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On stable distributions in Hilbert space

by

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The paper aims at giving the canonical form of the characteristic functional of a stable probability distribution in Hilbert space. Our formula is a generalization of the classical formula of Lévy-Khintchine [3], [4] for one-dimensional stable distributions.

Let H be a separable, real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. A countable additive and normed measure defined on the field $\mathfrak{B}$ of Borelian subsets of H is called a *probability distribution* in H.

A sequence of distributions $\{p_n\}$ is said to be *weakly convergent* to p $(p_n \to p)$ if for every function f defined in H, continuous and bounded in H we have

$$\lim_{n\to\infty}\int f(h)p_n(dh) = \int f(h)p(dh). \tag{1}$$

The distribution $p*q$ defined by the formula

$$(p*q)(Z) = \int p(Z-h)q(dh) \quad \text{for every } Z\in\mathfrak{B} \tag{2}$$

is called the *convolution* of the distributions p and q.

The characteristic functional p of a distribution p is defined by the formula

$$\hat{p}(h) = \int e^{i(g,h)}p(dg), \quad h\in H. \tag{3}$$

A distribution p is uniquely defined by the characteristic functional (3). The n-th convolution-power of a distribution p shall be denoted by p^{n*}. A one-point distribution concentrated at a point $x\in H$ (Dirac measure) will be denoted by δ_x, i.e. $\delta_x(Z) = 1$ if $x\in Z$.

For every positive a and every distribution p in H we put by definition

$$(T_a p)(Z) = p(a^{-1}Z) \tag{4}$$

for every $Z\in\mathfrak{B}$, where obviously $cZ = \{cz: z\in Z\}$. For $a=0$ we put moreover $T_0 p = \delta_0$.

A distribution p is said to be *stable* if for every pair of positive numbers a and b there exist a positive number c and an element x of the space H such that

$$T_a p * T_b p = \delta_x * T_c p. \tag{5}$$

We shall prove the following

THEOREM. *A functional φ defined in H is the characteristic functional of a stable distribution in H if and only if either*

$$\varphi(h) = \exp[i(x_0, h) - \tfrac{1}{2}(Dh, h)], \tag{6}$$

where $x_0 \in H$ and D is an S-operator (i.e. φ is the characteristic functional of Gaussian distribution) (1) *or*

$$\varphi(h) = \exp\Big[i(x_0, h) + \int K(g, h) M(dg)\Big],$$

where $x_0 \in H$, $K(g, h) = \exp\big(i(g, h)\big) - 1 - i(g, h)(1 + \|g\|^2)^{-1}$ and M is a semi-finite measure in H, finite on the complement of every neighbourhood of zero in H and such that

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg) < \infty, \tag{7}$$

(8) *there exists a $0 < \lambda < 2$ such that $T_a M = a^\lambda M$ for every positive a.*

We precede the proof of our theorem by several lemmas. The fundamental part of the proof is contained in Lemma 3. We prove this lemma making use of the method given by Urbanik in [6]. The idea of the proof is based on reducing the problem to some result of F. Bohnenblust which we shall refer to later.

LEMMA 1. *If p is a stable distribution, then there exist a sequence of positive numbers $\{a_n\}$ and a sequence $\{x_n\}$ of elements of the space H such that*

$$p = \lim_{n\to\infty} (\delta_{x_n} * T_{a_n} p^{n*}).$$

Proof. By the definition of the stability of distribution we easily deduce that for every natural n there exist $a_n > 0$ and an element $x_n \in H$ such that $p = \delta_{x_n} * T_{a_n} p^{n*}$, which ends the proof.

LEMMA 2. *If for some sequence of positive numbers $\{a_n\}$ and a sequence $\{x_n\}$ of elements of the space H we have*

$$q = \lim_{n\to\infty} (\delta_{x_n} * T_{a_n} p^{n*}), \tag{9}$$

(1) An operator in H is called *S-operator* when it is self-adjoint, non-negative and with finite trace (see [5], § 4).

where $q \neq \delta_x$ for every $x \in H$, then

$$a_n \to 0, \tag{10}$$

$$\frac{a_n}{a_{n+1}} \to 1 \quad as \quad n \to \infty. \tag{11}$$

Proof. Suppose (10) does not hold. Then there would exist a subsequence $\{a_{k_n}\}$ of the sequence $\{a_n\}$ such that $\lim 1/a_{k_n} = a < \infty$. Then, however, we would have (putting $y_n = x_{k_n}/a_{k_n}$)

$$p^{k_n *} * \delta_{y_n} = T_{a_{k_n}^{-1}}(\delta_{x_{k_n}} * T_{a_{k_n}} p^{k_n *}) \to T_a q.$$

Hence $e^{i(y_n, h)}[\hat{p}(h)]^{k_n} \to \hat{q}(ah)$; then $e^{i(y_n/k_n, h)}\hat{p}(h) \to 1$ (see e.g. [2], § 14), whence $p = \delta_{x_0}$. Thus we would have $q = \delta_{y_0}$, which contradicts the assumption.

To prove (11) we assume that there exists a subsequence $\{a_{k_n}\}$ such that

$$\lim_{n\to\infty} \frac{a_{k_n}}{a_{k_n+1}} = a, \quad \text{where} \quad 0 \leqslant a \leqslant \infty, a \neq 1.$$

The case $a = \infty$ is impossible. In fact, putting $b_n = a_{k_n+1}/a_{k_n}$, we would have

$$T_{b_n}(\delta_{x_{k_n}} * T_{a_{k_n}} p^{k_n *}) = \delta_{b_n x_{k_n}} * T_{a_{k_n+1}} p^{k_n *} \to \delta_\theta.$$

On the other hand,

$$e^{i(b_n x_{k_n}, h)} \cdot \widehat{T_{a_{k_n+1}} p^{k_n *}}(h) = \frac{e^{i(b_n x_{k_n}, h)} \widehat{T_{a_{k_n+1}} p^{k_n+1 *}}(h)}{\hat{p}(a_{k_n+1} h)} \to e^{i(\bar{x}, h)}$$

for some $\bar{x} \in H$, because $\hat{p}(a_{k_n} h) \to 1$ and $\widehat{T_{a_{k_n+1}} p^{k_n+1 *}}(h) \to \hat{q}(h)$, and the left-hand side converges to 1. Hence we would have $q = \delta_y$, which is impossible. Thus the only case to consider is $a < \infty$. Then, putting $c_n = a_{k_n}/a_{k_n+1}$, we get

$$\delta_{x_{k_n}} * T_{a_{k_n}} p^{k_n+1 *} = [T_{c_n}(\delta_{x_{k_n+1}} * T_{a_{k_n+1}} p^{k_n+1 *}] * \delta_{x_{k_n} - c_n x_{k_n+1}}.$$

But

$$\delta_{x_{k_n}} * T_{a_{k_n}} p^{k_n+1 *} = (\delta_{x_{k_n}} * T_{a_{k_n}} p^{k_n *}) * T_{a_{k_n}} p \to q$$

and the expression in the square brackets tends to $T_a q$. Hence the sequence $\{x_{k_n} - c_n x_{k_n}\}$ converges to some element y. Thus

$$\hat{q}(h) = e^{i(y, h)} \hat{q}(ah).$$

Without restricting the generality of our argument we may assume $a < 1$, for in contrary case

$$e^{i(z,h)} q\left(\frac{1}{a} h\right) = \hat{q}(h), \quad \text{where} \quad z = -\frac{y}{a}.$$

By iteration we obtain $\hat{q}(h) = e^{i(z_n,h)} \hat{q}(a^n h)$ for some sequence $\{z_n\}$ of elements of H. Passing to the limit as $n \to \infty$ we get $\hat{q}(h) = e^{i(z_0,h)}$ for some $z_0 \in H$, which is impossible. Thus condition (11) has been proved.

LEMMA 3. *If the assumptions of Lemma 2 are satisfied, then there exists a $\lambda > 0$ and a function of two variables $z(x, y)$ defined for $x, y \geqslant 0$ with values in the space H such that for every pair of non-negative numbers a and b the equality*

$$\hat{q}(ah)\hat{q}(bh) = e^{i(z(a,b),h)} \hat{q}\left((a^\lambda + b^\lambda)^{1/\lambda} \cdot h\right) \tag{12}$$

holds for any $h \in H$.

Proof. In a way indicated in the paper of Urbanik [6] (see the proof of Theorem 4, in particular, p. 224-227) we shall reduce the proof of Lemma 3 to the following result of Bohnenblust ([1], p. 630-632):

If a continuous function $g(x, y)$ defined for $x, y \geqslant 0$ satisfies the conditions

$$g(x, y) = g(y, x), \tag{13}$$

$$g\left(g(x, y), z\right) = g\left(x, g(y, z)\right), \tag{14}$$

$$g(zx, zy) = zg(x, y), \tag{15}$$

$$g(x, y_1) < g(x, y_2) \quad \text{if} \quad y_1 < y_2, \tag{16}$$

$$g(0, y) = y, \tag{17}$$

then it is of the form

$$g(x, y) = (x^\lambda + y^\lambda)^{1/\lambda}, \tag{18}$$

where λ is a positive constant.

First, let us observe that by Lemma 2 for two arbitrary positive numbers x and y there exist two subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ of the sequence $\{a_n\}$ such that

$$\lim_{k\to\infty} \frac{a_{m_k}}{a_{n_k}} = \frac{y}{x}. \tag{19}$$

It may be assumed that limit

$$\lim_{k\to\infty} \frac{a_{n_k}}{a_{n_k} + a_{m_k}} = s \tag{20}$$

exists. Analogously as in paper [6], we shall prove that the limit s must be finite. In fact, supposing that the sequence

$$v_k = \frac{a_{n_k} + a_{m_k}}{a_{n_k}}$$

converges to zero and putting $w_k = a_{n_k}/a_{m_k}$, we have

$$\begin{aligned} &\delta_{x_{n_k+m_k}} * T_{a_{n_k+m_k}} p^{n_k+m_k*} \\ &\qquad = [T_{v_k}(\delta_{x_{n_k}} * T_{a_{n_k}} p^{n_k*})] * [T_{v_k w_k}(\delta_{x_{m_k}} * T_{a_{m_k}} p^{m_k*})] * \delta_{z_k} \end{aligned} \tag{21}$$

the sequence $\{z_k\}$ being chosen suitably. Since the left-hand side of the equality converges to q and the expressions in the square brackets converge to δ_θ, the sequence $\{z_k\}$ converges to some element z_0 and, after passing to the limit in (21) we would have $q = \delta_{z_0}$, which is impossible. Putting $s_k = 1/v_k$ we get

$$\begin{aligned} &[T_x(\delta_{x_{n_k}} * T_{a_{n_k}} p^{n_k*})] * [T_{xw_k}(\delta_{x_{m_k}} * T_{a_{m_k}} p^{m_k*})] \\ &\qquad = T_{xa_{n_k}} p^{n_k+m_k*} * \delta_{z_k} = [T_{xs_k}(\delta_{x_{n_k+m_k}} * T_{a_{n_k+m_k}} p^{n_k+m_k*})] * \delta_{z_k}. \end{aligned} \tag{22}$$

The expressions in the square brackets on the left-hand side of (22) tend to $T_x q$ and $T_y q$ respectively and the expression in the square brackets on the right-hand side tends to $T_{xs} q$. The probability distribution $T_{xs} q$ is infinitely decomposable (this follows from Lemma 2 and general theory; see e.g. [7]), thus its characteristic functional is different from zero everywhere. Hence it follows that $\{z_n\}$ converges to some element $z = z(x, y)$. Passing to the limit in (22) and then to the characteristic functionals of the distributions obtained, we get

$$\hat{q}(xh)\hat{q}(yh) = e^{i(z(x,y),h)} \hat{q}\left((xs)h\right) \quad \text{for} \quad h \in H. \tag{23}$$

Now, following Urbanik [6], we shall define the function $g(x, y)$ for $x \geqslant 0, y \geqslant 0$ as follows:

$$\begin{aligned} &g(x, 0) = x, \quad g(0, y) = y, \\ &g(x, y) = sx \quad \text{for} \quad x > 0, y > 0. \end{aligned} \tag{24}$$

Then equality (23) may be written in the form

$$\hat{q}(xh)\hat{q}(yh) = e^{i(z(x,y),h)} q\left(g(x, y)h\right), \quad h \in H. \tag{25}$$

Equality (25) holds for $x \geqslant 0$, $y \geqslant 0$.

The function defined by formula (24) is the only function satisfying (25). In fact, if there were two such functions g_1 and g_2 satisfying for some non-negative x_0 and y_0 the inequality $g_1(x_0, y_0) < g_2(x_0, y_0)$, then putting

$$u = \frac{g_1(x_0, y_0)}{g_2(x_0, y_0)}$$

we would have for some z

$$\hat{q}(h) = e^{i(z,h)}\hat{q}(uh).$$

Iterating, we have

$$\hat{q}(h) = e^{i(z_n,h)}\hat{q}(u^n h), \qquad n = 1, 2, \ldots,$$

the sequence $\{z_n\}$ being chosen suitably. Since $q(u^n h) \to 1$, the sequence $\{z_n\}$ converges to some z_0 and we would have

$$\hat{q}(h) = e^{i(z_0,h)}$$

which is impossible.

The function g is continuous. Indeed, if $x_n \to x$, $y_n \to y$, $g(x_n, y_n) \to z$, then $z < \infty$ must hold, for in the contrary case, putting

$$p_n = \frac{x_n}{g(x_n, y_n)}, \qquad q_n = \frac{y_n}{g(x_n, y_n)}$$

we would have

$$\hat{q}(h) = \hat{q}(p_n h)\hat{q}(q_n h)e^{i(\bar{z}_n,h)},$$

the sequence $\{\bar{z}_n\}$ being chosen suitably. Since $\hat{q}(p_n h) \to 1$ and $\hat{q}(q_n h) \to 1$ for every $h \in H$, the sequence $\{\bar{z}_n\}$ is convergent and we obtain $\hat{q}(h) = e^{i(z_0,h)}$, which is impossible.

If $z < \infty$, then we have

$$\hat{q}(x_n h)\hat{q}(y_n h) = e^{i(z_n,h)}\hat{q}\big(g(x_n, y_n)h\big), \qquad n = 1, 2, \ldots,$$

for some sequence $\{z_n\}$; after passing to the limit we obtain

$$\hat{q}(xh)\hat{q}(yh) = e^{i(z_0,h)}\hat{q}(zh).$$

But in view of the uniqueness of the function g it must be $z = g(x, y)$, which ends the proof of continuity of the function g. It is seen that the function g satisfies conditions (13), (14), (15) and (17). We prove condition (16) almost exactly as they are proved in the paper of Urbanik [6], p. 226-227. Thus the function g satisfies the assumptions of the theorem of Bohnenblust. Thus there exists a constant $\lambda > 0$ such that

$$g(x, y) = (x^\lambda + y^\lambda)^{1/\lambda}.$$

Formula (25) may thus be written in the form (12) which ends the proof of the lemma.

Proof of the theorem. It follows from Lemmas 1 and 2 that a stable probability distribution is infinitely decomposable. The general form of the characteristic functional of an infinitely decomposable distribution in a Hilbert space (the generalized formula of Lévy-Khintchine) has been given by Varadhan [7]. Namely, the functional φ defined in H is the characteristic functional of an infinitely decomposable distribution if and only if it is of the form

$$\varphi(h) = \exp\left[i(x_0, h) - \tfrac{1}{2}(Dh, h)\right] + \int K(g, h)M(dg), \tag{26}$$

where $x_0 \in H$, D is an S-operator, K is defined by the formula

$$K(g, h) = \exp\big(i(g, h)\big) - 1 - i(g, h)(1 + \|g\|^2)^{-1}$$

and M is a semi-finite measure in H, finite on the complement of every neighbourhood of zero in H and such that

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg) < \infty. \tag{27}$$

Representation (26) is unambigous.

Let thus p be a stable probability distribution in H. To exclude the trivial case and to employ our previous results we assume that $p \neq \delta_x$. Thus we have

$$\log\hat{p}(h) = i(x_0, h) - \tfrac{1}{2}(Dh, h) + \int K(g, h)M(dg). \tag{28}$$

Hence

$$\log\hat{p}(ah) = i(\bar{x}, h) - \frac{a^2}{2}(Dh, h) + \int K(g, h)(T_a M)(dg), \tag{29}$$

where $\bar{x} = x_0 + \bar{\bar{x}}$ and $\bar{\bar{x}}$ is defined by the formula

$$(\bar{\bar{x}}, h) = a(1-a)\int \frac{(g, h)\|g\|^2}{(1+\|g\|^2)(1+a^2\|g\|^2)} M(dg) \tag{30}$$

(the integral in formula (30) exists by (27) and by the fact that the measure M is finite on the complement of a neighbourhood of zero in H).

Thus we have for non-negative a and b

$$\log\hat{p}(ah) + \log\hat{p}(bh) = i(y, h) - \frac{a^2+b^2}{2}(Dh, h) + \int K(g, h)[T_a M + T_b M](dg) \tag{31}$$

for some $y \in H$.

On the other hand, by Lemma 3 we have for some $z_0 \in H$ and $\lambda > 0$

$$\log\hat{p}(ah) + \log\hat{p}(bh) = i(z_0, h) - \frac{(a^\lambda + b^\lambda)^{2/\lambda}}{2}(Dh, h) + \int K(g, h)[T_{(a^\lambda+b^\lambda)^{1/\lambda}}M](dg). \tag{32}$$

By the uniqueness of representation (26) we obtain

$$a^2 + b^2 = (a^\lambda + b^\lambda)^{2/\lambda} \tag{33}$$

and

$$T_a M + T_b M = T_{(a^\lambda + b^\lambda)^{1/\lambda}} M \tag{34}$$

for any pair of positive numbers a and b holds.

From formula (33) immediately follows that if the operator D is not a zero operator then it must be $\lambda = 2$. Now we shall formulate a corollary from equality (34). We shall namely show that for every positive a

$$T_a M = a^\lambda M. \tag{35}$$

To this aim observe that condition (34) for $a = b = 1$ implies

$$M(Z) = \tfrac{1}{2} M(2^{-1/\lambda} Z) \quad \text{for every } Z \in \mathfrak{B} \tag{36}$$

and more generally

$$M(Z) = \frac{1}{2^n} M(2^{-n/\lambda} Z) \quad \text{for} \quad Z \in \mathfrak{B}. \tag{37}$$

Thus formula (35) holds for $a = 2^{n/\lambda}$ $(n = 1, 2, \ldots)$. Putting in (37) the set $2^{n/\lambda} Z$ for the set Z we obtain

$$M(2^{n/\lambda} Z) = \frac{1}{2^n} M(Z) \quad \text{for} \quad Z \in \mathfrak{B}, \tag{38}$$

which proves relation (35) for $a = 2^{-n/\lambda}$ $(n = 1, 2, \ldots)$. Combining formulae (37) and (38) we obtain

$$2^{n-m} M = T_{2^{\lambda(n-m)}} M, \tag{39}$$

m, n being arbitrary integers, i.e. formula (35) for $a = 2^{\lambda(n-m)}$. Since numbers of the form $2^{\lambda(n-m)}$ lie densely on the real axis, we have proved relation (35) for all $a > 0$.

Now we will show that in the case $\lambda \geqslant 2$ the measure M is reduced to a measure concentrated at zero, i.e. $M = k\delta_0$ and thus the last term in the square brackets in representation (26) equals to zero. In fact, by (35) for $a > 0$ we have

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg) = \frac{1}{a^\lambda} \int_{\|g\| \leqslant 1/a} \|ag\|^2 M(dg) = a^{2-\lambda} \int_{\|g\| \leqslant 1/a} \|g\|^2 M(dg). \tag{40}$$

If $\lambda = 2$, formula (40) gives

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg) = \int_{\|g\| \leqslant 1/a} \|g\|^2 M(dg) \quad \text{for any } a > 0, \tag{41}$$

which proves that $M = k\delta_0$.

If, on the other hand, $M \neq k\delta_0$ and $\lambda > 2$, then the integral

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg)$$

is greater than zero and we have

$$\int_{\|g\| \leqslant 1} \|g\|^2 M(dg) = a^{2-\lambda} \int_{\|g\| \leqslant 1/a} \|g\|^2 M(dg) \to \infty \quad \text{as} \quad a \to 0, \tag{42}$$

which contradicts condition (27). Thus we see that there are only two possible forms of the characteristic functional of a stable probability distribution in H. Either, in the case $\lambda = 2$ it is simply the characteristic functional of the normal distribution

$$\hat{p}(h) = \exp[i(x_0, h) - \tfrac{1}{2}(Dh, h)] \tag{43}$$

or, in the case $0 < \lambda < 2$, it is of the form

$$\hat{p}(h) = \exp\Big[i(x_0, h) + \int K(g, h) M(dg)\Big], \tag{44}$$

where the measure M satisfies conditions (7) and (8). It can be easily proved that the probability distributions given by formulae (43) and (44) are stable, which completes the proof of our theorem.

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