^1(a, b;\ V, W)$ if and only if there exist elements v_0^1 and v_1^2 in V such that $bv_0^2+av_1^2 = w_1$, i.e. if and only if $w_1 \in C^2 V$.

We shall use without further comment the obvious fact that if $\Gamma \in \hat{C}\Pi(a, b;\ V, W)$ and (P, Q) is a homomorphism of (V, W) into a system (U, Z), then $(P, Q)\Gamma \in \hat{C}\Pi(a, b;\ U, Z)$.

Lemma 6.2. *Let (V, W) be a finite-dimensional indecomposable system of type Π which is spanned by a (proper) chain $\Delta \in C\Pi(a, b;\ V, W)$. If Π is I^n or II^n_θ, let $\tilde{\Delta}$ denote the extension of Δ to a chain of $C^{1,n}_{\leftrightarrow}(a, b;\ V, W)$ or $C^{1,n}_{\rightarrow}(a, b_\theta;\ V, W)$ respectively. Let s denote a non-negative rational integer. Then:*

(a) *If Π is I^n and $\Gamma \in C^{1,s}(a, b;\ V, W)$, then Γ is of the form*

$$\Gamma = \sum_{k=-s+1}^{n-1} a_k R^{1,s} S^k \tilde{\Delta}, \qquad a_k \text{ complex numbers.}$$

(b) *If Π is II^n_θ and $\Gamma \in C^{1,s}(a, b_\theta;\ V, W)$, then*

$$\Gamma = \sum_{k=0}^{n-1} a_k R^{1,s} S^k \tilde{\Delta}.$$

(c) *If Π is III^n and $\Gamma \in C^{1,s}(a, b;\ V, W)$, then*

$$\Gamma = \sum_{k=0}^{n-1-s} a_k R^{1,s} S^k \Delta$$

(here the range of summation is to be considered empty if $n-1 < s$, and $\Gamma = 0$ in this case).

Proof. Let $\tilde{\Delta}$ or Δ be $\big((x_k), (y_k)\big)$ and let Γ be $\big((v_k)_1^s, (w_k)_1^{s+1}\big)$. If $s = 0$, the statements of the lemma amount to the true assertion that w_1 is a linear combination of (y_k). We therefore assume that $s \geqslant 1$. Since each v_j is a linear combination of (x_k), we may put in cases (a) and (b)

$$v_j = \sum_{k=-j+1}^{n-j} a_{jk} x_{j+k}, \qquad j = 1, \ldots, s.$$

It is clear that in case (c) if $n = 1$ (and $s \geqslant 1$), then $\Gamma = 0$ as claimed. Hence we exclude this possibility too, and put in case (c)

$$v_j = \sum_{k=-j+1}^{n-1-j} a_{jk} x_{j+k}, \qquad j = 1, \ldots, s.$$

In case (a) the relations $bv_{j-1} = av_j$, $j = 2, \ldots, s$, and the linear independence of $(y_k)_2^n$ imply that $a_{j-1,k} = a_{jk}$ for $-j+2 \leqslant k \leqslant n-j$. Thus a_{jk} is independent of j, and if we denote the common value by a_k, we obtain statement (a).

In case (b) the relations $b_\theta v_{j-1} = av_j$ (or, for $\theta = \infty$, $av_{j-1} = bv_j$) and the linear independence of $(y_k)_1^n$ imply the former equations $a_{j-1,k} = a_{jk}$

together with the equations $a_{j,-j+1} = 0, j = 2, \ldots, s$. Thus, in this case $a_k = 0$ for $-s+1 \leqslant k \leqslant -1$, and we get statement (b).

In case (c) we obtain similarly the equations $a_{j-1,k} = a_{jk}$ for $2 \leqslant j \leqslant s$, $-j+2 \leqslant k \leqslant n-1-j$ together with $a_{j,-j+1} = a_{j-1,n-j} = 0$ for $j = 2, \ldots, s$, which implies (c).

In the following three lemmas the assumptions on (V, W), $\varDelta$ and $\tilde{\varDelta}$ are the same as in Lemma 6.2:

LEMMA 6.3. (a) *If $\Gamma \in C\mathrm{I}^m(a, b; V, W)$, then: $\Gamma = a\varDelta$ if (V, W) is of type I^m; $\Gamma \in \hat{C}\mathrm{I}^m(a, b; V, W)$ if (V, W) is of type I^n, $n < m$; $\Gamma = 0$ if (V, W) is of any other type.*

(b) *If $\Gamma \in C\mathrm{II}_\theta^m(a, b; V, W)$, then: $\Gamma = a\varDelta + \hat{\varDelta}$, where $\hat{\varDelta} \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$, if (V, W) is of type II_θ^m; $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$ if (V, W) is of type II_θ^n, $n \neq m$, or of type I^n; $\Gamma = 0$ if (V, W) is of any other type.*

(c) *If $\Gamma \in C\mathrm{III}^m(a, b; V, W)$, then: $\Gamma = a\varDelta$ if (V, W) is of type III^m; $\Gamma = 0$ if (V, W) is of type III^n, $n < m$; $\Gamma \in \hat{C}\mathrm{III}^m(a, b; V, W)$ if (V, W) is of any other type.*

Proof. Most parts of the lemma follow from Lemma 6.2 either directly or by showing that due to the vanishing of extreme terms in the range sequence of a chain Γ of type I^m or II_θ^m, certain coefficients in the representation of Γ in terms of $\tilde{\varDelta}$ vanish. The rest follows from a consideration of eigenvalues. We omit the simple but tedious details.

The following lemma collects some of the results of the preceding one for easier reference:

LEMMA 6.4. *Γ vanishes in each of the following cases:*

(a) *$\Gamma \in \hat{C}\mathrm{I}^m(a, b; V, W)$ and (V, W) is of type I^n, $n \geqslant m$, type II_θ^n or type III^n;*

(b) *$\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$ and (V, W) is of type II_η^n, $\eta \neq \theta$, or type III^n;*

(c) *$\Gamma \in \hat{C}\mathrm{III}^m(a, b; V, W)$ and (V, W) is of type III^n, $n \leqslant m$.*

In the next lemma we add some details to the part (b) of Lemma 6.3.

LEMMA 6.5. *Suppose that (V, W) is of type II_θ^n and $\Gamma \in C^{1,m}(a, b_\theta; V, W)$.*

(a) *If $m > n$, then $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$.*

(b) *If $m < n$ and $\Gamma \in C\mathrm{II}_\theta^m(a, b; V, W)$, then $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$ and*

$$\Gamma = \sum_{k=n-m}^{n-1} a_k R^{1,m} S^k \tilde{\varDelta}.$$

(c) *If $m = n$, then $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$ if and only if*

$$\Gamma = \sum_{k=1}^{n-1} a_k R^{1,m} S^k \tilde{\varDelta}.$$

Proof. (a) follows immediately from Lemma 6.2(b) since in the present case for each $k = 0, \ldots, n-1$ we have $R^{1,m} S^k \tilde{\varDelta} \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$.

(b) We apply again Lemma 6.2 (b) and notice that the $(m+1)$-st range element of Γ is 0 and it is the linear combination with coefficients $a_0, \ldots, a_{n-m-1}$ of the last $n-m$ non-zero range elements of $\varDelta$ (which are linearly independent). Hence $a_k = 0$ for $k < n-m$. On the other hand, for $k \geqslant n-m$, we have $R^{1,m} S^k \tilde{\varDelta} \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$.

(c) By the same arguments as above we reduce the question to the proof that $\varDelta \notin \hat{C}\mathrm{II}_\theta^n(a, b; V, W)$. Assume the contrary. Then $\varDelta = R^{1,n}\Gamma^1 + \Gamma^2$, where $\Gamma^1 \in C_{\to}^{0,n}(a, b_\theta; V, W)$ and $\Gamma^2 \in R^{1,n}(C_{\to}^{1,n-1}(a, b_\theta; V, W))$. By Lemma 6.2 (b),

$$R^{1,n+1} S^{-1} \Gamma^1 = \sum_{k=0}^{n-1} \gamma_k' R^{1,n+1} S^k \tilde{\varDelta}$$

and by part (b) above

$$R^{1,n-1} \Gamma^2 = \sum_{k=1}^{n-1} \gamma_k'' R^{1,n-1} S^k \tilde{\varDelta}.$$

It follows that

$$\varDelta = \sum_{k=1}^{n-1} (\gamma_{k-1}' + \gamma_k'') R^{1,n} S^k \tilde{\varDelta}$$

which is impossible since it would make the n-th range element of $\varDelta$ vanish.

THEOREM 6.6. *Let (V, W) be a system. For each finite-dimensional indecomposable type Π let $(\Gamma_\Pi^j)_{j \in J_\Pi}$ be a (possibly empty) family of chains of $C\Pi(a, b; V, W)$. Let (V_Π^j, W_Π^j) denote the subsystem spanned by Γ_Π^j. Then in order that each Γ_Π^j be a proper chain of $C\Pi(a, b; V, W)$ (i.e. that (V_Π^j, W_Π^j) be of type Π) and that the sum $\sum_\Pi \sum_j (V_\Pi^j, W_\Pi^j)$ be a direct sum which is quasi-spectral in (V, W), it is necessary and sufficient that for every type Π the family $(\Gamma_\Pi^j)_{j \in J_\Pi}$ be linearly independent modulo $\hat{C}\Pi(a, b; V, W)$.*

Proof. Necessity. Suppose that for a certain type Π we have

$$\sum_{j \in J} a_j \Gamma_\Pi^j = \hat{\Gamma} \in \hat{C}\Pi(a, b; V, W),$$

where the a_j are complex numbers and J is a finite non-empty subset of J_Π. Then Γ is of the form $\Gamma = R^{1,s}\Gamma^1 + R^{1,s}\Gamma^2$, where Γ^1 and Γ^2 belong to the two spaces of chains which figure in the definition of $\hat{C}\Pi(a, b; V, W)$. Let (U, Z) denote the smallest subsystem of (V, W) containing $\sum_{j \in J} (V_\Pi^j, W_\Pi^j)$ Γ^1 and Γ^2. Then $\hat{\Gamma} \in \hat{C}\Pi(a, b; U, Z)$ and (U, Z) is an extension of finite type $\sum_{j\in J}(V^j_\Pi, W^j_\Pi)$. Since, by Proposition 5.3 (g), the last sum is a direct sum which is quasi-spectral in (V, W), we have a decomposition

$$(U, Z) = \sum_{j\in J}\cdot(V^j_\Pi, W^j_\Pi) \dotplus (U^1, Z^1).$$

Let (P_j, Q_j) denote the projection of (U, Z) onto (V^j_Π, W^j_Π) associated with this decomposition. Then $\hat{\Gamma}$ is invariant under the projection $\left(\sum_{j\in J} P_j, \sum_{j\in J} Q_j\right)$ and hence

$$\hat{\Gamma} = \sum_{j\in J}\left((P_j, Q_j)\hat{\Gamma}\right).$$

If Π is the type I^m or III^m, then by Lemma 6.4 (a), (c), $(P_j, Q_j)\hat{\Gamma} = 0$. If Π is II^m_θ, then by Lemma 6.5 (c),

$$(P_j, Q_j)\hat{\Gamma} \in R^{1,m}\left(C^{1,m-1}_{\rightarrow}(a, b_\theta; V^j_\Pi, W^j_\Pi)\right).$$

Thus in all cases we have $\sum_{j\in J} a_j w^j_m = 0$, where $\Gamma^j_\Pi = \left((v^j_k), (w^j_k)\right)$, and if Π is I^1 we have also $\sum_{j\in J} a_j v^j_1 = 0$. Since the w^j_m (or v^j_1) are non-zero elements in distinct direct summands, it follows that $a_j = 0$, $j\in J$.

Sufficiency. By Proposition 5.3 (g) it suffices to show that the chains are proper and that every finite subsum of $\sum_\Pi\sum_j(V^j_\Pi, W^j_\Pi)$ is a direct sum which is quasi-spectral in (V, W). If (V^1, W^1) is a subsystem of (V, W) which is an extension of finite type of such a finite subsum, then (V^1, W^1) is finite-dimensional, the chains Γ^j_Π involved in the subsum belong to $C\Pi(a, b; V^1, W^1)$ and, since $\hat{C}\Pi(a, b; V^1, W^1) \subset \hat{C}\Pi(a, b; V, W)$, they are linearly independent modulo $\hat{C}\Pi(a, b; V^1, W^1)$. Thus it is enough to prove the sufficiency statement under the additional assumption that (V, W) is finite-dimensional. We shall restrict ourselves to this case and show that if for every type Π the family of cosets $\left(\Gamma^j_\Pi + \hat{C}\Pi(a, b; V, W)\right)_{j\in J_\Pi}$ is a basis of $Q\Pi(a, b; V, W)$, then the chains Π^j_Π are proper and

$$(V, W) = \sum_\Pi\cdot\sum_j\cdot(V^j_\Pi, W^j_\Pi).$$

Since every linearly independent family can be completed to a basis, this will prove our statement.

Let

$$(V, W) = \sum_\Omega\cdot\sum_{k\in K_\Omega}\cdot(U^k_\Omega, Z^k_\Omega), \tag{6.6.1}$$

(U^k_Ω, Z^k_Ω) of type Ω, be *any* decomposition of (V, W) into indecomposable subsystems. By Theorem 4.3 such decompositions exist and the types Ω

range over I^n, II^n_θ and III^n. By Proposition 2.6 each (U^k_Ω, Z^k_Ω) is spanned by a chain Δ^k_Ω of $C\Omega(a, b; V, W)$. Let (P^Ω_k, Q^Ω_k) denote the projection of (V, W) onto (U^k_Ω, Z^k_Ω) associated with decomposition (6.6.1). Projecting, we get for each Π and $j\in J_\Pi$

$$\Gamma^j_\Pi = \sum_\Omega\sum_k (P^\Omega_k, Q^\Omega_k)\Gamma^j_\Pi.$$

By Lemma 6.3 this may be written

$$\Gamma^j_\Pi = \sum_k a^\Pi_{jk}\Delta^k_\Pi + \hat{\Gamma}^j_\Pi, \tag{6.6.2}$$

where a^Π_{jk}, $k\in K_\Pi$, are complex numbers and $\hat{\Gamma}^j_\Pi \in \hat{C}\Pi(a, b; V, W)$. Since $(\Gamma^j_\Pi)_{j\in J_\Pi}$ represents a basis of $Q\Pi(a, b; V, W)$, (6.6.2) implies that $\dim Q\Pi(a, b; V, W)$ is finite and does not exceed the cardinality of K_Π. By the necessity part of the theorem, which we have already proved, $(\Delta^k_\Pi)_{k\in K_\Pi}$ represents a linearly independent family of $Q\Pi(a, b; V, W)$. Therefore it represents a basis of $Q\Pi(a, b; V, W)$. Thus the index set K_Π may henceforth be identified with J_Π, and relations (6.6.2) may be inverted to yield

$$\Delta^k_\Omega = \sum_j \beta^\Omega_{kj}\Gamma^j_\Omega + \hat{\Delta}^k_\Omega, \tag{6.6.3}$$

where β^Ω_{kj}, $j\in J_\Omega$, are complex numbers and $\hat{\Delta}^k_\Omega \in \hat{C}\Omega(a, b; V, W)$. The chains $\hat{\Delta}^k_\Omega$ may be expressed in the form

$$\hat{\Delta}^k_\Omega = \sum_\Pi\sum_j (P^\Pi_j, Q^\Pi_j)\hat{\Delta}^k_\Omega. \tag{6.6.4}$$

We shall infer from (6.6.3) and (6.6.4) that for every type Ω and $k\in J_\Omega$ we have

$$\Delta^k_\Omega \subset (X, Y), \quad \text{where} \quad (X, Y) = \sum_\Pi\sum_j (V^j_\Pi, W^j_\Pi). \tag{6.6.5}$$

This together with (6.6.1) will imply that

$$(V, W) = \sum_\Pi\sum_j (V^j_\Pi, W^j_\Pi).$$

Since $\dim V^j_\Pi \leqslant \dim U^j_\Pi$, we get

$$\dim V = \sum_\Pi\sum_j \dim U^j_\Pi \geqslant \sum_\Pi\sum_j \dim V^j_\Pi \geqslant \dim V;$$

and it will follow that we have equalities everywhere and that $\dim V^j_\Pi = \dim U^j_\Pi$. In a similar fashion we obtain the analogous equalities for the range spaces and conclude that the chains Γ^j_Π are proper and that the sum $\sum_\Pi\sum_j (V^j_\Pi, W^j_\Pi)$ is a direct sum.

We finish the proof by verifying (6.6.5). If Ω is the type I^1, then (6.6.5) follows from (6.6.3) because $\hat{C}\mathrm{I}^1(a, b;\ V, W) = 0$. Suppose for the purpose of induction that (6.6.5) holds for the types I^n, $n < m$. For Ω the type I^m, (6.6.4) reduces by Lemma 6.4 (a) to

$$\hat\Delta^k_{\mathrm{I}^m} = \sum_{n<m}\sum_j (P_j^{\mathrm{I}^n}, Q_j^{\mathrm{I}^n})\,\hat\Delta^k_{\mathrm{I}^m}.$$

By the inductive assumption this is contained in (X, Y), hence so is $\Delta^k_{\mathrm{I}^m}$.

Suppose that (6.6.5) does not hold for all the chains of the type II^n_θ, θ fixed. Denote by Δ^k_n and $\hat\Delta^k_n$ the extensions of $\Delta^k_{\mathrm{II}^n_\theta}$ and $\hat\Delta^k_{\mathrm{II}^n_\theta}$ respectively to chains of $C^{1,n}_{\rightarrow}(a, b_\theta;\ V, W)$. Since only a finite number of chains are involved and $R^{1,\infty}S^i\Delta^k_n$ vanishes for $i \geqslant n$, we may define t to be the maximum of the integers i such that there exists a chain Δ^k_n with $R^{1,\infty}S^i\Delta^k_n \not\subset (X, Y)$ (by our assumption $t \geqslant 0$). Among the chains Δ^k_n with $R^{1,\infty}(S^t\Delta^k_n) \not\subset (X, Y)$ let Δ^r_m be one with m the minimal possible value of n. Applying Lemma 6.4 (b) to (6.6.4), we obtain

(6.6.6) $$\hat\Delta^r_m = \sum_n\sum_j (P_j^{\mathrm{I}^n}, Q_j^{\mathrm{I}^n})\,\hat\Delta^r_m + \sum_n\sum_j (P_j^{\mathrm{II}^n_\theta}, Q_j^{\mathrm{II}^n_\theta})\,\hat\Delta^r_m.$$

By what we have shown, the first summand on the right-hand side is contained in (X, Y). The second summand is according to Lemmas 6.2 (b) and 6.5 of the form

(6.6.7) $$\sum_{n<m}\sum_j\sum_{i=0}^{n-1} a_{nji}R^{1,\infty}S^i\Delta^j_n + \sum_{n\geqslant m}\sum_j\sum_{i=1}^{n-1} a_{nji}R^{1,\infty}S^i\Delta^j_n.$$

From (6.6.6) and (6.6.7) it follows that $R^{1,\infty}S^t\hat\Delta^r_m$ is the sum of a chain contained in (X, Y) and a linear combination of chains of the form $R^{1,\infty}S^{i+t}\Delta^j_n$, where $i \geqslant 0$ if $n < m$ and $i \geqslant 1$ if $n \geqslant m$. By the choice of t and m, these last chains are also contained in (X, Y). Hence $R^{1,\infty}S\,\hat\Delta^r_m$ and $R^{1,\infty}S^t\Delta^r_m$ are contained in (X, Y), which contradicts the choice of Δ^r_m.

Finally, we get a contradiction if we suppose that (6.6.5) is not valid for the types III^n by choosing $\Delta^r_{\mathrm{III}^m}$ as one of the chains not contained in (X, Y) with m the maximal possible value of n. For in this case (6.6.4), in view of Lemma 6.4(c), assumes the form

$$\hat\Delta^r_{\mathrm{III}^m} = \sum_\Pi\sum_j (P_j^\Pi, Q_j^\Pi)\,\hat\Delta^r_{\mathrm{III}^m} + \sum_{n>m}\sum_j (P_j^{\mathrm{III}^n}, Q_j^{\mathrm{III}^n})\,\hat\Delta^r_{\mathrm{III}^m},$$

where in the first summand on the right-hand side Π ranges over the types I^n and II^n_θ, for which (6.6.5) is already known, whereas the second summand is contained in (X, Y) by virtue of the maximality of m.

We now give a number of complements to Theorem 6.6.

In the course of proof of Theorem 6.6 we have actually shown that if (V, W) is finite-dimensional, then the chains Γ^j_Π are proper and

$$(V, W) = \sum_\Pi{}^{\cdot}\sum_j{}^{\cdot}(V^j_\Pi, W^j_\Pi)$$

if and only if for every type Π the family of cosets $\big(\Gamma^j_\Pi + \hat{C}\Pi(a, b;\ V, W)\big)_{j\in J_\Pi}$ is a basis of $Q\Pi(a, b;\ V, W)$. It follows that the number of summands of type Π in any decomposition of (V, W) into a direct sum of indecomposable systems is $\dim Q\Pi(a, b;\ V, W)$ (which is therefore independent of the choice of a, b). Hence any two such decompositions are isomorphic; namely, there exists a one-to-one correspondence between the summands of these decompositions such that corresponding summands are isomorphic. This last conclusion follows also from the Kurosh-Ore lattice theorem [8] as the similar theorem for groups.

It is clear that for a general system (V, W) the families $(\Gamma^j_\Pi)_{j\in J_\Pi}$ represent bases of $Q\Pi(a, b;\ V, W)$ for every finite-dimensional indecomposable type Π if and only if the chains Γ^j_Π are proper and the sum $\sum_\Pi\sum_j(V^j_\Pi, W^j_\Pi)$ is a maximal quasi-spectral direct sum of finite-dimensional indecomposable subsystems. Using the results 2.6, 3.1(e), 4.3, 5.3(b)(d) and 5.5, one deduces the following theorem:

THEOREM 6.7. *The families $(\Gamma^j_\Pi)_{j\in J_\Pi}$ represent bases of $Q\Pi(a, b;\ V, W)$ for every finite-dimensional type Π if and only if the chains Γ^j_Π are proper, the subsystems (V^j_Π, W^j_Π) spanned by them form a quasi-spectral direct sum $\sum_\Pi{}^{\cdot}\sum_j{}^{\cdot}(V^j_\Pi, W^j_\Pi)$ and the quotient $(V, W)/\sum_\Pi{}^{\cdot}\sum_j{}^{\cdot}(V^j_\Pi, W^j_\Pi)$ has no finite-dimensional spectral subsystems.*

Although the isomorphism type of such a "basic" subsystem $\sum\sum(V^j_\Pi, W^j_\Pi)$ is uniquely determined by (V, W), because the cardinality of the set of summands of type Π must again be $\dim Q\Pi(a, b;\ V, W)$, it is well known already for ordinary systems that a system may have in general more than one such subsystem. However, we have:

PROPOSITION 6.8. *Let $(V, W)_{\mathrm{I}}$ denote the sum of all the subsystems of (V, W) of the types I^m. For each positive integer m, let $(\Gamma^j_m)_{j\in J_m}$ be a (possibly empty) family of chains of $C\mathrm{I}^m(a, b;\ V, W)$. Let (V^j_m, W^j_m) denote the subsystem spanned by Γ^j_m. Then the family $(\Gamma^j_m)_{j\in J_m}$ represents a basis of $Q\mathrm{I}^m(a, b;\ V, W)$ for every m if and only if each (V^j_m, W^j_m) is of type I^m and*

$$(V, W)_{\mathrm{I}} = \sum_m{}^{\cdot}\sum_j{}^{\cdot}(V^j_m, W^j_m).$$

Proof. Suppose that each (V_m^j, W_m^j) is of type I^m and

$$(V, W)_{\mathrm{I}} = \sum_m{}^{\cdot} \sum_j{}^{\cdot} (V_m^j, W_m^j).$$

Then by Theorem 6.6 each family $(\Gamma_m^j)_{j\in J_m}$ is linearly independent modulo $\hat{C}\mathrm{I}^m(a, b; V, W)$. If for a certain value n of m there exists a chain Γ_n of $C\mathrm{I}^n(a, b; V, W)$ such that Γ_n is still linearly independent of $(\Gamma_n^j)_{j\in J_n}$ modulo $\hat{C}\mathrm{I}^n(a, b; V, W)$, then, again by Theorem 6.6, Γ_n spans a subsystem (V_n, W_n) of type I^n such that

$$(V_n, W_n) \cap \sum_m \sum_j (V_m^j, W_m^j) = (0, 0).$$

But then $(V_n, W_n) \not\subset (V, W)_{\mathrm{I}}$ against the definition of $(V, W)_{\mathrm{I}}$.

The only part of the converse not implied by Theorem 6.6 is that

$$(V, W)_{\mathrm{I}} \subset \sum_m \sum_j (V_m^j, W_m^j)$$

(the reverse inclusion being obvious). To show this we prove by induction on n that $\sum_{m\leqslant n} \sum_j (V_m^j, W_m^j)$ includes all the chains of $C\mathrm{I}^n(a, b; V, W)$. For $n = 1$, a chain Γ_1 of $C\mathrm{I}^1(a, b; V, W)$ is congruent to a linear combination of chains of (Γ_1^j) modulo $\hat{C}\mathrm{I}^1(a, b; V, W)$. But the last space vanishes, so

$$\Gamma_1 \subset \sum_j (V_1^j, W_1^j).$$

For $n > 1$, a chain Γ_n of $C\mathrm{I}^n(a, b; V, W)$ is congruent to a linear combination of chains of (Γ_n^j) modulo $\hat{C}\mathrm{I}^n(a, b; V, W)$. But a chain in $\hat{C}\mathrm{I}^n(a, b; V, W)$ is a sum of two chains which essentially belong to $C\mathrm{I}^{n-1}(a, b; V, W)$, hence by the inductive assumption such a chain is included in $\sum_{m\leqslant n-1} \sum_j (V_m^j, W_m^j)$. Thus

$$\Gamma_n \subset \sum_{m\leqslant n} \sum_j (V_m^j, W_m^j)$$

as required.

We can use Theorem 6.6 to complete Lemma 3.6 as follows:

Proposition 6.9. *Let (V^1, W^1) be a subsystem of (V, W). Then (V^1, W^1) is spectral in (V, W) in each of the following cases:*

(V^1, W^1) is of type I^m and $(V, W)/(V^1, W^1)$ is of type I^n, $n \geqslant m$, type II_θ^n or type III^n;

(V^1, W^1) is of type II_θ^m and $(V, W)/(V^1, W^1)$ is of type II_η^n, $\eta \neq \theta$, or type III^n;

(V^1, W^1) is of type III^m and $(V, W)/(V^1, W^1)$ is of type III^n, $n \leqslant m$.

Proof. In order not to introduce additional notation we present the proof only for (V^1, W^1) of type II_θ^m; the same argument is however valid in all cases. Let Γ be a chain of $C\mathrm{II}_\theta^m(a, b; V^1, W^1)$ which spans (V^1, W^1). Suppose that $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$. Then there exist chains

$$\Gamma^1 \in C_{\rightarrow}^{0,m}(a, b_\theta; V, W) \quad \text{and} \quad \Gamma^2 \in C_{\rightarrow}^{1,m-1}(a, b_\theta; V, W)$$

such that $\Gamma = R^{1,m}\Gamma^1 + R^{1,m}\Gamma^2$. Let (P, Q) be the natural homomorphism of (V, W) onto $(V, W)/(V^1, W^1)$. Then

$$(P, Q)\Gamma^1 \in C_{\rightarrow}^{0,m}(a, b_\theta; V/V^1, W/W^1) \quad \text{and}$$
$$(P, Q)\Gamma^2 \in C_{\rightarrow}^{1,m-1}(a, b_\theta; V/V^1, W/W^1).$$

From Lemma 6.3(b) it follows that $(P, Q)\Gamma^1$ and $(P, Q)\Gamma^2$ vanish. Hence Γ^1 and Γ^2 are contained in (V^1, W^1) and consequently $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V^1, W^1)$. But since Γ must be proper and (V^1, W^1) is spectral in itself, this contradicts Theorem 6.6. Therefore $\Gamma \notin \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$, and Theorem 6.6 implies that (V^1, W^1) is spectral in (V, W).

Remark. Using chains it is easy to construct examples showing that if the types of the finite-dimensional indecomposable systems (V^1, W^1) and $(V, W)/(V^1, W^1)$ are not as in Proposition 6.9, then (V^1, W^1) is not necessarily spectral in (V, W).

To finish this section we will state a theorem which is convenient in some applications (see [1]) and which results immediately from Theorem 6.6.

Theorem 6.10. *Consider for a subsystem (V^1, W^1) of (V, W) the following property:*

(A) *For any finite-dimensional type Π and for any set $\{\Gamma^j\} \subset C\Pi(a, b; V^1, W^1)$ if the Γ^j are linearly independent modulo $\hat{C}\Pi(a, b; V^1, W^1)$, then they are linearly independent modulo $\hat{C}\Pi(a, b; V, W)$.*

Then, for an arbitrary subsystem (V^1, W^1), property (A) *is necessary for quasi-spectrality, whereas for finite-dimensional (V^1, W^1) it is also sufficient for spectrality.*

7. Divisible systems

Definition 7.1. A system (V, W) is said to be *divisible* if and only if $cV = W$ for every $c \in \boldsymbol{C}^2 - \{(0, 0)\}$. A system is said to be *reduced* if and only if it has no divisible subsystem except $(0, 0)$.

The structure of divisible systems and the possibility of decomposing an arbitrary system into a direct sum of a divisible subsystem and a reduced subsystem will be discussed in Section 9. Here we shall be content with listing the elementary properties needed in the following sections.

The following proposition is an easy consequence of the definitions. Its proof will be omitted.

PROPOSITION 7.2. (a) *The only systems which are both divisible and reduced are the zero systems.*

(b) *Systems of the types* I^m *are divisible.*

(c) *An arbitrary sum of divisible subsystems of a given system is divisible. In particular, the sum of all divisible subsystems of* (V, W) *is the largest divisible subsystem of* (V, W). *This (unique) maximal divisible subsystem of* (V, W) *will be denoted by* $\mathrm{Div}(V, W)$.

(d) *A quotient system of a divisible system is divisible.*

(e) *If* (V^1, W^1) *is a divisible subsystem of* (V, W) *and* $(V, W)/(V^1, W^1)$ *is divisible, then so is* (V, W).

(f) *The quotient* $(V, W)/\mathrm{Div}(V, W)$ *is reduced.*

(g) *A quasi-spectral subsystem of a divisible system is divisible.*

Remark. It will follow from the theorem on the structure of divisible systems (Theorem 9.10) that systems of the types II_θ^m and III^m are reduced.

LEMMA 7.3. *If* (V^1, W^1) *is a divisible subsystem of* (V, W) *and* $(V, W)/(V^1, W^1)$ *is spanned by a proper chain* $\big((v_k^2+V^1)_p^q, (w_k^2+W^1)_p^{q+1}\big)$ *of* $C^{p,q}(a, b; V/V^1, W/W^1)$, *where* p *is finite, or of* $C_{\rightarrow}^{p,q}(a, b; V/V^1, W/W^1)$, *then* (V^1, W^1) *is spectral in* (V, W).

Proof. We shall apply the method of correcting transformations via Proposition 3.5.

In the first case the result is obvious if $q = p-1$. Otherwise we have the relations

$$bv_{k-1}^2 - av_k^2 = w_k^1 \in W^1 \quad \text{for} \quad k \in [p+1, q].$$

We define $x_p = 0$. Since (V^1, W^1) is divisible, we have $aV^1 = W^1$, and thus we can solve successively the equations $ax_k = bx_{k-1} - w_k^1$, $k \in [p+1, q]$ for x_k in V^1. Then $(v_k^2 - x_k)_p^q$ is the domain sequence of a chain in $C^{p,q}(a, b; V, W)$.

In the second case we have the additional relation $bv_q^2 = w_{q+1}^1 \in W^1$. Since $bV^1 = W^1$ and q is finite in this case, we can solve in V^1 the equation $bx_q = w_{q+1}^1$ and then the equations $bx_k = w_{k+1}^1 + ax_{k+1}$, $k \in [p, q-1]$ in decreasing order of k. Then $(v_k^2 - x_k)_p^q$ is the domain sequence of a chain in $R\big(C_{\rightarrow}^{p,q}(a, b; V, W)\big)$.

Remark. As we shall show later, the requirement made in the first case that p be finite is actually superfluous.

LEMMA 7.4. *A divisible system is spectral in extensions by the types* II_θ^m *and* III^m. *Therefore a divisible subsystem of* (V, W) *is quasi-spectral in* (V, W) *if and only if it is spectral in all extensions by the types* I^m *contained in* (V, W). *The maximal divisible subsystem* $\mathrm{Div}(V, W)$ *of a system* (V, W) *is always quasi-spectral in* (V, W).

Proof. The first statement is an immediate corollary of the preceding lemma and Proposition 2.6. The second statement then follows by criterion 4° of Proposition 5.2. This in turn implies that $\mathrm{Div}(V, W)$ is quasi-spectral because by Proposition 7.2 (b) and (f), $(V, W)/\mathrm{Div}(V, W)$ does not contain a subsystem of type I^m.

8. Systems of type $\mathrm{II}_\theta^\infty$

To describe the structure of divisible systems we need a class of ordinary quasi-spectrally irreducible systems which correspond to the Prüfer modules over $C[\lambda]$. We shall follow Rédei's characterization of the Prüfer modules in introducing these systems (see [3]). However, since our characterization is among systems, which are not necessarily ordinary, the verification that the definition leads to well defined isomorphism types is somewhat more involved than in the module case.

Definition 8.1. A system (V, W) is said to be *of type* II_c^∞ if and only if it contains subsystems of all the types II_c^m, $m = 1, 2, \ldots$, but no proper subsystem of (V, W) has this property.

As in the case of systems of the types II_c^m, if a basis a, b of C^2 is assumed given, we shall abbreviate $\mathrm{II}_{b_\theta}^\infty$ to $\mathrm{II}_\theta^\infty$. Thus a system (V, W) is of type $\mathrm{II}_\theta^\infty$ if and only if it contains subsystems of all the types II_θ^m, $m = 1, 2, \ldots$, but no proper subsystem of (V, W) has this property.

LEMMA 8.2. *Let* (V, W) *be a system spanned by a chain* $\Gamma = \big((v_k)_p^q, (w_k)_p^{q+1}\big)$ *of* $R\big(C_{\rightarrow}^{p,q}(a, b_\theta; V, W)\big)$, $q < +\infty$.

(a) *If* Γ *is not proper and does not vanish, then* (V, W) *contains a spectral subsystem of some type* I^m.

(b) *If* $p > -\infty$, *then* (V, W) *is a direct sum of subsystems of the types* I^{m_j} *and at most one subsystem of type* II_θ^n. *Such a spectral subsystem of type* II_θ^n *exists if and only if* w_p *is linearly independent of all the other* w_k.

Proof. We assume that $\theta \neq \infty$ (if $\theta = \infty$, replace a, b_θ by b, a). (a) Clearly, the non-vanishing of Γ implies that $p \leqslant q$ and the domain sequence of Γ does not vanish. Diminishing q if necessary, we may assume that $v_q \neq 0$. Since linear dependence of $(v_k)_p^q$ implies, on operating with a, linear dependence of $(w_k)_p^q$, we have a dependence relation

$$\sum_{k=r}^{q} a_k w_k = 0 \quad \text{with} \quad r > -\infty,\ p \leqslant r \leqslant q \text{ and } a_r \neq 0.$$

Let $\tilde{\Gamma}$ denote the extension of Γ to a chain in $C_{\rightarrow}^{p,q}(a, b_\theta; V, W)$. Then the chain

$$\sum_{k=r}^{q} a_k R^{1,q-r+1} S^{k-1}\tilde{\Gamma}$$

belongs to $C\mathrm{I}^{q+r-1}(a, b_\theta; V, W)$ and does not vanish because its domain element with index $q-r+1$ is $a_r v_q$. Let m be the smallest positive integer n such that $C\mathrm{I}^n(a, b_\theta; V, W) \neq 0$. Then clearly $\hat{C}\mathrm{I}^m(a, b_\theta; V, W) = 0$. Thus, if Δ is a non-zero chain in $C\mathrm{I}^m(a, b_\theta; V, W)$, it spans according to Theorems 6.6 and 5.5 a spectral subsystem of type I^m.

(b) As shown in Section 2, we can span (V, W) by a chain in $R(C_{\rightarrow}^{p,q}(c, b_\theta; V, W))$ for any c not proportional to b_θ. It follows that $b_\eta V = W$ for all $\eta \neq \theta$, whereas codim $b_\theta V$ is 0 or 1 depending on whether w_p is or is not linearly dependent on all the other w_k. Considering then the direct decomposition of (V, W) by Theorem 4.3 (Kronecker's theorem) we check immediately our statement.

PROPOSITION 8.3. *A system (V, W) is of type $\mathrm{II}_\theta^\infty$ if and only if it is spanned by a proper chain of $R(C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W))$.*

Proof. Without loss of generality we assume that $\theta \neq \infty$.

Suppose that (V, W) is spanned by a proper chain $\Gamma = \big((v_k)_{-\infty}^0, (w_k)_{-\infty}^1\big)$ of $R(C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W))$. Denote by (V^s, W^s) the subsystem spanned by $R^{-s,0}\Gamma$, $s \geqslant -1$. Then $(V^{-1}, W^{-1}) = (0, 0)$, while for $s \geqslant 0$, (V^s, W^s) is of type II_θ^{s+1}. Thus (V, W) contains subsystems of all the types II_θ^m. A proper subsystem of (V, W) cannot contain all the subsystems (V^s, W^s), $s \geqslant 0$, since their sum is (V, W). Thus to show that (V, W) is of type $\mathrm{II}_\theta^\infty$, it is sufficient to show that (V, W) contains no other subsystems of the types II_θ^m. We actually show that every proper subsystem (X, Y) of (V, W) which satisfies $aX = Y$ coincides with one of the subsystems (V^s, W^s). Let $s \geqslant -1$ be the maximal integer such that $(V^s, W^s) \subset (X, Y)$ (a maximum exists since (X, Y) is proper and contains (V^{-1}, W^{-1})). If $V^s \neq X$, there exists a vector

$$v = \sum_{k=-t}^{0} a_k v_k$$

in X with $t > s$ and $a_{-t} = 1$. Since Γ is proper, $T(a)$ is an isomorphism of V onto W and $T(a)^{-1}T(b_\theta)$ is an operator of V into itself under which X is invariant. Therefore the vector

$$\big(T(a)^{-1}T(b_\theta)\big)^{t-s-1}v = v_{-s-1} + \sum_{k=-s}^{0} a_{k-t+s+1}v_k$$

(where the last sum is zero if $s = -1$) belongs to X. Since the second term on the right-hand side belongs to $V^s \subset X$, we have $v_{-s+1} \in X$. Hence $(V^{s+1}, W^{s+1}) \subset (X, Y)$ against the choice of s. Thus $X = V^s$, and since (X, Y) is exact, $(X, Y) = (V^s, W^s)$.

Suppose now that (V, W) is of type $\mathrm{II}_\theta^\infty$. Let (X^m, Y^m), $m = 1, 2, \ldots$, be subsystems of the respective types II_θ^m. From their description by chains (Proposition 2.6) we infer that for $m \geqslant 2$, $\big(T(a)^{-1}b_\theta X^m, b_\theta X^m\big)$ is of type II_θ^{m-1}. Therefore, by Definition 8.1, the subsystem

$$\sum_{m=2}^{\infty}\big(T(a)^{-1}b_\theta X^m, b_\theta X^m\big)$$

must be equal to (V, W). Thus $b_\theta V = W$. Hence, starting from a null vector v_0 of b_θ, say in X^1, we can construct a non-zero chain $\Gamma \in R(C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W))$. It suffices to show that Γ is proper since then the subsystems spanned by $R^{-m+1,0}\Gamma$, $m = 1, 2, \ldots$, are of the respective types II_θ^m. Hence their sum is (V, W), which means that Γ spans (V, W).

Suppose to the contrary that Γ is improper. Then by Lemma 8.2 (a), (V, W) contains a spectral subsystem (X, Y) of type I^r. We reach a contradiction by showing that $(V, W)/(X, Y)$, and hence a direct complement of (X, Y) in (V, W), contains subsystems of type $\mathrm{II}_\theta^{m_j}$ with arbitrarily large m_j and thus subsystems of all the types II_θ^m, $m = 1, 2, \ldots$ The subsystem $\big((X^m, Y^m)+(X, Y)\big)/(X, Y)$ is spanned by a chain of $R\big(C_{\rightarrow}^{-m+1,0}(a, b_\theta; V/X, W/Y)\big)$ — the image of $R^{-m+1,0}\Gamma$ under the natural homomorphism of (V, W) onto $(V, W)/(X, Y)$. Therefore by Lemma 8.2 (b)

$$\big((X^m, Y^m)+(X, Y)\big)/(X, Y) = \sum_{j=1}^{t}\cdot(U^j, Z^j),$$

where the subsystems (U^j, Z^j) are of the types I^{m_j} except possibly for one which is of type $\mathrm{II}_\theta^{m_j}$. Comparing dimensions we get

$$m-r = \dim X^m - \dim X \leqslant \dim\big((X^m+X)/X\big) = \sum_{j=1}^{t}\dim U^j \leqslant t\max_j m_j,$$

and on the other hand

$$r-1 = \dim X^m - (\dim Y^m - \dim Y) \geqslant \dim\big((X^m+X)/X\big) - \dim\big((Y^m+Y)/Y\big)$$

$$= \sum_{j=1}^{t}\dim U^j - \sum_{j=1}^{t}\dim Z^j \geqslant \sum_{j=1}^{t} m_j - \sum_{j=1}^{t}(m_j-1) - 1 = t-1.$$

Therefore $\max m_j \geqslant (m-r)/r$. Since m can be taken arbitrarily large and since a system of type I^{m_j}, $m_j \geqslant 2$, contains a subsystem of type $\mathrm{II}_\theta^{m_j-1}$, this proves our assertion.

PROPOSITION 8.4. *A system of type $\mathrm{II}_\theta^\infty$ is divisible and quasi-spectrally irreducible. Its only eigenvalue is θ and $\dim \mathcal{N}(b_\theta | V) = 1$.*

Proof. Since, by Proposition 8.3, (V, W) is spanned by a proper chain in $R(C_{\rightarrow}^{-\infty,0}(c, d; V, W))$ where c, d is any basis of C^2 such that $\dot{d} = \dot{b}_\theta$, the divisibility and the statement about eigenvalues become clear. If (X, Y) were a non-trivial quasi-spectral subsystem of (V, W), then by Proposition 7.2 (g) it would be divisible. But then $cX = Y$. As we have seen in the first part of the proof of Proposition 8.3, this implies that (X, Y) is of some type II_θ^m; hence not divisible.

Our aim now is to generalize Theorem 6.6 so as to incorporate subsystems of the types $\mathrm{II}_\theta^\infty$.

Definition 8.5. Let (V, W) be a system and a, b a basis of C^2. We write:

$$C\mathrm{II}_\theta^\infty(a, b; V, W) = R\big(C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W)\big)$$

and

$$\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W) = R^{-\infty,0}\Big(C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W) + \sum_{t=0}^{\infty} C_{\leftrightarrow}^{-t,0}(a, b_\theta; V, W)\Big).$$

Then $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ is a subspace of $C\mathrm{II}_\theta^\infty(a, b; V, W)$, and we denote the quotient space $C\mathrm{II}_\theta^\infty(a, b; V, W)/\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ by $Q\mathrm{II}_\theta^\infty(a, b; V, W)$. A chain of $C\mathrm{II}_\theta^\infty(a, b; V, W)$ will be said to be of *type* $\mathrm{II}_\theta^\infty$.

Again we replace a, b_θ by b, a if $\theta = \infty$. Proposition 8.3 says that a system (V, W) is of type $\mathrm{II}_\theta^\infty$ if and only if it is spanned by a proper chain of $C\mathrm{II}_\theta^\infty(a, b; V, W)$. Note that the spaces $C_{\rightarrow}^{-t,0}(a, b_\theta; V, W)$ appearing in the definition of $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ form a non-decreasing sequence. Thus a chain Γ belongs to $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ if and only if there exists a non-negative integer t such that

$$\Gamma \in R^{-\infty,0}\big(C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W) + C_{\leftrightarrow}^{-t,0}(a, b_\theta; V, W)\big).$$

Lemma 8.6. *Let (V, W) be a system and let $(\Gamma^j)_{j\in J}$ be a family of chains of $C\mathrm{II}_\theta^\infty(a, b; V, W)$, θ fixed. Let (V^j, W^j) denote the subsystem spanned by Γ^j. Then in order that each Γ^j be a proper chain of $C\mathrm{II}_\theta^\infty(a, b; V, W)$ and that the sum $\sum_{j\in J}(V^j, W^j)$ be a direct sum which is quasi-spectral in (V, W) it is necessary and sufficient that $(\Gamma^j)_{j\in J}$ be linearly independent modulo $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$.*

Proof. Using Proposition 5.3 (g) we may assume that j runs over a finite set.

The proof of the necessity statement of Theorem 6.6 suits the present case with almost no modification. If $\sum a_j\Gamma^j = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 \in R^{-\infty,0}\big(C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W)\big) \quad \text{and} \quad \Gamma_2 \in R^{-\infty,0}\big(C_{\leftrightarrow}^{-t,0}(a, b_\theta; V, W)\big),$$

t a non-negative integer, then the smallest subsystem of (V, W) containing $\sum(V^j, W^j)$, Γ_1 and Γ_2 is an extension of finite type of $\sum(V^j, W^j)$ since it is sufficient to adjoin Γ_2. Instead of Lemma 6.4 or 6.5, we use the fact that the projections of Γ_2 into the subsystems (V^j, W^j) vanish — otherwise they would give rise to more than one eigenvalue contradicting Proposition 8.4.

Suppose now that the chains $\Gamma^j = \big((v_k^j)_{-\infty}^0, (w_k^j)_{-\infty}^1\big)$ are linearly independent modulo $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ and that there exists a non-trivial dependence relation of the form

$$\sum_j \sum_{k=-t}^{0} a_k^{(j)} w_k^j = 0, \tag{8.6.1}$$

where $t \geqslant 0$ and not all the coefficients $a_{-t}^{(j)}$ vanish. Let $\tilde{\Gamma}^j$ denote the extension of Γ^j to a chain of $C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W)$. Consider the chains

$$\Gamma_1 = R^{-\infty,-t-1} \sum_j a_{-t}^{(j)} \Gamma^j$$

and

$$\Gamma_2 = -R^{-t,0} \sum_j \sum_{k=-t+1}^{0} a_k^{(j)} S^{t+k} \tilde{\Gamma}^j.$$

Then $\Gamma_1 \in C^{-\infty,-t-1}(a, b_\theta; V, W)$, $\Gamma_2 \in R^{-t,0}\big(C_{\rightarrow}^{-t,-1}(a, b_\theta; V, W)\big)$ and, because of (8.6.1), the range elements with index $-t$ of these two chains coincide. Thus there exists a chain Γ_3 of $C^{-\infty,0}(a, b_\theta; V, W)$ such that $R^{-\infty,-t-1}\Gamma_3 = \Gamma_1$ and $R^{-t,0}\Gamma_3 = \Gamma_2$. Obviously

$$\Gamma_3 \in R^{-\infty,0}\big(C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W)\big)$$

and

$$\sum a_{-t}^{(j)} \Gamma^j - \Gamma_3 \in R^{-\infty,0}\big(C_{\leftrightarrow}^{-t,0}(a, b_\theta; V, W)\big).$$

Therefore $\sum a_{-t}^{(j)}\Gamma^j \in \hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$ against the assumption of linear independence. This shows that (8.6.1) is impossible, and since a dependence relation among the domain elements of the chains gives rise to a relation of the form (8.6.1), it follows that the chains are proper and that $\sum(V^j, W^j)$ is a direct sum.

Propositions 8.3, 8.4 and 7.2 (c) now show that $(X, Y) = \sum(V^j, W^j)$ is divisible. Hence to show that this sum is quasi-spectral in (V, W), it suffices by Lemma 7.4 to show that it is spectral in every extension (U, Z) contained in (V, W) such that $(U, Z)/(X, Y)$ is of type I^m.

Let $\Delta = \big((u_k + X)_1^m, (z_k + Y)_1^{m+1}\big)$ be a chain of $C\mathrm{I}^m(a, b_\theta; U/X, Z/Y)$ spanning $(U, Z)/(X, Y)$. We assume, as we may, that $\theta \neq \infty$. Following the procedure in the second case of Lemma 7.3, we construct a sequence

$(x_k^1)_1^m$ of elements of X such that $(u_k - x_k^1)_1^m$ is the domain sequence of a chain Δ^1 of $C\mathrm{II}_\theta^m(a, b; U, Z)$. If $a(u_k - x_1^1) = 0$, then Δ^1 actually belongs to $C\mathrm{I}^m(a, b_\theta; U, Z)$, and by Proposition 3.5, (X, Y) is spectral in (U, Z). Otherwise we have

$$a(u_1 - x_1^1) = \sum_j \sum_{l=-t}^{0} a_l^{(j)} w_l^j, \tag{8.6.2}$$

where not all the coefficients $a_{-t}^{(j)}$ vanish. If $m \geqslant t+1$, then using the extension $\tilde{\Gamma}^j$ introduced before, we get by (8.6.2) that the chain

$$\Delta^1 - R^{1,m} \sum_j \sum_{l=-t}^{0} a_l^{(j)} S^{l-1} \tilde{\Gamma}^j$$

belongs to $C\mathrm{I}^m(a, b_\theta; U, Z)$. Therefore if $(x_k^2)_1^m$ denotes the domain sequence of

$$R^{1,m} \sum_j \sum_{l=-t}^{0} a_l^{(j)} S^{l-1} \tilde{\Gamma}^j,$$

then $(x_k^1 + x_k^2)_1^m$ furnishes the values of the required correcting transformation and (X, Y) is spectral in (U, Z) in this case too.

We conclude the proof by showing that $m \leqslant t$ is against the assumption that the chains Γ^j are linearly independent modulo $\hat{C}\mathrm{II}_\theta^\infty(a, b; V, W)$. Denote by $\tilde{\Delta}^1$ the extension of Δ^1 to a chain of $C_{\to}^{1,m}(a, b_\theta; V, W)$. Because of (8.6.2) there exists a chain Δ^2 of $C^{-\infty,0}(a, b_\theta; V, W)$ such that

$$R^{-t,0}\Delta^2 = R^{-t,0} S^{t+1} \tilde{\Delta}^1 \tag{8.6.3}$$

and

$$R^{-\infty,-t-1}\Delta^2 = R^{-\infty,-t-1} \sum_j \sum_{l=-t}^{0} a_l^{(j)} S^{t+l} \tilde{\Gamma}^j. \tag{8.6.4}$$

Since $m \leqslant t$, it follows from (8.6.3) that $\Delta^2 \in R^{-\infty,0}\big(C_{\to}^{-\infty,-1}(a, b_\theta; V, W)\big)$, while by (8.6.4),

$$\Delta^3 = R^{-\infty,0} \sum_j \sum_{l=-t}^{0} a_l^{(j)} S^{t+l} \tilde{\Gamma}^j - \Delta^2 \in R^{-\infty,0}\big(C_{\leftrightarrow}^{-t,0}(a, b_\theta, V, W)\big).$$

Since

$$\sum a_{-t}^{(j)} \Gamma^j = -R^{-\infty,0} \sum_j \sum_{l=-t+1}^{0} a_l^{(j)} S^{t+l} \tilde{\Gamma}^j + \Delta^2 + \Delta^3$$

and the first summand on the right-hand side belongs to $R^{-\infty,0}\big(C^{-\infty,-1}(a, b_\theta; V, W)\big)$, we obtain

$$\sum_j a_{-t}^{(j)} \Gamma^j \in \hat{C}\mathrm{II}_\theta^\infty(a, b; V, W).$$

From now on the letter Π will stand not only for one of the types I^m, II_θ^m (m finite) and III^m but also for $\mathrm{II}_\theta^\infty$.

Lemma 8.7. *Let $(\Gamma^j)_{j \in J}$ be a family of chains of $C\Pi(a, b; V, W)$ which is linearly independent modulo $\hat{C}\Pi(a, b; V, W)$. Let (P, Q) be the natural homomorphism of (V, W) onto a quotient system $(V, W)/(X, Y)$. Then the chains $(P, Q)\Gamma^j$ (which belong to $C\Pi(a, b; V/X, W/Y)$) are linearly independent modulo $\hat{C}\Pi(a, b; V/X, W/Y)$ in each of the following cases:*

(a) *Π is the type II_θ^m and (X, Y) is divisible;*

(b) *Π is the type III^m and (X, Y) is divisible;*

(c) *Π is the type $\mathrm{II}_\theta^\infty$ and (X, Y) is either a finite-dimensional divisible subsystem or a direct sum of subsystems of the fixed type II_η^∞, where $\eta \neq \theta$.*

Proof. We may assume, of course, that the set of indices J is finite. We shall show that in each case

$$(P, Q)\Gamma \in \hat{C}\Pi(a, b; V/X, W/Y), \tag{8.7.1}$$

where $\Gamma = \sum a_j \Gamma^j$ implies $\Gamma \in \hat{C}\Pi(a, b; V, W)$ and hence $a_j = 0$ for every j.

In case (a), (8.7.1) assumes the form

$$(P, Q)\Gamma = R^{1,m}\Gamma_1 + \Gamma_2 \tag{8.7.2}$$

where $\Gamma_1 \in C_{\to}^{0,m}(a, b_\theta; V/X, W/Y)$ and $\Gamma_2 \in R^{1,m}\big(C_{\to}^{1,m-1}(a, b_\theta; V/X, W/Y)\big)$. Starting with the elements of index m, we subtract suitable elements of X from given representations in V of the domain sequences of Γ_1 and Γ_2 so as to obtain representations

$$\Gamma_1 = (P, Q)\Gamma_3 \quad \text{and} \quad \Gamma_2 = (P, Q)\Gamma_4,$$

where $\Gamma_3 \in C_{\to}^{0,m}(a, b_\theta; V, W)$ and $\Gamma_4 \in R^{1,m}\big(C_{\to}^{1,m-1}(a, b_\theta; V, W)\big)$. This can be done as in the proof of Lemma 7.3 since (X, Y) is divisible and thus $Y = b_\theta X$. Then, by (8.7.2),

$$\Gamma = R^{1,m}\Gamma_3 + \Gamma_4 + \Gamma_5, \tag{8.7.3}$$

where Γ_5 is a chain of $R\big(C_{\to}^{1,m}(a, b_\theta; X, Y)\big)$. The subsystem (X, Y) being divisible, Γ_5 can be extended to the left and belongs in fact to $R^{1,m}\big(C_{\to}^{0,m}(a, b_\theta; X, Y)\big)$. Hence (8.7.3) shows that $\Gamma \in \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$.

The proof in case (b) is similar to the one we have just given and will be omitted.

In case (c), (8.7.1) means that $(P, Q)\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 \in R^{-\infty,0}\big(C_{\to}^{-\infty,-1}(a, b_\theta; V/X, W/Y)\big)$$

and for some $t \geqslant 0$,

$$\Gamma_2 \in R^{-\infty,0}\big(C_{\to}^{-t,0}(a, b_\theta; V/X, W/Y)\big).$$

In the present case, given representations of the chains in (V, W) can be modified by elements of X and Y so as to yield

$$\Gamma_1 = (P, Q)\Gamma_3 \quad \text{and} \quad \Gamma_2 = (P, Q)\Gamma_4,$$

where $\Gamma_3 \in R^{-\infty,0}\big(C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W)\big)$ and

$$\Gamma_4 \in R\big(C_{\rightarrow}^{-\infty,0}(a, b_\theta; V, W)\big),$$

where $R^{-\infty,-t-1}\Gamma_4 \subset (X, Y)$. Hence $\Gamma = \Gamma_3 + \Gamma_4 + \Gamma_5$ with $\Gamma_5 \subset (X, Y)$.

If (X, Y) is of finite dimension, it follows that the infinite chains Γ_4 and Γ_5 lie in finite-dimensional systems, hence cannot be proper. Lemma 8.6 applied to a family composed of a single chain, implies that improper chains of type $\mathrm{II}_\theta^\infty$ belong to $\hat{C}\mathrm{II}_\theta^\infty$; hence our statement.

If

$$(X, Y) = \sum_{i \in I} \cdot (X^i, Y^i),$$

where (X^i, Y^i) is spanned by proper chains $\Delta^i \in R\big(C_{\rightarrow}^{-\infty,0}(a, b_\eta; V, W)\big)$, with $\eta \neq \theta$, we prove again that Γ_4 and Γ_5 are not proper. To this effect write

$$(X^i, Y^i) = \sum_{m=1}^{\infty} (X_m^i, Y_m^i),$$

where (X_m^i, Y_m^i) is spanned by $R^{-m,0}\Delta^i$. The transformation $T(b_\theta)^{-1}T(a)$ is a well defined operator of X into itself under which all the spaces X_m^i are invariant. Write further

$$\Gamma_4 = \big((v_k^4)_{-\infty}^0, (v_k^4)_{-\infty}^1\big), \qquad \Gamma_5 = \big((v_k^5)_{-\infty}^0, (w_k^5)_{-\infty}^1\big).$$

Since $R^{-\infty,-t-1}\Gamma_4 \subset (X, Y)$ and $\Gamma_5 \subset (X, Y)$, we have

$$v_{-t-1-l}^4 = [T(b_\theta)^{-1}T(a)]^l v_{-t-1}^4 \quad \text{and} \quad v_{-l}^5 = [T(b_\theta)^{-1}T(a)]^l v_0^5$$

for $l = 0, 1, 2, \ldots$ Since there exists a finite m and a finite set of indices $K \subset I$ such that v_{-t-1}^4 and v_0^5 belong to $\sum_{i \in K} \cdot X_m^i$, it follows that all v_{-t-1-l}^4 and v_{-l}^5 lie in the finite-dimensional space $\sum_{i \in K} \cdot X_m^i$ and Γ^4 and Γ^5 cannot be proper.

Remark. It is an easy consequence of the following theorem that actually if Π is one of the types I^m, II_θ^m, III^m or $\mathrm{II}_\theta^\infty$, then the linear independence of chains of $C\Pi(a, b; V, W)$ modulo $\hat{C}\Pi(a, b; V, W)$ is preserved in passing to a quotient modulo any direct sum of subsystems having types of the above list which are distinct from the type Π.

THEOREM 8.8. *Let (V, W) be a system. For each type Π among the types I^m, II_θ^m, III^m and $\mathrm{II}_\theta^\infty$, m a positive integer, $\theta \in \tilde{C}$ let $(\Gamma_\Pi^j)_{j \in J_\Pi}$ be*

a (possibly empty) family of chains of $C\Pi(a, b; V, W)$. Let (V_Π^j, W_Π^j) denote the subsystem spanned by Γ_Π^j. Then in order that each Γ_Π^j be a proper chain of $C\Pi(a, b; V, W)$ and that the sum $\sum_\Pi \sum_j (V_\Pi^j, W_\Pi^j)$ be a direct sum which is quasi-spectral in (V, W), it is necessary and sufficient that for every type Π the family $(\Gamma_\Pi^j)_{j \in J_\Pi}$ be linearly independent modulo $\hat{C}\Pi(a, b; V, W)$.

Proof. By Theorem 6.6 and Lemma 8.6 only the sufficiency statement (without the properness of the chains) remains to be proved. In particular, we know that if the chains satisfy the independence condition, then (V_Π^j, W_Π^j) is of type Π. Again we may assume that we have only a finite number s of chains and subsystems to deal with, which we denote now, without specifying the type, by Γ^j and (V^j, W^j), $j = 1, \ldots, s$. We prove the assertion by induction on s supposing its truth for fewer than s subsystems. By the cited results we may assume that at least one of the subsystems is of infinite-dimensional type $\mathrm{II}_\theta^\infty$ but not all the subsystems are of this type (hence the case $s = 1$ is taken care of). We suppose that the indices j have been so chosen that if at least one of the subsystems is of some type I^m, then the subsystems (V^j, W^j), $j = 1, \ldots, r$ $(1 \leqslant r < s)$ are all the subsystems of types I^m, $m = 1, 2, \ldots$, which appear; otherwise the subsystems (V^j, W^j), $j = 1, \ldots, r$, are taken to be all the subsystems of type $\mathrm{II}_\theta^\infty$, θ fixed. The sum

$$(X, Y) = \sum_{j=1}^{r} (V^j, W^j)$$

is obviously a direct sum which is quasi-spectral in (V, W). Furthermore, the sums $(X, Y) + (V^j, W^j)$, $j = r+1, \ldots, s$, are direct sums. If (X, Y) and (V^j, W^j) are both of finite dimension, this follows from Theorem 6.6. Otherwise, Lemma 8.7 is applicable and by this lemma the non-zero domain and range elements of Γ^j are linearly independent modulo X and Y respectively. The directness and quasi-spectrality of

$$\sum_{j=1}^{s} (V^j, W^j) = (X, Y) + \sum_{j=r+1}^{s} (V^j, W^j)$$

will now follow from Propositions 3.1 (e) and 5.3 (b) if we only show that the right-hand side of

$$\sum_{j=1}^{s} (V^j, W^j)/(X, Y) = \sum_{j=r+1}^{s} \big((X, Y) + (V^j, W^j)\big)/(X, Y)$$

is a direct sum which is quasi-spectral in $(V, W)/(X, Y)$. Let (P, Q) denote the natural homomorphism of (V, W) on $(V, W)/(X, Y)$. Then according to Lemma 8.7, the chains $(P, Q)\Gamma^j$, which span the subsystems $((X, Y)+(V^j, W^j))/(X, Y)$, satisfy the linear independence assumption. Consequently, what we need follows from our induction hypothesis.

In contradistinction to Theorem 6.7, a quotient of (V, W) modulo a sum of subsystems spanned by chains which represent bases of $Q\Pi(a, b; V, W)$ for *all* the types Π considered in Theorem 8.8 may well contain quasi-spectral subsystems of the types $\mathrm{II}_\theta^\infty$. This happens for instance in the example given at the beginning of Section 5. We do have however a generalization of Proposition 6.8.

PROPOSITION 8.9. *Let $(V, W)_{\mathrm{I,II}^\infty}$ denote the sum of all subsystems of (V, W) of types I^m, $m = 1, 2, \ldots$, and $\mathrm{II}_\theta^\infty$, $\theta \in \tilde{C}$. For each type Π considered above, let $(\Gamma_\Pi^j)_{j\in J_\Pi}$ be a family of chains of $C\Pi(a, b; V, W)$. Let (V_Π^j, W_Π^j) denote the subsystem spanned by Γ_Π^j. Then for every Π the family $(\Gamma_\Pi^j)_{j\in J_\Pi}$ represents a basis of $Q\Pi(a, b; V, W)$ if and only if each Γ_Π^j is a proper chain of $C\Pi(a, b; V, W)$ and*

$$(V, W)_{\mathrm{I,II}^\infty} = \sum_\Pi{}^{\cdot} \sum_j{}^{\cdot} (V_\Pi^j, W_\Pi^j).$$

The subsystem $(V, W)_{\mathrm{I,II}^\infty}$ is quasi-spectral in the system (V, W) and $(V, W)/(V, W)_{\mathrm{I,II}^\infty}$ contains no subsystem of type I^m or $\mathrm{II}_\theta^\infty$.

Proof. Suppose that each family $(\Gamma_\Pi^j)_{j\in J_\Pi}$ represents a basis of $Q\Pi(a, b; V, W)$, and write

$$\sum_\Pi \sum_j (V_\Pi^j, W_\Pi^j) = (X, Y).$$

Then by Proposition 6.8, $(V, W)_{\mathrm{I}} \subset (X, Y)$. Moreover, since the basis of C^2 which Proposition 6.8 referred to was arbitrary, it follows from the proof of this proposition that (X, Y) contains all the chains of $C\mathrm{I}^m(a, b_\theta; V, W)$, $m = 1, 2, \ldots$, $\theta \in \tilde{C}$. To prove that $(V, W)_{\mathrm{I,II}^\infty} \subset (X, Y)$ we show that (X, Y) contains all the chains of $C\mathrm{II}_\theta^\infty(a, b; V, W)$, $\theta \in \tilde{C}$. Otherwise, let s be the smallest non-negative integer such that there exists a chain $\Gamma = ((v_k)_{-\infty}^0, (w_k)_{-\infty}^1)$ of some space $C\mathrm{II}_\theta^\infty(a, b; V, W)$ with $v_{-s} \notin X$. The chain Γ is of the form

$$\Gamma = \sum_{j\in J} a_j \Gamma_{\mathrm{II}_\theta^\infty}^j + R^{-\infty,0}\Gamma_1 + \Gamma_2,$$

where J is a finite subset of $J_{\mathrm{II}_\theta^\infty}$, $\Gamma_1 = ((u_k)_{-\infty}^\infty, (z_k)_{-\infty}^\infty) \in C_{\rightarrow}^{-\infty,-1}(a, b_\theta; V, W)$ and $\Gamma_2 \in R^{-\infty,0}(C_{\leftrightarrow}^{-t,0}(a, b_\theta; V, W))$ for some $t \geqslant 0$. The chains $\Gamma_{\mathrm{II}_\theta^\infty}^j$ and the chain Γ_2, which is essentially in $C\mathrm{I}^{t+1}(a, b_\theta; V, W)$, are contained in (X, Y). Hence $v_{-s} - u_{-s} \in X$. But $R^{-\infty,0}S^{-1}\Gamma_1$ is a chain of $C\mathrm{II}_\theta^\infty(a, b; V, W)$. Hence by the choice of s, $u_{-s} \in X$ (this is true for $s = 0$ because $u_0 = 0$). Thus $v_{-s} \in X$, against the choice of Γ.

From Theorem 8.8 it follows that the chains are proper and that $\sum_\Pi \sum_j (V_\Pi^j, W_\Pi^j)$ is a quasi-spectral direct sum. Therefore $(X, Y) \subset (V, W)_{\mathrm{I,II}^\infty}$. Thus $(V, W)_{\mathrm{I,II}^\infty}$ equals (X, Y) and is quasi-spectral in (V, W).

Conversely, suppose that the chains are proper and

$$(V, W)_{\mathrm{I,II}^\infty} = \sum_\Pi{}^{\cdot} \sum_j{}^{\cdot} (V_\Pi^j, W_\Pi^j).$$

As we have just proved, $(V, W)_{\mathrm{I,II}^\infty}$ is quasi-spectral in (V, W). Therefore Theorem 8.8 implies that the families $(\Gamma_\Pi^j)_{j\in J_\Pi}$ satisfy the independence condition. These families in fact represent bases of the spaces $Q\Pi(a, b; V, W)$ since otherwise there would exist, again by 8.8, a subsystem of type I^m or $\mathrm{II}_\theta^\infty$ which has intersection $(0, 0)$ with

$$(V, W)_{\mathrm{I,II}^\infty} = \sum \sum (V_\Pi^j, W_\Pi^j).$$

The same impossible situation would occur if $(V, W)/(V, W)_{\mathrm{I,II}^\infty}$ had a subsystem of type I^m or $\mathrm{II}_\theta^\infty$. Being quasi-spectral, $(V, W)_{\mathrm{I,II}^\infty}$ is spectral in extensions by type I^m and being divisible, it is spectral according to Lemma 7.3 in extensions by type $\mathrm{II}_\theta^\infty$.

Remark. There is a complete analog of Proposition 8.9 for the sum $(V, W)_{\mathrm{I,II}_\theta^\infty}$ of all the subsystems of (V, W) of types I^m, $m = 1, 2, \ldots$, and $\mathrm{II}_\theta^\infty$, θ fixed. The proof, which is obtained from that of Proposition 8.9 simply by keeping θ fixed, shows also that $(V, W)_{\mathrm{I,II}_\theta^\infty}$ contains all the chains of types I^m and $\mathrm{II}_\theta^\infty$.

9. Eigenvalue systems. Structure of divisible systems. Reduction

We now generalize to systems the concepts of torsion and torsion free modules over the polynomial ring $C[\lambda]$.

Definition 9.1. A system (V, W) is said to be *eigenvalue free* if and only if it has no eigenvalues.

Let (V, W) be a non-singular system, and let a, b be any basis of C^2 such that $T(a)$ is an isomorphism of V onto W. Consider the module V over $C[\lambda]$ defined as in Section 1 by $p(\lambda)v = p(T(a)^{-1}T(b))v$. It is obvious that this module is torsion free if and only if (V, W) is eigenvalue free. Thus, eigenvalue free systems may be regarded as a generalization of torsion free modules over $C[\lambda]$.

Since we do not have a ring of coefficients for systems, we define the analog of torsion modules in terms of the analog of torsion free modules rather than directly, as usually done in the module case.

A quotient system $(V, W)/(X, Y)$ is eigenvalue free if and only if $cv \in Y$ where $v \in V$ and $c \in C^2 - \{(0,0)\}$ implies that $v \in X$. It is therefore obvious that if $((V^j, W^j))_{j \in J}$ is a family of subsystems of a system (V, W) such that every quotient $(V, W)/(V^j, W^j)$ is eigenvalue free, then

$$(V, W)/\bigcap_{j \in J}(V^j, W^j)$$

is eigenvalue free. As the set $\mathfrak{M}$ of all subsystems (X, Y) of (V, W) such that $(V, W)/(X, Y)$ is eigenvalue free is not empty $((V, W) \in \mathfrak{M})$, it is clear that $\mathfrak{M}$ has a (unique) smallest element with respect to inclusion. This justifies the following definition:

Definition 9.2. The *eigenvalue part of a system* (V, W) is the smallest subsystem (X, Y) of (V, W) such that $(V, W)/(X, Y)$ is eigenvalue free. This subsystem will be denoted by $\operatorname{Eig}(V, W)$. The domain and range space of $\operatorname{Eig}(V, W)$ will be denoted by $\operatorname{Eig} V$ and $\operatorname{Eig} W$ respectively. A system (V, W) is said to be an *eigenvalue system* if and only if $\operatorname{Eig}(V, W) = (V, W)$.

We shall show in this section that the study of eigenvalue parts of systems can be reduced to that of reduced primary modules over $C[\lambda]$ — an extensively investigated subject. Therefore our attention will be restricted mainly to those properties of eigenvalue parts which have bearing on this result.

Proposition 9.3. *Let (V, W) be a system. Then:*

(a) $\operatorname{Eig} V$ *contains every eigenvector of* V. $\operatorname{Eig}(V, W)$ *contains every chain of every space* $C_{\to}^{p,q}(a, b; V, W)$. *Hence it contains all the subsystems of (V, W) of the types* I^m, II_θ^m *and* $\mathrm{II}_\theta^\infty$.

(b) *If* $\operatorname{Eig}(V, W)$ *is eigenvalue free, then* $\operatorname{Eig}(V, W) = (0, 0)$.

(c) *If (X, Y) is a subsystem of* $\operatorname{Eig}(V, W)$, *then* $\operatorname{Eig}((V, W)/(X, Y)) = \operatorname{Eig}(V, W)/(X, Y)$.

(d) $\operatorname{Eig}(V, W)$ *is an eigenvalue system.*

(e) *A quotient system of an eigenvalue system is an eigenvalue system.*

(f) *A spectral subsystem of an eigenvalue system is an eigenvalue system.*

Proof. (a) Suppose that a chain $\Gamma = ((v_k)_p^\infty, (w_k)_p^\infty)$ of $C_{\to}^{p,q}(a, b; V, W)$ is not contained in $\operatorname{Eig}(V, W)$. Then $(v_k)_p^\infty$ is not contained in $\operatorname{Eig} V$. Since, by definition of $C_{\to}^{p,q}(a, b; V, W)$, $q < \infty$ and $v_k = 0$ for $k > q$, we have $q \geqslant p$ and there exists a maximal integer t in $[p, \infty]$ such that $v_t \notin \operatorname{Eig} V$. But then we have $bv_t = av_{t+1} \in \operatorname{Eig} W$, against the fact that $(V, W)/\operatorname{Eig}(V, W)$ is eigenvalue free. Thus $\Gamma \subset \operatorname{Eig}(V, W)$.

The other statements in (a) are an immediate consequence of what we have just proved.

(b) If $\operatorname{Eig}(V, W)$ is eigenvalue free, then by (a), (V, W) has no eigenvectors; i.e., $(V, W)/(0, 0)$ is eigenvalue free.

(c) If $(X, Y) \subset \operatorname{Eig}(V, W)$, then $((V, W)/(X, Y))/(\operatorname{Eig}(V, W)/(X, Y))$ is eigenvalue free since it is isomorphic to $(V, W)/\operatorname{Eig}(V, W)$. Hence by its definition $\operatorname{Eig}((V, W)/(X, Y))$ is contained in $\operatorname{Eig}(V, W)/(X, Y)$. Write

$$\operatorname{Eig}((V, W)/(X, Y)) = (U, Z)/(X, Y).$$

Then $(V, W)/(U, Z) \cong ((V, W)/(X, Y))/((U, Z)/(X, Y))$ is eigenvalue free. Therefore $\operatorname{Eig}(V, W) \subset (U, Z)$ and hence $\operatorname{Eig}(V, W)/(X, Y) \subset (U, Z)/(X, Y)$.

(d) To prove (d), which means $\operatorname{Eig}\operatorname{Eig}(V, W) = \operatorname{Eig}(V, W)$, we have to show that $(X, Y) \subset \operatorname{Eig}(V, W)$ and $\operatorname{Eig}(V, W)/(X, Y)$ is eigenvalue free imply $(X, Y) = \operatorname{Eig}(V, W)$. But by (c), $\operatorname{Eig}((V, W)/(X, Y)) = \operatorname{Eig}(V, W)/(X, Y)$. Hence by (b) this is a zero system.

(e) Let (X, Y) be a subsystem of an eigenvalue system (V, W). Then $(X, Y) \subset (V, W) = \operatorname{Eig}(V, W)$. Hence, by (c),

$$\operatorname{Eig}((V, W)/(X, Y)) = \operatorname{Eig}(V, W)/(X, Y) = (V, W)/(X, Y).$$

(f) This follows from (e) because a spectral subsystem is isomorphic to a quotient modulo a direct complement.

Lemma 9.4. *If θ is an eigenvalue of the system (V, W), then (V, W) contains a quasi-spectral subsystem of one of the types* I^m, II_θ^m *or* $\mathrm{II}_\theta^\infty$.

Proof. We assume that (V, W) does not contain a quasi-spectral subsystem of one of the types II_θ^m and show that it contains a quasi-spectral subsystem of one of the types I^m or $\mathrm{II}_\theta^\infty$.

From our assumption it follows by Theorem 6.6 that $C\mathrm{II}_\theta^m(a, b; V, W) = \hat{C}\mathrm{II}_\theta^m(a, b; V, W)$ for every m. Applying a shift operator, we see that this is equivalent to

$$R(C_{\to}^{-t,0}(a, b_\theta; V, W)) = R^{-t,0}(C_{\to}^{-t-1,0}(a, b_\theta; V, W)) + R^{-t,0}(C_{\to}^{-t,-1}(a, b_\theta; V, W)) \tag{9.4.1}$$

for every non-negative integer t. We prove by induction on t that (9.4.1) implies

$$R(C_{\to}^{-t,0}(a, b_\theta; V, W)) \subset R^{-t,0}(C_{\to}^{-t-1,0}(a, b_\theta; V, W)) \quad \text{for all } t \geqslant 0. \tag{9.4.2}$$

For $t = 0$ (9.4.2) follows from (9.4.1) because $R^{0,0}(C_{\to}^{0,-1}(a, b_\theta; V, W)) = 0$. Suppose that $s > 0$ and that (9.4.2) is valid for $t = s-1$. Let $\Gamma \in R(C_{\to}^{-s,0}(a, b_\theta; V, W))$. By (9.4.1) we have

$$\Gamma = R^{-s,0}\Gamma^1 + R^{-s,0}\Gamma^2,$$

where $\Gamma^1 \in C_{\to}^{-s-1,0}(a, b_\theta; V, W)$ and $\Gamma^2 \in C_{\to}^{-s,-1}(a, b_\theta; V, W)$. Then

$$R^{-s+1,0}S^{-1}\Gamma^2 \in R\big(C_{\to}^{-s+1,0}(a, b_\theta; V, W)\big).$$

Hence by the induction hypothesis, there exists a chain $\Gamma^3 \in C_{\to}^{-s,0}(a, b_\theta; V, W)$ such that $R^{-s+1,0}S^{-1}\Gamma^2 = R^{-s+1,0}\Gamma^3$. Since the domain elements with index 1 of both $S^{-1}\Gamma^2$ and Γ^3 vanish, we actually have $R^{-s+1,1}S^{-1}\Gamma^2 = R^{-s+1,1}\Gamma^3$. Applying the operator S to the last equation, we obtain $R^{-s,0}\Gamma^2 = R^{-s,0}S\Gamma^3$. Since $S\Gamma^3 \in C_{\to}^{-s-1,0}(a, b_\theta; V, W)$, we have

$$\Gamma = R^{-s,0}\Gamma^1 + R^{-s,0}\Gamma^2 \in R^{-s,0}\big(C_{\to}^{-s-1,0}(a, b_\theta; V, W)\big),$$

as desired.

Since θ is an eigenvalue of (V, W), there exists a non-zero chain of $R\big(C_{\to}^{0,0}(a, b_\theta; V, W)\big)$. From (9.4.2) it follows that it can be extended to a non-zero chain of $C\mathrm{II}_\theta^\infty(a, b; V, W)$. By the remark following Proposition 8.9 this implies that $(V, W)_{\mathrm{I},\mathrm{II}_\theta^\infty} \neq (0, 0)$, and therefore (V, W) contains a quasi-spectral subsystem of one of the types I^m or of type $\mathrm{II}_\theta^\infty$.

Proposition 9.5. *For every system (V, W) we have* $\operatorname{Eig}\operatorname{Div}(V, W) = (V, W)_{\mathrm{I},\mathrm{II}^\infty}$.

Proof. If (X, Y) is a subsystem of (V, W) of one of the types I^m or $\mathrm{II}_\theta^\infty$, then (X, Y) is divisible. Hence, by the definition of $\operatorname{Div}(V, W)$ (Proposition 7.2 (c)), $(X, Y) \subset \operatorname{Div}(V, W)$. But then it follows from 9.3 (a) that $(X, Y) \subset \operatorname{Eig}\operatorname{Div}(V, W)$. Since $(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ is the sum of all subsystems (X, Y) of the above types, we have $(V, W)_{\mathrm{I},\mathrm{II}^\infty} \subset \operatorname{Eig}\operatorname{Div}(V, W)$. A fortiori $(V, W)_{\mathrm{I},\mathrm{II}^\infty} \subset \operatorname{Div}(V, W)$, and the quotient $\operatorname{Div}(V, W)/(V, W)_{\mathrm{I},\mathrm{II}_\theta^\infty}$ is well defined. If we show that this quotient is eigenvalue free, we shall obtain the reverse inclusion $\operatorname{Eig}\operatorname{Div}(V, W) \subset (V, W)_{\mathrm{I},\mathrm{II}^\infty}$.

According to Proposition 8.9. $(V, W)/(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ and hence also its subsystem $\operatorname{Div}(V, W)/(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ contains no subsystem of type I^m or $\mathrm{II}_\theta^\infty$. By part (d) of Proposition 7.2, $\operatorname{Div}(V, W)/(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ is divisible. Hence by part (g) of the same proposition it does not contain non-divisible quasi-spectral subsystems. In particular, $\operatorname{Div}(V, W)/(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ has no quasi-spectral subsystem of any of the types II_θ^m. The desired conclusion, that $\operatorname{Div}(V, W)/(V, W)_{\mathrm{I},\mathrm{II}^\infty}$ is eigenvalue free, now follows from Lemma 9.4.

The preceding is more than we need of the general properties of eigenvalue parts in order to describe divisible systems. However, we still have to introduce one isomorphism type of eigenvalue free systems.

Let $\boldsymbol{C}(\lambda)$ denote the set of all rational functions in the complex variable λ. Unless otherwise stated, $\boldsymbol{C}(\lambda)$ will stand also for this set endowed with its usual structure as a complex vector space. Let a, b be a basis

of $\boldsymbol{C}^2$. Consider the system $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$ having $\boldsymbol{C}(\lambda)$ as domain and range space and the mapping $\big(\varepsilon a+\zeta b, f(\lambda)\big) \to (\varepsilon a+\zeta b)f(\lambda) = (\varepsilon+\zeta\lambda)f(\lambda)$ as system operation. Although the system $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$ actually depends on the basis a, b, its isomorphism type does not. If $c = \alpha a+\beta b$, $d = \gamma a+\delta d$ is another basis of $\boldsymbol{C}^2$, then the pair (P, Q) of linear transformations of $\boldsymbol{C}(\lambda)$ onto itself defined by

$$Pf(\lambda) = \frac{1}{\alpha+\beta\lambda} f\left(\frac{\gamma+\delta\lambda}{\alpha+\beta\lambda}\right), \quad Qf(\lambda) = f\left(\frac{\gamma+\delta\lambda}{\alpha+\beta\lambda}\right)$$

is an isomorphism of $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{c,d}$ onto $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$.

Definition 9.6. The common isomorphism type of all the systems $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$, a, b a basis of $\boldsymbol{C}^2$, will be denoted by $\mathfrak{R}$. A system is said to be a *full rational system* if and only if it is of type $\mathfrak{R}$. A system is said to be a *rational system* if and only if it is a subsystem of a full rational system.

A full rational system (V, W) is clearly eigenvalue free and divisible. Hence it is non-singular with $T(c)$ an isomorphism of V onto W for every $c \in \boldsymbol{C}^2-\{(0, 0)\}$. It is easy to see that every module corresponding to (V, W) by the procedure described at the beginning of this section is isomorphic to $\boldsymbol{C}(\lambda)$ regarded as a module over $\boldsymbol{C}[\lambda]$.

Clearly, every rational system is eigenvalue free (but not necessarily divisible). Every system of type III^m, $m = 1, 2, \ldots$, is rational.

Lemma 9.7. *An eigenvalue free divisible system (V, W) is a direct sum of full rational subsystems. The cardinal number of summands in such a decomposition is uniquely determined by (V, W).*

Proof. The proof can be reduced to the module case, but it is almost as simple to give it directly for systems. Let a, b be a basis of $\boldsymbol{C}^2$. Since (V, W) is eigenvalue free and divisible, for every $\theta \in \tilde{\boldsymbol{C}}$ the operator

$$T(a)^{-1}T(b_\theta) = \begin{cases} T(a)^{-1}T(b)-\theta\mathrm{I} & \text{if } \theta \neq \infty, \\ \mathrm{I} & \text{if } \theta = \infty \end{cases}$$

is a well defined linear automorphism of V. These operators clearly commute. Therefore, if $f(\lambda) \in \boldsymbol{C}(\lambda)$, we can define the operator $f\big(T(a)^{-1}T(b)\big)$ on V uniquely by substituting $T(a)^{-1}T(b)$ for λ in any representation of $f(\lambda)$ as a quotient of products of linear polynomials. We make V into a vector space over the field $\boldsymbol{C}(\lambda)$ by defining $f(\lambda)v = f\big(T(a)^{-1}T(b)\big)v$. If $(v_j)_{j\in J}$ is a basis of this vector space, the space decomposes into a direct sum $V = \sum\cdot \boldsymbol{C}(\lambda)v_j$, where the summands $\boldsymbol{C}(\lambda)v_j$ are a fortiori subspaces of V as a vector space over $\boldsymbol{C}$. Since $T(a)$ is an isomorphism of V onto W, we have a corresponding decomposition over the complex numbers $W = \sum\cdot a\boldsymbol{C}(\lambda)v_j$. One immediately verifies that the pair of linear transformations (P_j, Q_j) where $P_j f(\lambda) = f(\lambda)v_j$, $Q_j f(\lambda) = af(\lambda)v_j$ is an isomorphism of $(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda))_{a,b}$ onto the *subsystem* $(\boldsymbol{C}(\lambda)v_j, a\boldsymbol{C}(\lambda)v_j)$. Thus $(V, W) = \sum\cdot(\boldsymbol{C}(\lambda)v_j, a\boldsymbol{C}(\lambda)v_j)$ is a decomposition of (V, W) into a direct sum of full rational subsystems.

Conversely, let

$$(V, W) = \sum_{j\in J}\cdot(V^j, W^j)$$

be any decomposition of (V, W) into subsystems of type $\mathfrak{R}$ and let $0 \neq v_j \in V^j$. Then $(v_j)_{j\in J}$ is a basis of V made as above into a vector space over $\boldsymbol{C}(\lambda)$ using any basis a, b of $\boldsymbol{C}^2$. Hence the cardinal number of summands in the decomposition equals the dimension of V over $\boldsymbol{C}(\lambda)$ (which therefore does not depend on a, b) and is uniquely determined by (V, W).

As a corollary of the preceding lemma we obtain a characterization of the isomorphism type $\mathfrak{R}$.

PROPOSITION 9.8. *A system (V, W) is of type $\mathfrak{R}$ if and only if it is non-zero, eigenvalue free and divisible, and no proper subsystem of (V, W) has the same properties. A system of type $\mathfrak{R}$ is quasi-spectrally irreducible.*

Proof. We already know that a system (V, W) of type $\mathfrak{R}$ is non-zero, eigenvalue free and divisible and that a vector space over $\boldsymbol{C}(\lambda)$ attached to it as in the proof of Lemma 9.7 is of dimension 1. If (X, Y) is a subsystem of (V, W) with the same properties, then X becomes a non-zero $\boldsymbol{C}(\lambda)$-subspace of V. Hence $X = V$ and $Y = \boldsymbol{C}^2 X = \boldsymbol{C}^2 V = W$.

Conversely, if (V, W) satisfies the requirements of the proposition, then by Lemma 9.7 it is a non-empty direct sum of subsystems of type $\mathfrak{R}$, and by the minimality requirement there must be just one summand.

Finally, if (X, Y) is a quasi-spectral subsystem of a system (V, W) of type $\mathfrak{R}$, then, by Proposition 7.2 (g), (X, Y) is divisible. Since (X, Y) is clearly also eigenvalue free, it follows from the above characterization that it must be a trivial subsystem of (V, W).

LEMMA 9.9. *A divisible system is spectral in extensions by rational systems.*

Proof. From Lemma 3.2 and Proposition 3.1 (c) it follows that it is sufficient to consider the case of an extension by type $\mathfrak{R}$.

Let us represent a system of type $\mathfrak{R}$ by $(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda))_{a,b}$. If $\boldsymbol{C}[\lambda]$ denotes the complex vector space of polynomials in λ, then $(\boldsymbol{C}[\lambda], \boldsymbol{C}[\lambda])$ is a subsystem of $(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda))_{a,b}$ which is spanned by the proper chain $((\lambda^k)_0^\infty, (\lambda^k)_0^\infty)$ of $C^{0,\infty}(a, b; \boldsymbol{C}[\lambda], \boldsymbol{C}[\lambda])$. From the unique representation of rational functions in λ by partial fractions and the fact that for $\theta \neq \infty$ the pair of sequences of cosets

$$\Big(\big((\lambda-\theta)^{k-1}+\boldsymbol{C}[\lambda]\big)_{-\infty}^{0}, \ \big((\lambda-\theta)^{k-1}+\boldsymbol{C}[\lambda]\big)_{-\infty}^{1}\Big)$$

is a proper chain of $R\big(C_{\to}^{-\infty,0}(a, b_\theta; \boldsymbol{C}(\lambda))/\boldsymbol{C}[\lambda], \boldsymbol{C}(\lambda)/\boldsymbol{C}[\lambda]\big)$, one sees that $(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda))_{a,b}/(\boldsymbol{C}[\lambda], \boldsymbol{C}[\lambda])$ is a direct sum of subsystems of the types $\mathrm{II}_\theta^\infty$, θ ranging over the finite complex numbers. Hence, if (V, W) is an extension by type $\mathfrak{R}$ of a divisible system (K, L), then $(V, W)/(K, L)$ contains a subsystem $(V^1, W^1)/(K, L)$ spanned by a proper chain of $C^{0,\infty}(a, b; V^1/K, W^1/L)$ such that

$$\big((V, W)/(K, L)\big)/\big((V^1, W^1)/(K, L)\big)$$

is a direct sum of subsystems of the types $\mathrm{II}_\theta^\infty$.

From the chain representation of $(V^1, W^1)/(K, L)$ and Lemma 7.3 it follows that (K, L) is spectral in (V^1, W^1):

$$(V^1, W^1) = (K, L)\dot{+}(M, N). \tag{9.9.1}$$

The quotient $\big((V, W)/(M, N)\big)/\big((V^1, W^1)/(M, N)\big)$, being isomorphic to $\big((V, W)/(K, L)\big)/\big((V^1, W^1)/(K, L)\big)$, has a decomposition

$$\begin{aligned}\big((V, W)/(M, N)\big)/\big((V^1, W^1)/(M, N)\big) \\ = \sum_{\theta\in C}\cdot\big((U^\theta, Z^\theta)/(M, N)\big)/\big((V^1, W^1)/(M, N)\big),\end{aligned} \tag{9.9.2}$$

where the summands on the right-hand side are of the types $\mathrm{II}_\theta^\infty$. Since $(V^1, W^1)/(M, N)$ is isomorphic to (K, L), it is divisible. Thus by Lemma 7.3, $(V^1, W^1)/(M, N)$ is spectral in every $(U^\theta, Z^\theta)/(M, N)$. According to Proposition 3.1 (e), this fact together with (9.9.2) imply the existence of a decomposition

$$(V, W)/(M, N) = (V^1, W^1)/(M, N)\dot{+}(V^2, W^2)/(M, N). \tag{9.9.3}$$

From (9.9.1) and (9.9.3) one easily concludes that

$$(V, W) = (K, L)\dot{+}(V^2, W^2),$$

which proves our lemma.

Lemma 9.9 justifies the remark made after Lemma 7.3 since a system spanned by a proper chain of $C^{-\infty,a}$ is isomorphic to a subsystem of $\mathfrak{R}$.

THEOREM 9.10. *A system (V, W) is divisible if and only if it is a direct sum of subsystems of the types I^m, $\mathrm{II}_\theta^\infty$ and $\mathfrak{R}$. The cardinal number of summands of each type in such a decomposition is uniquely determined by (V, W).*

Proof. In Propositions 7.2 (b), 8.4 and 9.8 we have seen that systems of the types mentioned in the theorem are divisible. Hence it follows from Proposition 7.2 (c) that the condition of the theorem is sufficient for divisibility.

Conversely, let (V, W) be a divisible system. Its subsystem $(V, W)_{\mathrm{I,II}\infty}$ is also divisible. According to Proposition 9.5, $(V, W)/(V, W)_{\mathrm{I,II}\infty}$ is eigenvalue free. Since by Lemma 7.2 (d) this quotient is also divisible, it follows from Lemma 9.7 that

$$(9.10.1)\qquad (V, W)/(V, W)_{\mathrm{I,II}\infty} = \sum\cdot(U^j, Z^j)/(V, W)_{\mathrm{I,II}\infty},$$

where the summands on the right-hand side are of type $\mathfrak{R}$. By Lemma 9.9, $(V, W)_{\mathrm{I,II}\infty}$ is spectral in every (U^j, Z^j). This implies, by 3.1 (e), that

$$(V, W) = (V, W)_{\mathrm{I,II}\infty} \dotplus \sum\cdot(X^j, Y^j),$$

where the summands (X^j, Y^j) are again of type $\mathfrak{R}$. Since, by Proposition 8.9, $(V, W)_{\mathrm{I,II}\infty}$ is a direct sum of subsystems of the types I^m and $\mathrm{II}_\theta^\infty$, (V, W) has a decomposition of the required kind.

To prove the uniqueness statement, consider any decomposition

$$(9.10.2)\qquad (V, W) = \sum\cdot(V^i, W^i) \dotplus \sum\cdot(X^j, Y^j)$$

of (V, W), where the subsystems (V^i, W^i) are of the types I^m or $\mathrm{II}_\theta^\infty$ while the subsystems (X^j, Y^j) are of type $\mathfrak{R}$. Then the quotient $(V, W)/\sum\cdot(V^i, W^i)$, being isomorphic to $\sum\cdot(X^j, Y^j)$, is eigenvalue free. Hence $\sum\cdot(V^i, W^i)$ is contained in $\mathrm{Eig}(V, W)$, which by Proposition 9.5 coincides with $(V, W)_{\mathrm{I,II}\infty}$. Since the reverse inclusion follows from the definition of $(V, W)_{\mathrm{I,II}\infty}$, we have $\sum\cdot(V^i, W^i) = (V, W)_{\mathrm{I,II}\infty}$. Proposition 8.9 now implies that the cardinal number of summands in (9.10.2) of each of the types I^m and $\mathrm{II}_\theta^\infty$ is uniquely determined by (V, W). Lemma 9.7 applied to $(V, W)/(V, W)_{\mathrm{I,II}\infty}$ shows that the same holds for the cardinal number of summands of type $\mathfrak{R}$.

We can now characterize the types $\mathrm{II}_\theta^\infty$ (or equivalently, the types II_c^∞) by requirements similar to those used in Definition 2.1 for finite-dimensional isomorphism types.

PROPOSITION 9.11. *A system (V, W) is of type $\mathrm{II}_\theta^\infty$ if and only if it is a divisible system which has the single eigenvalue θ, and no proper subsystem of (V, W) has the same properties.*

Proof. We already know (see Proposition 8.4 and its proof) that a system of type $\mathrm{II}_\theta^\infty$ satisfies the above requirements. Conversely, let (V, W) satisfy these requirements. By Theorem 9.10, (V, W) is a direct sum of subsystems of types I^m, II_η^∞ or $\mathfrak{R}$. No subsystem of a type I^m can actually appear in such a decomposition, because this would imply that every extended complex number is an eigenvalue of (V, W). Hence, since θ is an eigenvalue of (V, W), at least one summand in the decomposition is of type $\mathrm{II}_\theta^\infty$. The minimality assumption implies that it must be the only summand.

Our next aim is to show that if (X, Y) is a divisible quasi-spectral subsystem of (V, W), then (X, Y) is spectral in (V, W). Before giving the lemmas leading to this result, which are of independent interest, let us remark that the usual proof of the analog for modules over principal ideal domains actually shows more. One easily proves that if X is a divisible submodule of V and U is any maximal element in the set of submodules of V which have zero intersection with X, then $V = X \dotplus U$. Hence, by Zorn's lemma, X is always a direct summand of V. The following examples show that a divisible subsystem is not necessarily spectral, and even if it is, a maximal subsystem among those which form a direct sum with it is not always a direct complement. This accounts for the more involved proof of the mentioned result in the case of systems.

Let (V, W) be a direct sum of two subsystems of type I^2 spanned by chains Γ^1 and Γ^2 of $C^{1,2}_{\leftrightarrow}(a, b; V, W)$. Then $\Delta = R^{1,3}(S^{-1}\Gamma^1 + \Gamma^2)$ is a proper chain of $C\mathrm{I}^3(a, b; V, W)$, and thus it spans a divisible subsystem (X, Y) of type I^3. Since

$$R^{1,3}S^{-1}\Gamma^1 \in R^{1,3}\big(C^{2,3}_{\leftrightarrow}(a, b; V, W)\big) \quad\text{and}\quad R^{1,3}\Gamma^2 \in R^{1,3}\big(C^{1,2}_{\leftrightarrow}(a, b; V, W)\big),$$

we have $\Delta \in \hat{C}\mathrm{I}^3(a, b; V, W)$ and (X, Y) is not spectral in (V, W).

Let (V, W) be a system defined by the following requirements:

$$V = X \dotplus U \dotplus [m], \qquad W = Y \dotplus Z,$$

where (X, Y) is a subsystem of type I^3 spanned by a chain $\big((x_k)_1^3, (y_k)_1^4\big)$ of $C\mathrm{I}^3(a, b; V, W)$, (U, Z) is a subsystem of type III^4 spanned by a chain $\big((u_k)_1^3, (z_k)_1^4\big)$ of $C\mathrm{III}^4(a, b; V, W)$ and $m \neq 0$, $am = z_1$, $bm = y_3 + z_3$. Then (X, Y) is a divisible spectral subsystem of (V, W) (actually $(X, Y) = \mathrm{Div}(V, W)$) — a direct complement is the direct sum of subsystems of types II^2_∞, II^1_1 and II^1_0 spanned by chains with domain-elements $(x_3 - u_2 + u_3, -u_1 + m)$, $(-x_3 + u_1 + u_2 - m)$ and $(-x_2 - u_2 + m)$ respectively. On the other hand, the subsystem (U, Z), which is not a direct complement of (X, Y) in (V, W), is easily seen to be maximal among the subsystems of (V, W) which form a direct sum with (X, Y).

PROPOSITION 9.12. *The eigenvalue part of a system (V, W) is quasi-spectral in (V, W). It equals the sum of all the subsystems of (V, W) of the types I^m, II_θ^m, and $\mathrm{II}_\theta^\infty$ (or equivalently, of the types I^1 and II_θ^m).*

Proof. Since a system of type I^m, $m \geqslant 2$, is a sum of subsystems of the types II^1_∞ and II_0^{m-1}, while a system of type $\mathrm{II}_\theta^\infty$ is a sum of subsystems of the types II_θ^m, $m = 1, 2, \ldots$, the two sums mentioned in the proposition give the same subsystem (U, Z).

We construct an increasing transfinite sequence $\big((V^\nu, W^\nu)\big)_{0\leqslant\nu\leqslant\sigma}$ of subsystems of (U, Z) which are quasi-spectral in (V, W) such that:

1° $(V^0, W^0) = (0, 0)$;

2° if $\nu \leqslant \sigma$ is a limit ordinal, then

$$(V^\nu, W^\nu) = \sum_{\mu<\nu} (V^\mu, W^\mu);$$

3° if $\nu \leqslant \sigma$ has an immediate predecessor $\nu-1$, then

$$(V^\nu, W^\nu)/(V^{\nu-1}, W^{\nu-1})$$

is of one of the types I^m, II_θ^m or $\mathrm{II}_\theta^\infty$;

4° $(V, W)/(V^\sigma, W^\sigma)$ is eigenvalue free.

Starting from $(V^0, W^0) = (0, 0)$ and supposing that $\big((V^\nu, W^\nu)\big)_{0\leqslant\nu<\varrho}$ is already defined so as to satisfy our requirements, we define

$$(V^\varrho, W^\varrho) = \sum_{\nu<\varrho} (V^\nu, W^\nu)$$

in case ϱ is a limit ordinal. Then (V^ϱ, W^ϱ) is contained in (U, Z) and, by Proposition 5.3 (f), it is quasi-spectral in (V, W). If $\varrho-1$ exists and $(V, W)/(V^{\varrho-1}, W^{\varrho-1})$ is not yet eigenvalue free, then by Lemma 9.4, it contains a quasi-spectral subsystem $(V^\varrho, W^\varrho)/(V^{\varrho-1}, W^{\varrho-1})$ of one of the considered types. By Proposition 5.3 (b), (V^ϱ, W^ϱ) is quasi-spectral in (V, W).

If $(V^\varrho, W^\varrho)/(V^{\varrho-1}, W^{\varrho-1})$ is of type I^m or II_θ^m, then $(V^{\varrho-1}, W^{\varrho-1})$ is spectral in (V^ϱ, W^ϱ) and a direct complement is of the type I^m or II_θ^m respectively. Hence $(V^\varrho, W^\varrho) \subset (U, Z)$.

If $(V^\varrho, W^\varrho)/(V^{\varrho-1}, W^{\varrho-1})$ is of type $\mathrm{II}_\theta^\infty$, then it is a sum of subsystems of types II_θ^m, and by the above argument $(V^\varrho, W^\varrho) \subset (U, Z)$ in this case too. Since the sequence is strictly increasing, a subsystem (V^σ, W^σ) satisfying 4° is eventually reached.

We now have $(V^\sigma, W^\sigma) \subset (U, Z)$ by construction, $(U, Z) \subset \mathrm{Eig}(V, W)$ by Proposition 9.3 (a) and $\mathrm{Eig}(V, W) \subset (V^\sigma, W^\sigma)$ by 4° and the definition of $\mathrm{Eig}(V, W)$. Hence, these three subsystems coincide and, like (V^σ, W^σ), they are quasi-spectral in (V, W).

COROLLARY 9.13. (a) $\mathrm{Eig}(V, W)$ *is the largest eigenvalue subsystem of* (V, W).

(b) *If* $(X, Y) \subset (V, W)$, *then* $\mathrm{Eig}(X, Y) \subset \mathrm{Eig}(V, W)$.

(c) *If* $(X, Y) \subset (V, W)$ *and both* (X, Y) *and* $(V, W)/(X, Y)$ *are eigenvalue systems, then so is* (V, W).

Proof. According to Proposition 9.3 (d), $\mathrm{Eig}(V, W)$ is an eigenvalue system. By 9.3 (a) it contains every subsystem of (V, W) of one of the types I^1 or II_θ^m. Hence by Proposition 9.12 it contains every eigenvalue subsystem of (V, W).

Part (b) follows from part (a) on noting that $\mathrm{Eig}(X, Y)$ is an eigenvalue subsystem of (V, W).

Under the assumptions of part (c) we have $(X, Y) = \mathrm{Eig}(X, Y)$. Hence by part (b), $(X, Y) \subset \mathrm{Eig}(V, W)$. Proposition 9.3 (c) now yields

$$(V, W)/(X, Y) = \mathrm{Eig}\big((V, W)/(X, Y)\big) = \mathrm{Eig}(V, W)/(X, Y).$$

Thus $(V, W) = \mathrm{Eig}(V, W)$.

LEMMA 9.14. *If* (V, W) *is a non-zero eigenvalue free system, then it contains a non-zero rational subsystem* (V^1, W^1) *such that* $(V, W)/(V^1, W^1)$ *is eigenvalue free.*

Proof. The case $V = 0$ is trivial, so we assume $V \neq 0$ and choose a fixed $v_0 \neq 0$ in V. Denote the subsystem $([v_0], \boldsymbol{C}^2[v_0])$, which is of type III^2, by (X, Y), and define (V^1, W^1) by

$$(V^1, W^1)/(X, Y) = \mathrm{Eig}\big((V, W)/(X, Y)\big).$$

Then $(V, W)/(V^1, W^1)$ is obviously eigenvalue free, and we shall show that (V^1, W^1) can be isomorphically embedded in $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$.

The system $(V^1, W^1)/(X, Y)$ does not contain subsystems of any type I^m since if $(X^1, Y^1)/(X, Y)$ were such a subsystem, we would have $\dim X^1 = \dim Y^1 = m+1$ against the fact that (V, W) is eigenvalue free. Since $(V^1, W^1)/(X, Y)$ is an eigenvalue system, it follows from Proposition 9.12 that it is the sum of all its subsystems $(U^j, Z^j)/(X, Y)$ which are *finite sums* of subsystems of the types II_θ^m; hence $(V^1, W^1) = \sum (U^j, Z^j)$. We assert that there exists one and only one isomorphism (P_j, Q_j) of (U^j, Z^j) into $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$ which sends v_0 to a preassigned element of $\boldsymbol{C}(\lambda)$, say 1. Since the family $\big((U^j, Z^j)\big)$ is directed under the relation of inclusion $\subset$, the isomorphisms (P_j, Q_j) agree on intersections and together yield an isomorphism of (V^1, W^1) into $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$.

To prove our assertion note that the finite-dimensional system $(U^j, Z^j)/(X, Y)$ is an eigenvalue system (Proposition 9.12) and hence has no spectral subsystem of any type III^m. Since it has also no subsystem of any type I^m, it must be non-singular. Therefore $\dim U^j/X = \dim Z^j/Y$, and $\dim Z^j - \dim U^j = 1$. As a finite-dimensional eigenvalue free system, (U^j, Z^j) decomposes into a direct sum of subsystems of the types III^m. Since $\dim Z^j - \dim U^j = 1$, there is exactly one summand, and (U^j, Z^j) is of some type III^m, $m \geqslant 2$. Let $\big((u_k)_1^{m-1}, (z_k)_1^m\big)$ be a chain of $C\mathrm{III}^m$ which spans (U^j, Z^j). If (P_j, Q_j) is an isomorphism of (U^j, Z^j) into $\big(\boldsymbol{C}(\lambda), \boldsymbol{C}(\lambda)\big)_{a,b}$ and $P_j u_1 = f(\lambda)$, we must obviously have $P_j u_k = \lambda^{k-1} f(\lambda)$, $1 \leqslant k \leqslant m-1$, and $Q_j z_k = \lambda^{k-1} f(\lambda)$, $1 \leqslant k \leqslant m$. Since there is a unique representation

$$v_0 = \sum_{k=1}^{m-1} a_k u_k,$$

the condition $P v_0 = 1$ implies

$$f(\lambda) = 1/\sum_{k=1}^{m-1} a_k \lambda^{k-1}.$$

This shows that (P_j, Q_j) if it exists, is uniquely determined. Since this determination obviously yields an isomorphism which satisfies our requirement, our assertion follows.

Remark. The analogous construction for modules over principal ideal domains yields of course a pure submodule, but it is easy to give examples (see [2]) in which (V^1, W^1) is not quasi-spectral.

THEOREM 9.15. *If (X, Y) is a divisible quasi-spectral subsystem of (V, W), then (X, Y) is spectral in (V, W).*

Proof. The argument follows Sąsiada's characterization of algebraically compact groups ([3], p. 83). There exists an increasing transfinite sequence $((V^\nu, W^\nu))_{0\leqslant\nu\leqslant\tau}$ of subsystems of (V, W) and an ordinal σ, $0\leqslant\sigma\leqslant\tau$, such that:

1° $(V^0, W^0) = (X, Y)$.

2° If $\nu\leqslant\tau$ is a limit ordinal, then

$$(V^\nu, W^\nu) = \sum_{\mu<\nu}(V^\mu, W^\mu).$$

3° If $\nu\leqslant\sigma$, then (V^ν, W^ν) is quasi-spectral in (V, W).

4° If $\nu\leqslant\sigma$ has an immediate predecessor $\nu-1$, $(V^\nu, W^\nu)/(V^{\nu-1}, W^{\nu-1})$ is of one of the types I^m, II^m_θ or $\mathrm{II}^\infty_\theta$.

5° If $\sigma\leqslant\nu\leqslant\tau$, then $(V, W)/(V^\nu, W^\nu)$ is eigenvalue free.

6° If $\sigma<\nu\leqslant\tau$ and ν has an immediate predecessor $\nu-1$, then $(V^\nu, W^\nu)/(V^{\nu-1}, W^{\nu-1})$ is a rational system.

7° $(V^\tau, W^\tau) = (V, W)$.

To see this let $((V^\nu, W^\nu)/(X, Y))_{0\leqslant\nu\leqslant\sigma}$ be a transfinite sequence constructed as in the proof of Proposition 9.12 but for the quotient $(V, W)/X, Y)$ instead of (V, W). Since (X, Y) is quasi-spectral in (V, W), Proposition 5.3 (b) implies that 3° holds. Since $(V, W)/(V^\sigma, W^\sigma)$ is isomorphic to $((V, W)/(X, Y))/((V^\sigma, W^\sigma)/(X, Y))$, it is eigenvalue free. Evidently 1° and 4° hold too. Suppose now that $(V^\sigma, W^\sigma)\neq(V, W)$ and that $((V^\nu, W^\nu))_{0\leqslant\nu<\varrho}$, $\sigma<\varrho$, is already properly defined. If ϱ is a limit ordinal, we define

$$(V^\varrho, W^\varrho) = \sum_{\nu<\varrho}(V^\nu, W^\nu).$$

Since (V^ϱ, W^ϱ) is also the sum of the increasing subsequence $((V^\nu, W^\nu))_{\sigma\leqslant\nu<\varrho}$ and 5° holds for $\sigma\leqslant\nu<\varrho$, it holds for $\nu=\varrho$ as well. If $\varrho-1$ exists and $(V, W)/(V^{\varrho-1}, W^{\varrho-1})$ is not yet a zero system, then by Lemma 9.14 we take $(V^\varrho, W^\varrho)/(V^{\varrho-1}, W^{\varrho-1})$ to be a rational subsystem of $(V, W)/(V^{\varrho-1}, W^{\varrho-1})$ such that

$$(V, W)/(V^\varrho, W^\varrho)\cong((V, W)/(V^{\varrho-1}, W^{\varrho-1}))/((V^\varrho, W^\varrho)/(V^{\varrho-1}, W^{\varrho-1}))$$

is eigenvalue free. Since the sequence is strictly increasing, it terminates with a subsystem (V^τ, W^τ) satisfying 7°.

Suppose now that, starting with $(X^0, Y^0) = (0, 0)$, we have defined an increasing sequence $((X^\nu, Y^\nu))_{\nu<\varrho}$, $0<\varrho\leqslant\tau$, of subsystems of (V, W) such that

$$(V^\nu, W^\nu) = (X, Y)\dotplus(X^\nu, Y^\nu) \tag{9.15.1}$$

for $\nu<\varrho$.

If $\varrho-1$ exists, then $(V^{\varrho-1}, W^{\varrho-1})/(X^{\varrho-1}, Y^{\varrho-1})$ being isomorphic to (X, Y), is a divisible subsystem of its extension $(V^\varrho, W^\varrho)/(X^{\varrho-1}, Y^{\varrho-1})$. If $\varrho\leqslant\sigma$, then by 3° and Proposition 5.3 (c), (d), this is a quasi-spectral subsystem, and by 4° the extension is by one of the types I^m, II^m_θ or $\mathrm{II}^\infty_\theta$. If $\sigma<\varrho$, then by 6° the extension is by a rational system. Applying quasi-spectrality, Lemma 7.3 or Lemma 9.9, we see that in all cases we have a decomposition

$$\begin{aligned}(V^\varrho, W^\varrho)&/(X^{\varrho-1}, Y^{\varrho-1})\\ &= (V^{\varrho-1}, W^{\varrho-1})/(X^{\varrho-1}, Y^{\varrho-1})\dotplus(X^\varrho, Y^\varrho)/(X^{\varrho-1}, Y^{\varrho-1}),\end{aligned} \tag{9.15.2}$$

where $(X^\varrho, Y^\varrho)\supset(X^{\varrho-1}, Y^{\varrho-1})$. From (9.15.1) for $\nu=\varrho-1$ and (9.15.2) it follows that (9.15.1) holds for $\nu=\varrho$. If ϱ is a limit ordinal the same conclusion is valid with

$$(X^\varrho, Y^\varrho) = \sum_{\nu<\varrho}(X^\nu, Y^\nu)$$

because the summands form an increasing sequence. The subsystems (X^ν, Y^ν) can therefore be defined up to $\nu=\tau$, and (9.15.1) for $\nu=\tau$ proves our theorem.

COROLLARY 9.16. (a) *Every system (V, W) has a decomposition*

$$(V, W) = \mathrm{Div}(V, W)\dotplus(U, Z),$$

where (U, Z) is a reduced subsystem the isomorphism type of which is uniquely determined by (V, W).

(b) *An indecomposable system is either eigenvalue free or an eigenvalue system of one of the types I^m, II^m_θ or $\mathrm{II}^\infty_\theta$.*

Proof. (a) By Lemma 7.4 and Theorem 9.15, $\mathrm{Div}(V, W)$ is spectral in (V, W). Since a direct complement (U, Z) is isomorphic to $(V, W)/\mathrm{Div}(V, W)$, its isomorphism type is determined by (V, W). By Proposition 7.2 (f), (U, Z) is reduced.

(b) If (V, W) is not eigenvalue free, then by Lemma 9.4 it contains a quasi-spectral subsystem of one of the types I^m, II^m_θ or $\mathrm{II}^\infty_\theta$. From Theorem 5.5 or Proposition 8.4 and Theorem 9.15 it follows that such a subsystem is spectral.

The following counterpart to Proposition 9.5 shows what form the decomposition of 9.16 (a) assumes for eigenvalue systems.

PROPOSITION 9.17. *For every system (V, W) we have*

$$\operatorname{Div}\operatorname{Eig}(V, W) = \big(\operatorname{Eig}(V, W)\big)_{\mathrm{I},\mathrm{II}^\infty} = (V, W)_{\mathrm{I},\mathrm{II}^\infty}.$$

Proof. By 9.16 (a), $\operatorname{Div}\operatorname{Eig}(V, W)$ is spectral in $\operatorname{Eig}(V, W)$. Hence by 9.3 (d), (f), it is an eigenvalue system, namely,

$$\operatorname{Div}\operatorname{Eig}(V, W) = \operatorname{Eig}\operatorname{Div}\operatorname{Eig}(V, W).$$

The first equality of the proposition now follows from 9.5; the second from the definition of the operation $(\ )_{\mathrm{I},\mathrm{II}^\infty}$ in 8.9 and 9.3 (a).

In the investigation of the isomorphism problem of general systems it suffices by 9.16 (a) and 9.10 to consider reduced systems. If (V, W) is an eigenvalue system, then by 9.17 the reductions are of the form

$$(V, W) = (V, W)_{\mathrm{I},\mathrm{II}^\infty} \dotplus (U, Z).$$

Here the reduced subsystem (U, Z), being isomorphic to a quotient of (V, W), is also an eigenvalue system. Thus, in studying eigenvalue systems one may restrict attention to reduced eigenvalue systems. A further simplification is afforded by the following considerations.

Definition 9.18. Let θ be an element of the extended complex plane $\tilde{C}$. A system (V, W) is said to be *θ-free* if and only if θ is not an eigenvalue of (V, W). The *θ-part* of a system (V, W) is the smallest subsystem (X, Y) of (V, W) such that $(V, W)/(X, Y)$ is θ-free. This subsystem will be denoted by $(V, W)_\theta$. A system (V, W) is said to be a *θ-system* if and only if $(V, W)_\theta = (V, W)$.

The above concepts are relative to a basis a, b of C^2. It is clear how this dependence on a basis can be avoided by speaking of $\dot{c}$-free systems, etc., where $\dot{c}$ is a point of the complex projective line. The existence of $(V, W)_\theta$ is verified in the same manner as the existence of $\operatorname{Eig}(V, W)$ (cf. the remarks made before Definition 9.2).

The results 9.3, 9.5, 9.12, 9.13 and 9.17 can all be viewed as stating properties of the operation of taking the eigenvalue part. The reader will have no difficulty in formulating the analogous results for the operation of taking the θ-part. The proofs are obtained by considering a fixed element θ of $\tilde{C}$ instead of letting it range over $\tilde{C}$. For convenience, Lemma 9.4 was already formulated for a given θ, as the implied result for systems which are not eigenvalue free was evident.

PROPOSITION 9.19. *Let (V, W) be a system which contains no subsystem of any type I^m. Then*

$$\operatorname{Eig}(V, W) = \sum_{\theta\in\tilde{C}}{}^{\cdot}(V, W)_\theta.$$

This decomposition is unique in the following sense. If

$$\operatorname{Eig}(V, W) = \sum_{\theta\in\tilde{C}}(X, Y)_\theta,$$

where $(X, Y)_\theta$ is for every $\theta\in\tilde{C}$ a θ-system, then $(X, Y)_\theta = (V, W)_\theta$ for every $\theta\in\tilde{C}$.

Proof. The equality

$$\operatorname{Eig}(V, W) = \sum_{\theta\in\tilde{C}}(V, W)_\theta$$

follows from Proposition 9.12 and its analog for θ-parts. We show that the domains form a direct sum, the argument for the range spaces being identical. Suppose that $\sum_{i=1}^{n} v_i = 0$, where v_i belongs to the domain of $(V, W)_{\theta_i}$, $i = 1, \ldots, n$, and $\theta_1, \ldots, \theta_n$ are distinct. Since (V, W) contains no subsystems of types I^m, v_i belongs to the domain of a finite sum of subsystems (U_{ij}, Z_{ij}), $j = 1, \ldots, k_i$, of the respective types $\mathrm{II}_{\theta_i}^{m_{ij}}$. Consider a decomposition of the finite-dimensional system $\sum_{i=1}^{n}\sum_{j=1}^{k_i}(U_{ij}, Z_{ij})$ into a direct sum of indecomposable subsystems. From Lemma 6.3 (b) and the lack of subsystems of types I^m, it follows that $\sum_{j=1}^{k_i}(U_{ij}, Z_{ij})$ is contained in the sum of the subsystems of types $\mathrm{II}_{\theta_i}^m$ appearing in the decomposition. Hence the v_i's belong to the domains of distinct components and must vanish.

The uniqueness statement follows on noting that, by the analog of 9.13 (a) for θ-parts, $(X, Y)_\theta \subset (V, W)_\theta$.

Remark. If (V, W) is an arbitrary system, then

$$\operatorname{Eig}(V, W) = \sum_{\theta\in\tilde{C}}(V, W)_\theta$$

and for every $\theta\in\tilde{C}$, $(V, W)_\theta \cap \sum_{\eta\neq\theta}(V, W)_\eta = (V, W)_{\mathrm{I}}$. If

$$\operatorname{Eig}(V, W) = \sum_{\theta\in\tilde{C}}(X, Y)_\theta$$

and $(X, Y)_\theta$ is for every θ a θ-system containing $(V, W)_{\mathrm{I}}$, then $(X, Y) = (V, W)_\theta$ for every $\theta\in\tilde{C}$. This can be easily reduced to Proposition 9.19 by taking quotients modulo $(V, W)_{\mathrm{I}}$.

From Proposition 9.19 it follows that a reduced eigenvalue system has a unique decomposition into a direct sum of θ-systems, which are clearly also reduced. If (V, W) is a reduced θ-system, then it is the sum of its subsystems of the types II_θ^m. Therefore if $\eta \in \tilde{C}$ and $\eta \neq \theta$, then $b_\eta V = W$. Moreover,

$$(V, W)_\eta = (V, W) \cap (V, W)_\eta = (V, W)_\theta \cap (V, W)_\eta = (0, 0),$$

and η is not an eigenvalue of (V, W). Thus (V, W) is a non-singular system, and (provided it does not vanish) it has the single eigenvalue θ. All questions relating to the structure of (V, W) are equivalent to questions about the $C[\lambda]$-module V defined by

$$p(\lambda)v = p\big(T(b_\eta)^{-1}T(b_\theta)\big)v, \quad p(\lambda) \in C[\lambda], \ v \in V.$$

The latter is a reduced λ primary module. Taking into account the former reduction we see that the study of the structure of an arbitrary eigenvalue system is reduced to the study of such modules. Note though that even a reduced eigenvalue system may be singular (this is the case if and only if every θ in $\tilde{C}$ is an eigenvalue) and not correspond to a module over $C[\lambda]$. The reader will observe that if (V, W) is a reduced θ-system, then a subsystem spanned by chains representing bases of $Q\mathrm{II}_\theta^m(a, b; V, W)$, $m = 1, 2, \ldots$, corresponds to a basic submodule of a λ-primary module attached to (V, W). However, as the present work shows, the scope of the method of chains is much larger.

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Reçu par la Rédaction le 5. 6. 1967

STUDIA MATHEMATICA, T. XXX. (1968)

Eigenschaften Greenscher Funktionen nicht-selbstadjungierter allgemeiner elliptischer Operatoren

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In [10] wurden Differenzierbarkeitseigenschaften Greenscher Funktionen spezieller elliptischer selbstadjungierter Operatoren angegeben, wobei sich die Betrachtungen auf das Dirichletsche Randwertproblem beschränkten. Weitere Aussagen über Greensche Funktionen allgemeiner elliptischer Differentialoperatoren mit glatten Koeffizienten, die zumeist lokaler Natur sind, und über Eigenwertverteilungen elliptischer Operatoren findet man bei Berezanskij [4], III, § 5, sowie bei Agmon [2].

Die in [11] entwickelten Methoden gestatten nun, Aussagen über Greensche Funktionen nicht-selbstadjungierter Differentialoperatoren mit allgemeinen Randbedingungen zu machen, die notwendigen und hinreichenden Charakter haben. Wie üblich wird die Bezeichnung „nicht-selbstadjungiert" im Sinne von „nicht notwendig selbstadjungiert" verwendet, die selbstadjungierten Operatoren sind also nicht ausgeschlossen.

Abschnitt 1 beginnt mit einer Zusammenstellung bekannter Aussagen über elliptische Differentialoperatoren

$$Au = \sum_{|\alpha| \leqslant 2m} a_\alpha(x) D^\alpha u$$

mit normalen (mit A verträglichen) Randbedingungen

$$B_j u = \sum_{|\alpha| \leqslant m_j} b_{j\alpha}(x) D^\alpha u, \quad j = 1, \ldots, m; \, m_j < 2m,$$

$$D(A) = \{u \,|\, u \in W_2^{2m}(\Omega), \, B_j u|_{\partial\Omega} = 0, \, j = 1, \ldots, m\}.$$

Ω ist hierbei ein beschränktes Gebiet im R_n mit glattem Rand $\partial\Omega$. Ferner wird gezeigt (Satz 1), daß $(AA^* + E)^{-1}$ und $(A^*A + E)^{-1}$ nicht zu $\mathfrak{S}_{n/4m}, p$, $0 < p < \infty$, aber zu $\mathfrak{S}_{n/4m,\infty}$ gehören. Die Opera-