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Topology and non-linear functional equations

by

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Dedicated to Professors
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Introduction. Let X and Y be topological spaces, G an open subset of X whose closure in X is denoted by $\mathrm{cl}(G)$, boundary in X by $\mathrm{bdry}(G)$. If f is a continuous mapping of $\mathrm{cl}(G)$ into Y and y is a point of $Y-f(\mathrm{bdry}(G))$, the degree of the mapping f on G over y is, in principle whenever it is defined, an algebraic count of the number of times f assumes the value y in G. For the case in which X and Y are oriented Euclidean spaces of the same finite dimension, the concept of the topological degree was defined by Brouwer in 1912. In the classical paper of Leray-Schauder [16] in 1934, the Brouwer degree was extended to the case when $X = Y$ and the mapping f is of the form $f = I-C$, where I is the identity mapping of X and C is compact in the sense that $C(\mathrm{cl}(G))$ is a relatively compact subset of X.

It is the purpose of the present paper to present an extension of the theory of the topological degree to a much wider class of mappings defined in Banach spaces. This extended degree theory (as we show in detail elsewhere, [8]) can be applied to obtain a number of fixed point and mapping theorems of interest which fall outside the framework of the classical theory of compact non-linear mappings in a Banach space X.

To focus on the properties of our generalized degree, we note the following basic properties of the Leray-Schauder degree, written as $\deg_{LS}(f, G, y)$ (proved in the most explicit form by Nagumo [17]):

(1) *Let X be a Banach space, G an open subset of X, f a continuous mapping of $\mathrm{cl}(G)$ into X of the form $f = I-C$, C compact. Let $y \in X-f(\mathrm{bdry}(G))$. Then $\deg_{LS}(f, G, y)$ is defined, and if $\deg_{LS}(f, G, y) \neq 0$, then there exists a point x in G such that $f(x) = y$.*

(2) *The degree is additive in G, i.e. if $G = G_1 \cup G_2$ with G_1 and G_2 open sets such that $y \in X-f(G')$, where $G' = (G_1 \cap G_2) \cup \mathrm{bdry}(G_1) \cup \mathrm{bdry}(G_2)$, then*

$$\deg_{LS}(f, G, y) = \deg_{LS}(f, G_1, y) + \deg_{LS}(f, G_2, y).$$

(3) *The degree is invariant during homotopies in which it remains defined, i.e. if W is an open subset of $X\times[0,1]$, C a continuous map of* $\mathrm{cl}(W)$ *into a relatively compact subset of X, and if for each t in $[0,1]$, we let*

$$G_t = \{x \mid x\in X, (x,t)\in W\},$$

$C_t\colon \mathrm{cl}(G_t)\to X$ *be given by* $C_t(x) = C(x,t)$, *and if we set* $f_t = I+C_t\colon \mathrm{cl}(G_t)\to X$, *then for any continuous curve* $P=\{y(t): 0\leqslant t\leqslant 1\}$ *in X such that* $y(t)\in X-f_t(\mathrm{bdry}(G_t))$ *for all t in $[0,1]$, then*

$$\deg_{\mathrm{LS}}(f_t, G_t, y(t))$$

is well-defined and independent of t in $[0,1]$.

(4) *If f is a homeomorphism of G into X of the form $f=I+C$ with C compact and y lies in* $f(G)-f(\mathrm{bdry}(G))$, *then*

$$\deg_{\mathrm{LS}}(f,G,y) = \pm 1.$$

(5) *The degree* $\deg_{\mathrm{LS}}(f,G,y)$ *depends only upon the behaviour of f on* $\mathrm{bdry}(G)$.

(Further properties of the Leray-Schauder degree are given at the begining of Section 1, below).

The main objective of the extended theory of the topological degree which we develop below is to extend the concept of degree to mappings f of the form $h+C$, with h a homeomorphism and C compact, or, more generally, with the homeomorphism h and the map C intertwined in a sense made precise below. More generally still, we shall consider uniform limits of sequences of mappings of these last forms, i.e. in principle to pass to the case of degenerate homeomorphisms h.

We must begin therefore with a precise statement of what we mean by a homeomorphism:

Definition 1. Let X and Y be topological spaces, G an open subset of X, h a continuous mapping of $\mathrm{cl}(G)$ into Y. Then h is said to be a *permissible homeomorphism* of $\mathrm{cl}(G)$ into Y if h is a homeomorphism of G on an open subset $h(G)$ of Y which maps $\mathrm{cl}(G)$ homeomorphically onto $\mathrm{cl}(h(G))$.

In the discussion which follows, all homeomorphisms h will be assumed to be permissible in the given context and we shall not use the explicit distinction between permissible homeomorphisms and more general homeomorphisms from $\mathrm{cl}(G)$ into Y.

We shall give several definitions of the degree for various classes of mappings in various representations, and these definitions are logically independent though we shall establish relations between them. The crucial point in every case is that we shall not define these degrees as integer-valued functionals of mappings but of *representations of mappings*. To put the point in greater detail, we shall take the viewpoint that the generalized degree should be taken in the most general circumstances as a quantity defined for a representation of a mapping f within a given class of representations, and not initially as being defined in terms of the mapping f itself. It is then the objective of the degree theory to establish results in the form of *theorems* concerning the generalized degree function that under certain general hypotheses, the degree and its properties depend only upon the mapping f or its properties and not upon the choice of representations.

One of our basic pieces of data in each definition is the following:

Definition 2. M will denote a set of (permissible) homeomorphisms from $\mathrm{cl}(G)$ into Y, where G ranges over a family of open subsets of the Banach space X. Let M_G denote the subset of maps h in M whose domain is $\mathrm{cl}(G)$ for a given open set G in X. Each M_G is considered as a metric space with the metric

$$d(h,h_1) = \sup_{x\in\mathrm{cl}(G)} \|h(x)-h_1(x)\|_Y$$

and we assume that $d(h,h_1)<\infty$ for all h, h_1 in M_G.

M is said to be *convex* if M_G is a convex family of maps from $\mathrm{cl}(G)$ to Y for each open set G, i.e. if h_0 and h_1 lies in M_G, and if $0<\lambda<1$, then the mapping h_λ given by

$$h_\lambda(x) = (1-\lambda)h_0(x)+\lambda h_1(x), \qquad x\in\mathrm{cl}(G),$$

should also lie in M_G.

First Definition of Degree. Let X and Y be Banach spaces, M a set of permissible homeomorphisms as in Definition 2, G an open set in X, f a continuous mapping of $\mathrm{cl}(G)$ into Y such that $f=h+C$, where h is a homeomorphism of $\mathrm{cl}(G)$ into Y lying in M and C maps $\mathrm{cl}(G)$ into a relatively compact subset of Y. Let y be a point of $Y-f(\mathrm{bdry}(G))$. Then we set

$$\deg_1([f,h],G,y) = \deg_{\mathrm{LS}}(fh^{-1}, h(G), y).$$

Note that this definition makes sense since $fh^{-1} = I+Ch^{-1}$ is a mapping of $\mathrm{cl}(h(G))$ into Y of the form $I+C_1$ with C_1 compact.

Theorem 1. *Let M be a class of homeomorphisms, and consider the first definition of degree. Then*:

(a) $\deg_1([f,h],G,y)$ *is independent of the choice of representation of f up to a change of sign.*

(b) *If M is convex,* $\deg_1([f,h],G,y)$ *is independent of the representation.*

(c) *In the general case,* $\deg_1$ *has the following properties*:

(1) *If* $\deg_1([f,h],G,y) \neq 0$, *there must exist* x *in* G *such that* $f(x) = y$.

(2) *The degree is additive on* G.

(3) *The degree is invariant under permissible homotopies of representations in the following sense: Let* G *be an open subset of* X, f *a continuous mapping of* $cl(G)\times[0,1]$ *into* Y, *and set* $f_t(x) = f(x,t)$ *for* x *in* $\mathrm{cl}(G)$. *Suppose that there exists a continuous mapping* C *of* $\mathrm{cl}(G)\times[0,1]$ *into a relatively compact subset of* Y *and a continuous function* h *from* $[0,1]$ *into the family of homeomorphisms* M *such that for each* t *in* $[0,1]$, $f_t(x) = h_t(x)+C(x,t)$ *for* x *in* $\mathrm{cl}(G)$. *Let* $P = \{y(t): 0 \leqslant t \leqslant 1\}$ *be a continuous curve in* Y *such that* $y(t)$ *lies in* $Y - f_t(\mathrm{bdry}(G))$. *Then*

$$\deg_1([f_t,h_t],G,y(t))$$

is independent of t *in* $[0,1]$.

(4) *If* $\deg_1([f,h],G,y)$ *is well-defined and* f *is a homeomorphism from* G *to an open subset of* Y *such that* y *lies in* $f(G)$, *then*

$$\deg_1([f,h],G,y) = \pm 1.$$

(5) *The degree* $\deg_1([f,h],G,y)$ *whenever it is defined depends only upon the behaviour of* f *and* h *on the boundary of* G.

(d) *In the case when* M *is convex*, $\deg_1([f,h],G,y)$ *can be written as* $\deg_1(f,G,y)$ *by part* (b). *This degree function has the following additional properties*:

(3)′ *Let* G *be an open subset of* X, *and for each* t *in* $[0,1]$, *let* f_t *be a mapping of* $\mathrm{cl}(G)$ *into* Y *representable in the form* h_t+C_t, *with* h_t *a homeomorphism in* M, C_t *compact. Suppose that the map* $t\to f_t$ *is a continuous curve of maps and that for a continuous curve* $t\to y(t)$ *in* Y, $y(t)\in Y-f_t(\mathrm{bdry}(G))$ *for each* t *in* $[0,1]$. *Then*,

$$\deg_1(f_t,G,y(t))$$

is independent of t *in* $[0,1]$.

(5)′ *The degree function* $\deg_1(f,G,y)$ *is dependent only upon the behaviour of* f *on* $\mathrm{bdry}(G)$ *and not in the rest of* $\mathrm{cl}(G)$.

The proof of Theorem 1 is given in Section 1.

The second definition of degree which we introduce is defined for the following more general type of representation of a mapping f from $\mathrm{cl}(G)$ in X to Y:

Definition 3. Let X and Y be Banach spaces, G an open subset of X, f a continuous mapping of $\mathrm{cl}(G)$ into Y. Let M be a set of permissible homeomorphisms from open subsets of X to Y. Then f is said to have an *intertwined representation with respect to* M if the following is true:

There exists a continuous mapping S of $\mathrm{cl}(G)\times\mathrm{cl}(G)$ into Y such that $f(u) = S(u,u)$ for all u in $\mathrm{cl}(G)$, and S satisfies the two conditions:

(i) For each v in $\mathrm{cl}(G)$, the map $S_v = S(\cdot,v)$ of $\mathrm{cl}(G)$ into Y is a homeomorphism belonging to M.

(ii) The mapping $v\to S_v$ maps $\mathrm{cl}(G)$ into a relatively compact subset of M.

S is said to be a *representation* of the given class for f.

Example. Suppose that M is closed under adding constants, i.e. if h is a map in M, so is h_w, where $h_w(x) = h(x)+w$ for each w in Y. Suppose that f is a map of the form $f = h+C$, with h in M and C compact. Then f has a representation in intertwined form, namely

$$f(u) = S(u,u), \quad \text{with } S(u,v) = h(u)+C(v).$$

For each fixed v, $h_{C(v)}(x) = h(x)+C(v)$ yields an element $h_{C(v)}$ of M and the map $v\to h_{C(v)}$ is obviously compact.

Second definition of degree. Let X and Y be Banach spaces, M a set of permissible homeomorphisms, G an open subset of X, f a continuous mapping of $\mathrm{cl}(G)$ into Y having an intertwined representation S in the sense of Definition 3 with respect to the class M. Then we define the degree of f with respect to S on G over any point y of $Y-f(\mathrm{bdry}(G))$ as follows:

Let $G_y = \{v \mid v\in G,\ y\in S_v(G)\}$, and for v in G_y, set $C_y(v) = S_v^{-1}(y)$. We set

$$\deg_M([f,S],G,y) = \deg_{LS}(I-C_y,G_y,0).$$

THEOREM 2. *Let* X *and* Y *be Banach spaces*, M *a family of permissible homeomorphisms from subsets of* X *to* Y, G *an open subset of* X, f *a continuous mapping of* $\mathrm{cl}(G)$ *into* Y *having an intertwined representation in the sense of Definition* 3. *Let* y *be a point of* $Y-f(\mathrm{bdry}(G))$. *Then*:

(a) $\deg_M([f,S],G,y)$ *is well-defined by the above definition, and in particular*, G_y *is an open subset of* X, C_y *can be extended to a continuous mapping of* $\mathrm{cl}(G_y)$ *into* X *whose image is relatively compact in* X, *and* C_y *as thus extended has no fixed points on* $\mathrm{bdry}(G_y)$.

(b) *The second degree function has the following properties*:

(1) *If* $\deg_M([f,S],G,y)$ *is well-defined and* $\neq 0$, *there exists a point* x *in* G *such that* $f(x) = y$.

(2) *The degree function is additive in* G.

(3) *The degree function* $\deg_M([f,S],G,y)$ *is invariant under permissible homotopies of representations: i.e. let* F *be a continuous mapping of* $\mathrm{cl}(G)\times[0,1]$ *into* Y, $f_t(x) = F(x,t)$ *for* x *in* $\mathrm{cl}(G)$, *and suppose that there exists a mapping* S *of* $\mathrm{cl}(G)\times\mathrm{cl}(G)\times[0,1]$ *into* Y *where* $S_t:\mathrm{cl}(G)\times$

$\times \mathrm{cl}(G) \to Y$ *defined by* $S_t(u, v) = S(u, v, t)$ *is an intertwined representation for* f_t. *Suppose that the map* $t \to S_t$ *is a continuous curve in the metric space of maps of* $\mathrm{cl}(G) \times \mathrm{cl}(G)$ *into* Y, *and that for another continuous curve* $P = \{y(t): 0 \leqslant t \leqslant 1\}$ *in* Y, $y(t) \in Y - f_t(\mathrm{bdry}(G))$ *for each* t *in* $[0, 1]$. *Then*

$$\deg_M([f_t, S_t], G, y(t))$$

is independent of t *in* $[0, 1]$.

(4) *If the representation* $S(u, v)$ *is independent of* v, *so that* f *is a homeomorphism in* M, $\deg_M([f, S], G, y) = +1$ *of* $y \in f(G)$.

The proof of Theorem 2 is given in Section 2.

THEOREM 3. *Let* X *and* Y *be Banach spaces,* M *a convex family of permissible homeomorphisms from subsets of* X *to* Y, G *an open subset of* X. *Let* f *be a continuous mapping of* $\mathrm{cl}(G)$ *into* Y *having an intertwined representation with respect to* M *in the sense of Definition 3,* y *a point in* $Y - f(\mathrm{bdry}(G))$. *Then:*

(a) *The second degree function,* $\deg_M([f, S], G, y)$, *is independent of the choice of the representation* S *of* f *with respect to* M, *and may be written as* $\deg_M(f, G, y)$.

(b) *The following stronger homotopy property holds for* $\deg_M(f, G, y)$: *Let* F *be a continuous mapping of* $\mathrm{cl}(W)$ *into* Y, *where* W *is an open subset of* $X \times [0, 1]$. *For each* t *in* $[0, 1]$, *let* $G_t = \{x \mid x \in X, (x, t) \in W\}$, *and let* f_t *be the mapping of* $\mathrm{cl}(G_t)$ *into* Y *given by* $f_t(x) = F(x, t)$. *Suppose that each* f_t *has an intertwined representation with respect to* M, *and that we are given a continuous curve* $\{y(t): 0 \leqslant t \leqslant 1\}$ *in* Y *such that the set* $W_0 = \{(x, t) \mid (x, t) \in \mathrm{cl}(W), F(x, t) = y(t)\}$ *is a compact subset of* W. *Suppose further that for each open set* G *in* X *and interval* $[a, b] \subset [0, 1]$ *such that* $\mathrm{cl}(G) \times [a, b] \subset W$, *the map* $t \to f_t$ *is continuous from* $[a, b]$ *to the space of mappings of* $\mathrm{cl}(G)$ *into* Y.

Then $\deg_M(f_t, G_t, y(t))$ *is independent of* t *in* $[0, 1]$.

(c) *Suppose that* $X = Y$, M *is convex and includes the identity map of* $\mathrm{cl}(G)$ *into* X, *and that* f *is of the form* $f = h + C$, *with* h *in* M *and* C *compact. Then*

$$\deg_1([f, h], G, y) = \deg_M(f, G, y).$$

We note that Theorem 3 states in general terms that if the set M of permissible homeomorphisms is convex, the second degree function is independent of the choice of the representation S and varies with homotopies of the mapping f, not of its representation S. In addition, the two degrees coincide for maps of the form $f = h + C$ when $X = Y$.

A still wider class of mappings for which a degree can be defined is that treated in the following theorem:

THEOREM 4. *Let* X *and* Y *be Banach spaces,* M *a convex family of homeomorphisms from subsets of* X *to* Y, G *an open subset of* X, f *a mapping of* $\mathrm{cl}(G)$ *into* Y *such that there exists a sequence* $\{f_k\}$ *of mappings of* $\mathrm{cl}(G)$ *into* Y *having intertwined representations in the sense of Definition 3, with* $f_k(x)$ *converging to* $f(x)$ *uniformly for* x *in* $\mathrm{cl}(G)$. *Then:*

(a) *For any* y *in* $Y - \mathrm{cl}(f(\mathrm{bdry}(G)))$, $\deg_M(f_k, G, y)$ *is well defined for* k *sufficiently large, and* $\deg_M(f_k, G, y)$ *converges as* $k \to \infty$ *to a limit, which we denote by* $\deg_M(f, G, y)$. *This limit is independent of the choice of the sequence* $\{f_k\}$.

(b) *The degree function thus defined has the following properties:*

(1) *If* $\deg_M(f, G, y)$ *is well-defined and* $\deg_M(f, G, y) \neq 0$, *there exists a sequence of points* $\{x_k\}$ *in* G *such that* $f(x_k) \to y$. *In particular, if* $f(\mathrm{cl}(G))$ *is closed in* Y, *there exists* x *in* G *such that* $f(x) = y$.

(2) *The degree is additive in* G.

(3) *If* $\{f_t, 0 \leqslant t \leqslant 1\}$, *is a continuous curve of mappings for which* $\deg_M(f_t, G, y(t))$ *is well-defined for a continuous curve* $\{y(t), 0 \leqslant t \leqslant 1\}$ *in* Y, *then* $\deg_M(f_t, G, y(t))$ *is independent of* t *in* $[0, 1]$.

(4) *If* $f = \lim_k h_k$, *where each* h_k *is a homeomorphism in* M, *then* $\deg_M(f, G, y) = +1$ *for any point* y *in* $f(G) - \mathrm{cl}(f(\mathrm{bdry}(G)))$.

The proofs of Theorems 3 and 4 are given in Section 3 below.

To note the simplest applications of Theorems 3 and 4, we consider the following classes of permissible homeomorphisms:

$M_1 = \{h: G$ *is an open set in* X, $h: \mathrm{cl}(G) \to X^*$, *with* X^* *the conjugate space of* X, *and* h *satisfies the firm monotonicity condition:*

$$(h(u) - h(v), u - v) \geqslant c(\|u - v\|), \qquad u, v \in \mathrm{cl}(G),$$

for a continuous function $c: R^+ \to R^+$ *with* $c(0) = 0$, $c(r) > 0$ *for* $r > 0$.$\}$

(Here, we use (w, u) to denote the value of the functional w at the point u.)

$M_2 = \{h: G$ *is an open subset in the complex Banach space* X, $h: \mathrm{cl}(G) \to X^*$, *and* h *satisfies the complex monotonicity condition*

$$|(h(u) - h(v), u - v)| \geqslant c(\|u - v\|), \qquad u, v \in \mathrm{cl}(G),$$

for a function $c(r)$ *as above*$\}$.

$M_3 = \{h: G$ *is an open subset in* X, $h: \mathrm{cl}(G) \to X$, *with* h *satisfying the firm accretiveness condition*

$$(h(u) - h(v), J(u - v)) \geqslant c(\|u - v\|), \qquad u, v \in \mathrm{cl}(G),$$

where J *is a duality mapping of* X *into* X^* *satisfying the conditions:* $(J(x), x) = \|x\| \|J(x)\|$; $\|J(x)\| = \zeta(\|x\|)$ *for a continuous strictly increasing function* $\zeta(r)$ *with* ζ *mapping* R^+ *onto* $R^+$$\}$.

$M_4 = \{h: G$ *is an open subset of* X, $h: \mathrm{cl}(G) \to X$, *with* $h = I - V$ *and* V *a strict contraction, i.e. there exists* $k < 1$ *such that for all* u *and* v *of* $\mathrm{cl}(G)$, $\|V(u) - V(v)\| \leqslant k\|u - v\|\}$.

THEOREM 5. *Let* X *be a Banach space,* h *a continuous mapping of* $\mathrm{cl}(G)$ *in* X *into* X *or* X^* *lying in one of the four classes* M_1, M_2, M_3, *or* M_4. *Suppose that* X *is reflexive for* M_1 *and* M_2, *and that* X^* *is uniformly convex (or* h *uniformly continuous on bounded sets) for* M_3. *Then all the mappings* h *thus obtained are permissible homeomorphisms in the sense of Definition* 1.

The proof of Theorem 5 and its detailed application to a wide variety of fixed point and mapping theorems for semi-monotone, semi-accretive and semi-contractive mappings is given in paper [8].

1. We begin now with the detailed argument concerning our generalized degree functions. We note first an additional property of the Leray-Schauder degree that is conspicuously not preserved in most of the generalizations given in the present paper, namely the product formula for the degree (Nagumo [17], Theorem 9):

(6) *Let* G *be an open subset of the B-space* X, f *a continuous map of* $\mathrm{cl}(G)$ *into* X *of the form* $f = I + C$, *with* $C(\mathrm{cl}(G))$ *rel. compact in* X. *Let* H *be an open subset of* X *containing* $f(\mathrm{cl}(G))$, *and consider the components* H_j *of the set* $H - f(\mathrm{bdry}(G))$. *Let* g *be a continous mpaping of* $\mathrm{cl}(H)$ *into* X *of the form* $g = I + C_1$, *with* $C_1(\mathrm{cl}(H))$ *rel, compact in* X, *and suppose that* y *is a point of* X *outside of* $g(\mathrm{bdry}(H)) \cup gf(\mathrm{bdry}(G))$. *Then:*

$$\deg_{\mathrm{LS}}(gf, G, y) = \sum_j \deg_{\mathrm{LS}}(g, H_j, y) \cdot \deg_{\mathrm{LS}}(f, G, z_j),$$

where for each j, z_j *is an arbitrary point of* H_j.

PROPOSITION 1. *Let* G *be an open subset of the Banach space* X, f *a continuous mapping of* $\mathrm{cl}(G)$ *into* Y *of the form* $f = h + C = h_1 + C_1$, *where* h *and* h_1 *are permissible homeomorphisms of* $\mathrm{cl}(G)$ *into* Y *and* $C(\mathrm{cl}(G))$ *and* $C_1(\mathrm{cl}(G))$ *are relatively compact in* Y. *Then*

$$|\deg_{\mathrm{LS}}(I + Ch^{-1}, h(G), y)| = |\deg_{\mathrm{LS}}(I + C_1 h_1^{-1}, h_1(G), y)|$$

for any y *in* $Y - f(\mathrm{bdry}(G))$. *In particular,* $|\deg_1([f, h], G, y)|$ *is independent of the representation of* f *in the form* $f = h + C$.

Proof. By hypothesis, $h + C = h_1 + C_1$. Hence on $h(\mathrm{cl}(G))$, $I + Ch^{-1} = h_1 h^{-1} + C_1 h^{-1}$. In particular, the homeomorphism of $h(\mathrm{cl}(G))$ into $h_1(\mathrm{cl}(G))$ given by $h_1 h^{-1}$ is a mapping of the form $I + C'$ with $C'(\mathrm{cl}(h(G))$ relatively compact in Y, and by property (4) of the Leray-Schauder degree, $\deg_{\mathrm{LS}}(h_1 h^{-1}, h(G), z) = \pm 1$ for any z in $h_1(G)$. On the other hand, we have

$$(I + Ch^{-1}) = (I + C_1 h_1^{-1})(h_1 h^{-1}).$$

Since $\deg_{\mathrm{LS}}$ is additive in G, we may assume without loss of generality that G is connected, and thereby $h(G)$ and $h_1(G)$ are connected as well. Both h and h_1 are permissible homeomorphisms and hence map $\mathrm{bdry}(G)$ onto $\mathrm{bdry}(h(G))$ and $\mathrm{bdry}(h_1(G))$, respectively. Thus $h_1 h^{-1}$ maps $\mathrm{bdry}(h(G))$ onto $\mathrm{bdry}(h_1(G))$, while $h_1(G)$ is connected and thereby contained in one component of $Y - (h_1 h^{-1})(\mathrm{bdry}(h(G)) = Y - \mathrm{bdry}(h_1(G))$. On the other hand, $h_1(G)$ must exhaust this component since otherwise this component would contain a point of $\mathrm{bdry}(h_1(G))$.

We now apply the product formula for the Leray-Schauder degree with $f = h_1 h^{-1}$, $g = (I + C_1 h_1^{-1})$. We obtain

$$\deg_{\mathrm{LS}}(I + Ch^{-1}, h(G), y) = \deg_{\mathrm{LS}}(I + C_1 h_1^{-1}, h_1(G), y) \deg_{\mathrm{LS}}(h_1 h^{-1}, h(G), z)$$

for any point z in the single component of $Y - (h_1 h^{-1})\big(\mathrm{bdry}(h(G))\big)$ which contains points of $(h_1 h^{-1})(h(G)) = h_1(G)$. Finally

$$\deg_{\mathrm{LS}}(I + Ch^{-1}, h(G), y) = \pm \deg_{\mathrm{LS}}(I + C_1 h_1^{-1}, h_1(G), y),$$

q.e.d.

PROPOSITION 2. *The first degree function,* $\deg_1([f, h], G, y)$, *defined for all maps of the form* $f = h + C$, *with* h *a homeomorphism and* C *compact, and for* $y \in Y - f(\mathrm{bdry}(G))$, *has the following properties:*

(1) *If* $\deg_1([f, h], G, y) \neq 0$, *there exists* x *in* G *such that* $f(x) = y$.

(2) *The degree is additive in* G.

(3) *The degree is invariant under permissible homotopies of representations.*

(4) *If* f *is a homeomorphism from* G *to an open subset of* Y, $y \in f(G)$, *then for any representation of* f,

$$\deg_1([f, h], G, y) = \pm 1.$$

(5) *The degree* $\deg_1([f, h], G, y)$ *depends only upon the behaviour of* f *and* h *on the boundary of* G.

Proof. (1) If $\deg_1([f, h], G, y) \neq 0$, we have $\deg_{\mathrm{LS}}(fh^{-1}, h(G), y) \neq 0$. There must exist z in $h(G)$ such that $fh^{-1}(z) = y$. If $x = h^{-1}(z)$, then x lies in G and $f(x) = y$, q.e.d.

(2) Suppose that $G = G_1 \cup G_2$ with G_1 and G_2 disjoint and $y \in Y - f(\mathrm{bdry}(G_1) \cup \mathrm{bdry}(G_2))$. Then $h(G) = h(G_1) \cup h(G_2)$ with $h(G_1)$ and $h(G_2)$ disjoint and $y \in Y - fh^{-1}(\mathrm{bdry}(h(G_1)) \cup \mathrm{bdry}(h(G_2)))$. Hence

$$\deg_{\mathrm{LS}}(I + Ch^{-1}, h(G), y) = \deg_{\mathrm{LS}}(I + Ch^{-1}, h(G_1), y) + \deg_{\mathrm{LS}}(I + Ch^{-1}, h(G_2), y),$$

i.e.

$$\deg_1([f, h], G, y) = \deg_1([f, h], G_1, y) + \deg_1([f, h], G_2, y),$$

q.e.d.

(3) Consider a family of mapping $f_t = h_t + C_t$ for t in $[0,1]$ with h_t a continuous curve of homeomorphisms of $\mathrm{cl}(G)$ into Y, $C_t(x) = C(x,t)$ continuous from $\mathrm{cl}(G)\times[0,1]$ to Y with a relatively compact image in Y. Suppose that for all t in $[0,1]$, $y \in Y - f_t(\mathrm{bdry}(G))$. Then

$$\deg_1([f_t, h_t], G, y) = \deg_{\mathrm{LS}}(I + C_t h_t^{-1}, h_t(G), y).$$

If we apply property (3) of invariance of the Leray-Schauder degree under permissible homotopies, it suffices to assume that G is connected and to prove that

$$W = \{(z,t)\,|\,(z,t)\in Y\times[0,1],\, z\in h_t(G)\}$$

is an open subset of $Y\times[0,1]$, that its boundary is given by

$$W' = \{(z,t)\,|\,(z,t)\in Y\times[0,1],\, z\in h_t(\mathrm{bdry}(G))\},$$

that the map $\varphi\colon (z,t)\to C_t(h_t^{-1}(z))$ is a continuous map of $\mathrm{cl}(W)$ into Y with relatively compact image, and that for $(z,t)\in W'$, $C_t(h_t^{-1}(z)) + z \neq y$. The last property follows from the others since for $(z,t)\in W'$, $z \in h_t(\mathrm{bdry}(G))$, $x = h_t^{-1}(z)\in\mathrm{bdry}(G)$, and $z + C_t(h_t^{-1}(z)) = h_t(x) + C_t(x) = f_t(x) \neq y$. The relative compactness of the image of φ follows from that of the mapping C, and since $\varphi = C\psi$, where ψ is the mapping of $W \cup W'$ given by $\psi(z,t) = (h_t^{-1}(z), t)$, into $X\times[0,1]$, it suffices to prove that ψ is continuous in order to prove that φ is continuous. Thus the rest of the proof follows from the following Lemma:

Lemma 1. *Let X and Y be Banach spaces, G a connected open subset of X, K a topological space, h a continuous mapping of K into the space M of permissible homeomorphisms of* $\mathrm{cl}(G)$ *into Y. Then:*

(a) *The set* $W = \{(y,k)\,|\,(y,k)\in Y\times K,\, y\in h_k(G)\}$ *is open in* $Y\times K$.

(b) *The boundary of W in $Y\times K$ is given by* $W' = \{(y,k)\,|\,y\in h_k(\mathrm{bdry}(G))\}$

(c) *The mapping ψ of* $\mathrm{cl}(W)$ *into $X\times K$ given by* $\psi(y,k) = (h_k^{-1}(y), k)$ *is continuous.*

Proof. Let $k\in K$. Since G is connected, $h_k(G)$ is connected and hence a subset of a single component Y_1 of $Y - h(\mathrm{bdry}(G))$. Y_1 must equal $h_k(G)$, since otherwise Y_1 would contain a point of $\mathrm{bdry}(h(G)) = h(\mathrm{bdry}(G))$ Hence, $h_k(G)$ contains any connected subset A of $Y - h(\mathrm{bdry}(G))$ whenever it contains a single point of A.

(a) Suppose $(z_0, k_0)\in W$. Then $z_0 \in h_{k_0}(G)$, and there exists a closed ball $B_r(z_0)$ of radius r about z_0 contained in $h_{k_0}(G)$. Since h is continuous from K to M, we may find a neighborhood N of k_0 in K such that for k in N and all x in $\mathrm{cl}(G)$,

$$\|h_k(x) - h_{k_0}(x)\| < \frac{r}{2}.$$

Let B' be the ball of radius $r/2$ about z_0. Then for k in N, $h_k(\mathrm{bdry}(G)) \subset Y - B'$, while if $x_0 = h_{k_0}^{-1}(z_0)$, $h_k(x_0)\in B'$. Hence $B' \subset h_k(G)$ for all k in N, and $(B'\times N)\subset W$. Hence W is open in $Y\times K$, q.e.d.

(b) $W' \subset \mathrm{bdry}(W)$. Hence, it suffices to show that if (z_0, k_0) does not lie in $W\cup W'$, it has a neighborhood which does not intersect W. Since $z_0 \in Y - h_{k_0}(\mathrm{cl}(G))$, there exists a ball $B_r(z_0)$ with $r > 0$ outside of $h_{k_0}(\mathrm{cl}(G))$. By the continuity of h from K to M, we may choose a neighborhood N of k_0 in K such that for k in N and all x in $\mathrm{cl}(G)$,

$$\|h_k(x) - h_{k_0}(x)\| < \frac{r}{2}.$$

Let B' be the ball about z_0 of radius $r/2$. Then for k in N, $h_k(\mathrm{bdry}(G)) \subset Y - B'$. We assert that $B'\times N$ does not meet W. Indeed, if it did, there would exist k in N and x in G such that $h_k(x)\in B'$. Hence $B' \subset h_k(G)$, and if $x' = h_k^{-1}(z_0)$, $h_{k_0}(x')\in B'$. It would follows that $z_0 \in h_{k_0}(G)$ which is a contradiction, q.e.d.

(c) Suppose that $\psi(z_0, k_0) = (x_0, k_0)$, i.e. $h_{k_0}(x_0) = z_0$, where k_0 lies in K, $x_0\in\mathrm{cl}(G)$. If $h_k(x) = z$, with $\|z - z_0\| < r$ and if k is chosen in a suitable neighborhood N of k_0, $\|h_k(x) - h_{k_0}(x)\| < r$ for all x in $\mathrm{cl}(G)$. Hence $\|h_{k_0}(x) - z_0\| < 2r$, and $\|x - x_0\| < \beta(2r)$ where $\beta(r)$ is the modulus of continuity of $h_{k_0}^{-1}$ at z_0, q.e.d.

Proof of Proposition 2 completed. The proof of (3) follows from Lemma 1. Since

$$\deg_1([f,h], G, y) = \deg_{\mathrm{LS}}(fh^{-1}, h(G), y)$$

while fh^{-1} is a homeomorphism if f is, property (4) follows from property (4) of the Leray-Schauder degree. Since h maps $\mathrm{bdry}(G)$ on $\mathrm{bdry}(h(G))$, the behaviour of fh^{-1} on $\mathrm{bdry}(h(G))$ follows from the behaviour of f and h on $\mathrm{bdry}(G)$. Hence Property (5) follows from Property (5) of the Leray-Schauder degree.

Proposition 3. *If the family of permissible homeomorphisms M is convex, then* $\deg_1([f,h], G, y)$ *depends only upon f, and the homotopy invariance property* (3)′ *holds in Theorem* 1.

Proof. Let M be convex and let $f = h + C = h_1 + C_1$ be representations with respect to M. If $y\in Y - f(\mathrm{bdry}(G))$, we set

$$h_t(x) = (1-t)h(x) + th_1(x), \qquad C_t(x) = (1-t)C(x) + tC_1(x).$$

Then for all t in $[0,1]$, h_t is an element of M, and $f = h_t + C_t$ is a permissible homotopy of representations. Hence by Proposition 2, $\deg_1([f,h_t], G, y)$ is independent of t in $[0,1]$. Thus

$$\deg_1([f,h], G, y) = \deg_1([f,h_1], G, y) = \deg_1(f, G, y)$$

depends only upon f.

Suppose that $\{f_t : t\in[0,1]\}$ is a homotopy of mappings with representations of the form $f_t = h_t + C_t$ for each t, $h_t \in M$. Suppose that $y \in Y - f_t(\mathrm{bdry}(G))$ for all t in $[0,1]$. Then there exists $d_0 > 0$ such that for all x in $\mathrm{bdry}(G)$ and all t in $[0,1]$, $\|f_t(x) - y\| \geqslant d_0$. We may find an increasing finite sequence $\{t_j\}$ in $[0,1]$ with $t_0 = 0$, $t_n = 1$ such that for each j,

$$\|f_{t_{j+1}}(x) - f_{t_j}(x)\| < d_0$$

for all x in $\mathrm{bdry}(G)$. It suffices to show that

$$\deg_1(f_{t_j}, G, y) = \deg_1(f_{t_{j+1}}, G, y).$$

Let $f_{t_j} = h_j + C_j$, $f_{t_{j+1}} = h_{j+1} + C_{j+1}$. We set

$$g_\lambda = (1-\lambda) f_{t_j} + \lambda f_{t_{j+1}}$$

for λ in $[0,1]$, where $g_\lambda = h_\lambda + C_\lambda$, with $h_\lambda = (1-\lambda) h_j + \lambda h_{j+1}$, $C_\lambda = (1-\lambda) C_j + \lambda C_{j+1}$. This gives a permissible homotopy of representations since each h_λ lies in M and $\|g_\lambda(x) - y\| > 0$ for $x \in \mathrm{bdry}(G)$, q.e.d.

The conclusions of Theorem 1 follow from Propositions 1, 2, and 3.

2. The proof of Theorem 2 is based upon the following two propositions:

PROPOSITION 4. *Let G be an open set in X, S a continuous map of $\mathrm{cl}(G)\times\mathrm{cl}(G)$ into Y where for each v in $\mathrm{cl}(G)$, $S_v = S(\cdot, v)$ is a permissible homeomorphism in M from $\mathrm{cl}(G)$ to Y and the map $v \to S_v$ from $\mathrm{cl}(G)$ to M is continuous and maps $\mathrm{cl}(G)$ into a relatively compact subset of M. Then:*

(a) *For a given y in Y, $G_y = \{v \mid v\in G,\ y \in S_v(G)\}$ is open in X.*

(b) *If $G_y^+ = \{v \mid v \in \mathrm{cl}(G),\ y\in S_v(\mathrm{cl}(G))\}$, $\mathrm{cl}(G_y) \subset G_y^+$.*

(c) *The map C_y of $\mathrm{cl}(G_y)$ into $\mathrm{cl}(G)$ given by $C_y(v) = S_v^{-1}(y)$, is continuous and maps $\mathrm{cl}(G_y)$ into a relatively compact subset of $\mathrm{cl}(G)$.*

(d) *C_y maps $\mathrm{bdry}(G_y)$ into $\mathrm{bdry}(G)\cdot C_y(v) = v$ if and only if $S(v,v) = y$.*

Proof. (a) Suppose $v_0 \in G_y$. Then $y \in S_{v_0}(G)$ and if $u_0 = S_{v_0}^{-1}(y)$, we have $S(u_0, v_0) = y$. Since $S_{v_0}(G)$ is open in Y and S_{v_0} is a homeomorphism, there exist balls $B_s(u_0)$ and $B_r(y)$ such that $B_r(y) \subset S_{v_0}(B_r(u_0)) \subset S_{v_0}(G)$. It follows that

$$S_{v_0}\big(\mathrm{bdry}(B_s(u_0))\big) \subset Y - B_r(y).$$

We may find a neghborhood N of v_0 in G such that for v in N and all u in $\mathrm{cl}(G)$,

$$\|S_v(u) - S_{v_0}(u)\| < \frac{r}{2}.$$

For such v, we have $S_v\big(\mathrm{bdry}(B_s(u_0))\big) \subset Y - B'$, where B' is the ball of radius $r/2$ about y while $S_v(u_0) \in B'$. Applying the principle developed in the proof of Lemma 1, $B' \subset S_v(B_s(u_0))$ for v in N, so that $N \subset G_y$. Hence G_y is open in G, q.e.d.

(b) Since $G_y \subset G_y^+$, it suffices to show that $\mathrm{bdry}(G_y) \subset G_y^+$. Let v_0 be a point of $\mathrm{bdry}(G_y)$. There exists a sequence $\{v_j\}$ in G_y with $v_j \to v_0$. Then $u_j = S_{v_j}^{-1}(y) \in G$. Since

$$\|S_{v_j}(u_j) - S_{v_0}(u_j)\| \to 0,$$

it follows that y lies in $S_{v_0}(\mathrm{cl}(G))$ and hence v_0 lies in G_y^+, q.e.d.

(c) Let $v_j \to v$ in $\mathrm{cl}(G_y)$, with $u_j = C_y(v_j)$, $u = C_y(v)$. Then

$$\|S(u_j, v_j) - S(u_j, v)\| \to 0, \quad \text{i.e.} \quad S_v(u_j) \to y.$$

Since S_v^{-1} is continuous, $u_j \to S_v^{-1}(y) = u$, and C_y is continuous on $\mathrm{cl}(G_y)$.

Suppose that $\{v_j\}$ is a sequence in $\mathrm{cl}(G_y)$. We pass to a subsequence and using the fact that the map of $\mathrm{cl}(G)$ into M given by $v \to S_v$ has a relatively compact image, we may assume that $S_{v_j} \to h$, where $h \in M$. If $u_j = C_y(v_j)$, it follows that $h(u_j) \to y$. Hence $y \in h(\mathrm{cl}(G))$, and $u_j \to h^{-1}(y)$. Thus C_y maps $\mathrm{cl}(G_y)$ into a relatively compact set in $\mathrm{cl}(G)$, q.e.d.

(d) Since $C_y^{-1}(G) = G_y$, it follows that $C_y(\mathrm{bdry}(G)) \subset \mathrm{bdry}(G)$. On the other hand, $C_y(v) = v$ for v in $\mathrm{cl}(G_y)$ if and only if $S(v,v) = y$.

PROPOSITION 5. *Let $S: \mathrm{cl}(G)\times\mathrm{cl}(G)\times[0,1] \to Y$ with G connected be a permissible homotopy in the sense of Theorem 2, and for each t in $[0,1]$, let $G_{y,t}$, $G_{y,t}^+$, and $C_{y,t}$ be defined as in Proposition 4 for the mapping $S_t(u,v) = S(u,v,t)$. Then:*

(a) *The set $W = \{(v,t) \mid (v,t) \in X\times[0,1],\ v \in G_{y,t}\}$ is open in $X\times[0,1]$.*

(b) *The boundary of W is $W' = \{(v,t) \mid v \in \mathrm{bdry}(G_{y,t})\}$.*

(c) *The map $C_y: \mathrm{cl}(W) \to X$ given by $C_y(v,t) = C_{y,t}(v)$ is continuous from $\mathrm{cl}(W)$ into X and has relatively compact image in X.*

Proof. The proof of (a) duplicates the proof of Proposition 4 (a), and that of (c), the proof of Proposition 4 (c). Hence it suffices to prove (b). Here, it suffices to show that if (v_0, t_0) does not lie in $W \cup W'$, there exists a neighborhood of (v_0, t_0) which does not meet W. Since $y \in Y - S_{v_0,t_0}(\mathrm{cl}(G))$, there exists a ball $B_r(y)$ in $Y - S_{v_0,t_0}(\mathrm{cl}(G))$. Let B' be the ball about y of radius $r/2$, and choose a neighborhood N of (v_0, t_0) in $X\times[0,1]$ such that for $(v,t) \in N$,

$$\|S_{v,t}(u) - S_{v_0,t_0}(u)\| < \frac{r}{2}$$

for all u in $\mathrm{cl}(G)$. Then $S_{v,t}(\mathrm{bdry}(G)) \subset Y - B'$, and we assert that $N \cap W = 0$. Indeed, otherwise if $(v,t) \in N \cap W$, we have $y \in S_{v,t}(G)$ and if $u = C_{y,t}(v)$, $S_{v_0,t_0}(u) \in B'$. Since G is connected, $B' \subset S_{v_0,t_0}(G)$, which is a contradiction, q.e.d.

Proof of Theorem 2. Since C_y is a compact continuous mapping of $\mathrm{cl}(G_y)$ into X and since by Proposition 4 (c), $C_y(v) \neq v$ on $\mathrm{bdry}(G_y)$ because $f(v) \neq y$ on $\mathrm{bdry}(G)$, it follows that

$$\deg_M([f, S], G, y) = \deg_{\mathrm{LS}}(I - C_y, G_y, 0)$$

is well-defined. If $\deg_M([f, S], G, y) \neq 0$, there exists v in G_y such that $C_y(v) = v$, and for this $v, f(v) = S(v, v) = y$. If $G = G_1 \cup G_2$ with G_1 and G_2 disjoint and $f(v) \neq y$ on $\mathrm{bdry}(G_1) \cup \mathrm{bdry}(G_2)$, and if we form the sets $G_{1,y}$ and $G_{2,y}$ analogous to G_y for G_1 and G_2, respectively, then $G_y = G_{1,y} \cup G_{2,y}$ and $G_{1,y} \cap G_{2,y} = \emptyset$. Hence

$$\deg_{\mathrm{LS}}(I - C_y, G_y, 0) = \deg_{\mathrm{LS}}(I - C_y, G_{1,y}, 0) + \deg_{\mathrm{LS}}(I - C_y, G_{2,y}, 0),$$

and the additivity on G of $\deg_M([f, S], G, y)$ follows.

By Proposition 5, if we have a permissible homotopy of representations, we obtain a permissible homotopy of $C_{y,t}$ and $G_{y,t}$ in the sense of the Leray-Schauder degree. Hence

$$\deg_{\mathrm{LS}}(I - C_{y,t}, G_{y,t}, 0) = \deg_M([f_t, S_t], G, y)$$

is independent of t in $[0, 1]$. Finally, if $f(u) = S(u, v)$ for all v in $\mathrm{cl}(G)$ then $C_y(v) = f^{-1}(y)$ is a constant map and $\deg_M([f, S], G, y) = +1$, q.e.d.

3. Proof of Theorem 3 (a). Let S_0 and S_1 be two representations for f with respect to the convex family M. For λ in $[0, 1]$, we set

$$S_\lambda(u, v) = (1-\lambda) S_0(u, v) + \lambda S_1(u, v).$$

Then S_λ is a permissible homotopy of representations in the sense of Theorem 2, and since $S_\lambda(u, u) = f(u) \neq y$ for u on $\mathrm{bdry}(G)$, we see that $\deg_M([f, S_\lambda], G, y)$ is independent of λ on $[0, 1]$, q.e.d.

Proof of Theorem 3 (b). Since the set W_0 of solutions (x, r) of the equation $f_t(x) = y$ is a compact subset of W, we may replace W by a finite collection of sets of the form $\mathrm{cl}(G_j) \times [a_j, b_j]$ contained in W and prove the invariance of $\deg_M(f_t, G, y)$ with G_t replaced by one of the G_j and $y(t)$ replaced by a fixed element of Y. We may moreover break up this homotopy into small steps and assume that

$$\|f_0(u) - f_1(u)\| < d_0,$$

where for all u in $\mathrm{bdry}(G)$, $\|f_0(u) - y\| \geqslant d_0$. We choose two representations S_0 and S_1 for f_0 and f, respectively. For λ in $[0, 1]$, we define mappings f_λ by $f_\lambda = (1-\lambda) f_0 + \lambda f_1$ and representations S_λ by $S_\lambda = (1-\lambda) S_0 + \lambda S_1$. For u in $\mathrm{bdry}(G)$,

$$\|f_\lambda(u) - y\| \geqslant \|f_0(u) - y\| - \|f_\lambda(u) - f_0(u)\| > 0.$$

Hence S_λ is a permissible homotopy in the sense of Theorem 2, and thereby

$$\deg_M(f_0, G, y) = \deg_M(f_1, G, y),$$

q.e.d.

Proof of Theorem 3 (c). Let $f = h + C$ with h in M and C compact. For λ in $[0, 1]$, we let $h_\lambda = (1-\lambda) h + \lambda I$ (I = identity map of $\mathrm{cl}(G)$ into X). We consider the map

$$f_\lambda = (I + Ch^{-1}) h_\lambda \colon h_\lambda^{-1}\big(h(\mathrm{cl}(G))\big) \to X.$$

This is a permissible homotopy in the sense of (b) above, and therefore

$$\deg_M\big(f_\lambda, h_\lambda^{-1}(h(G)), y\big) = \deg_M\big(I + Ch^{-1}, h(G), y\big) = \deg_1([f, h], G, y).$$

For $\lambda = 0$, we obtain $\deg_M(f, G, y) = \deg_1([f, h], G, y)$, q.e.d.

Proof of Theorem 4. Suppose that for u in $\mathrm{bdry}(G)$ we have $\|f(u) - y\| \geqslant d_0$. Then for k and j so large that

$$\|f_k(u) - f(u)\| < \frac{d_0}{2}, \qquad \|f_j(u) - f(u)\| < \frac{d_0}{2},$$

it follows that the segment joining f_j to f_k in the space of maps of $\mathrm{cl}(G)$ into Y consists of maps g such that $g(u) \neq y$ on $\mathrm{bdry}(G)$. Hence

$$\deg_M(f_k, G, y) = \deg_M(f_j, G, y)$$

for such j and k, and $\deg_M(f, G, y)$ is welldefined. Two sequences of approximating maps can be interspersed, and hence the limit $\deg_M(f, G, y)$ is independent of the approximating sequence. In particular, we may take a fixed element h in M, and define a canonical approximation $f_\varepsilon = f + \varepsilon h$. For $\varepsilon > 0$ and sufficiently small, $\deg_M(f, G, y) = \deg_M(f, G, y)$. In particular, if $\deg_M(f, G, y) \neq 0$, for $\varepsilon_k \to 0$, there exist points x_k in G such that $f_{\varepsilon_k}(x_k) = y$, i.e. $f(x_k) \to y$. The degree is obviously additive in G since it is additive in G for the map f_ε. If f_t is a continuous curve of maps for which $\deg_M\big(f_t, G, y(t)\big)$ is welldefined, there exists $d_0 > 0$ such that $\|f_t(u) - y(t)\| \geqslant d_0$ for all u in $\mathrm{bdry}(G)$. Hence for ε fixed and sufficiently small,

$$\deg_M\big(f_t, G, y(t)\big) = \deg_M\big(f_{t,\varepsilon}, G, y(t)\big)$$

is independent of t. Finally, if h_k is a homeomorphism in M, $\deg_M(h_k, G, y) = +1$ for y in $h_k(G)$. Hence $\deg_M(f, G, y) = +1$ for any y such that y lies in $h_k(G)$ for k sufficiently large. Property 4 then follows by replacing G by its component whose image under f contains y, q.e.d.

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