On convergence of sequences of periodic distributions

by

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1. In this paper we are concerned with distributions in the q-dimensional Euclidean space $\boldsymbol{R}^q$. The points of $\boldsymbol{R}^q$ are denoted by $x = (\xi_1, \ldots, \xi_q)$. If nothing else is said explicitly, the values of distributions are in a fixed Banach space.

A distribution f in $\boldsymbol{R}^q$ will be called *periodic*, iff (iff = if and only if) $f(x+p) = f(x)$ for every integral point $p \, \epsilon \boldsymbol{R}^q$, i.e., for every point $p = (\pi_1, \ldots, \pi_q)$ whose the coordinates π_j are integers. In the space of periodic distributions, we consider six kinds of convergence:

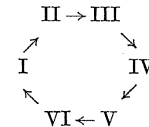I. Tempered strong convergence.

II. Strong convergence.

III. Weak convergence.

IV. Tempered weak convergence.

V. Weak convergence in Fourier coefficients.

VI. Strong convergence in Fourier coefficients.

We prove cyclically the implications

$$
\begin{array}{ccc}
& \text{II} \rightarrow \text{III} & \\
\nearrow & & \searrow \\
\text{I} & & \text{IV} \\
\nwarrow & & \swarrow \\
& \text{VI} \leftarrow \text{V} &
\end{array}
$$

so that it is seen that all the kinds of convergence are equivalent. In fact, the diagram of implications, given in section 10, is some more complicated, because it is needed to consider every kind of convergence in two different ways: as a convergence to a limit and as a convergence which says nothing about the limit. So we have to consider, in the whole, 12 kinds of convergence.

All proofs in this paper have a classical character, i.e., do not use any tools from topology or functional analysis.

We shall first establish, in section 2, the notation and recollect some more or less known facts about periodic distributions.

2. It is known that if f and g are distributions in $\boldsymbol{R}^q$, one of them being of bounded carrier and one of them (the same or the other one) real-valued, then the convolution $f*g$ does exist (see, e.g., [1]). Using the general notation $y_{(\lambda)}(x) = y(x+\lambda)$, we have for every $\lambda \epsilon \boldsymbol{R}^q$,

$$(f*g)_{(\lambda)} = f_{(\lambda)}*g = f*g_{(\lambda)} \quad \text{for every } \lambda \epsilon \boldsymbol{R}.$$

On the other hand, if f is any distribution and g a smooth function, we have for the ordinary product,

$$(fg)_{(\lambda)} = f_{(\lambda)}g_{(\lambda)}.$$

In the sequel, we shall use the following notation:

$$f*gh = f*(gh) \quad \text{and} \quad fg*h = (fg)*h.$$

This means that if we have to perform the multiplication and the convolution, we first perform the multiplication.

If $x = (\xi_1, \dots, \xi_q) \epsilon \boldsymbol{R}^q$ and $\lambda \epsilon \boldsymbol{R}$, then by

$$x < \lambda, \quad \lambda < x, \quad x \leqslant \lambda, \quad \lambda \leqslant x$$

we understand, respectively,

$$\xi_j < \lambda, \quad \lambda < \xi_j, \quad \xi_j \leqslant \lambda, \quad \lambda \leqslant \xi_j,$$

where $j = 1, \dots, q$. Thus, if $x \epsilon \boldsymbol{R}^q$, the interval $0 \leqslant x < 1$ denotes the set of points x with $0 \leqslant \xi_j < 1$, $j = 1, \dots, q$.

Let Π denote the characteristic function of the interval $0 \leqslant x < 1$. Then the convolution $f*\Pi$ exists for every distribution f in $\boldsymbol{R}^q$. In particular

$$1*\Pi = 1.$$

Let P denote the set of all integral points of $\boldsymbol{R}^q$. Then evidently

$$(1) \qquad \sum_{p \epsilon P} \Pi_{(p)} = 1,$$

where $\Pi_{(p)}(x) = \Pi(x+p)$.

Let l be a real-valued smooth function of bounded carrier such that $1*l = 1$ and let $\varLambda = \Pi*l$. Then $1*\varLambda = 1*\Pi*l = 1*l = 1$, thus

$$1*\varLambda = 1.$$

By (1), it is easy to verify that

$$(2) \qquad \sum_{p \epsilon P} \varLambda_{(p)} = 1.$$

A distribution f is periodic, iff

$$f_{(p)} = f \quad \text{for } p \epsilon P.$$

For periodic distributions f we have

$$f\varLambda_{(p)}*\Pi = (f_{(-p)}\varLambda)_{(p)}*\Pi = (f\varLambda)_{(p)}*\Pi = f\varLambda*\Pi_{(p)}.$$

Hence, by (2) and (1),

$$f*\Pi = \sum_{p \epsilon P}(f\varLambda_{(p)}*\Pi) = f\varLambda*\sum_{p \epsilon P}\Pi_{(p)} = f\varLambda*1,$$

thus

$$(3) \qquad f*\Pi = f\varLambda*1.$$

Given any $\lambda \epsilon \boldsymbol{R}^q$, we have

$$(f\varLambda*1)_{(\lambda)} = f\varLambda*1_{(\lambda)} = f\varLambda*1.$$

Thus we see that the convolution $f*\Pi$, where the distribution f is periodic, is a constant function.

If $E_p(x) = e^{2\pi ipx}$ $(p \epsilon P)$, then the products fE_p are also periodic distributions for f periodic. The elements

$$c_p = fE_{-p}*\Pi$$

are called *Fourier coefficients* of the periodic distribution f.

We shall prove generally that, if g is any distribution and φ is a real-valued smooth function of bounded carrier, then for every order k,

$$(4) \qquad g^{(k)}\varphi*1 = (-1)^k \cdot g\varphi^{(k)}*1,$$

where $(-1)^k = (-1)^{\varkappa_1 + \dots + \varkappa_q}$. In fact, this formula holds, in particular, if g is a smooth function. If g is an arbitrary distribution and g_n is a sequence of smooth functions, convergent to g, then the sequence $g_n^{(k)}\varphi$ converges to $g^{(k)}\varphi$ and the sequence $g_n\varphi^{(k)}$ converges to $g\varphi^{(k)}$. Hence $g_n^{(k)}\varphi*1$ converges to $g^{(k)}\varphi*1$ and $g_n\varphi^{(k)}*1$ converges to $g\varphi^{(k)}*1$ (see e.g., [2], Theorem 5). This implies (4).

Applying formula (4), we can easily estimate the coefficients c_p. In fact, we may write

$$c_p = fE_{-p}\varLambda*1.$$

Let I be the carrier of $\varLambda$. There is an order k and a continuous function F such that $F^{(k)} = f$ in I. Thus

$$c_p = F^{(k)}E_{-p}\varLambda*1 = (-1)^k \cdot F(E_{-p}\varLambda)^{(k)}*1 = (-1)^k \int_I F(E_{-p}\varLambda)^{(k)}.$$

But

$$(E_{-p}\varLambda)^{(k)} = \sum_{0 \leqslant m \leqslant k}\binom{k}{m}E_{-p}^{(m)}\varLambda^{(m-k)}.$$

If $m = (\mu_1, \dots, \mu_q)$ and $p = (a_1, \dots, a_q)$, then generally by a^m we understand the product $a_1^{\mu_1} \dots a_q^{\mu_q}$. If in this product there occurs any-

where 0^0, we read it as 1. Using this general notation, we can easily verify that $E_{-p}^{(m)} = (-2\pi i p)^m E_{-p}$. Hence we obtain

$$(5) \qquad |c_p| \leqslant M \sum_{0 \leqslant m \leqslant k} \binom{k}{m} (2\pi \overline{p})^m = M(1 + 2\pi \overline{p})^k,$$

where $\overline{p} = (|\pi_1|, \dots, |\pi_q|)$ for $p = (\pi_1, \dots, \pi_q)$ and $1 + 2\pi \overline{p}$ denotes the point of $\mathbf{R}^q$ whose jth coordinate is $1 + 2\pi |\pi_j|$, $j = 1, \dots, q$; furthermore, the constant M in (5) is equal to the product of the maximum of $|F|$ on $\overline{I}$, of the maximum of $|\varLambda^{(m)}|$ on $\overline{I}$ for all m $(0 \leqslant m \leqslant k)$, and of the volume of I.

Let φ be a real-valued smooth function of bounded carrier. If f is any distribution, then

$$(6) \qquad (\varphi \dot{*} f) E_p = \varphi E_p \dot{*} f E_p.$$

In fact, this equality holds for smooth functions f. If f is an arbitrary distribution, then there is a sequence of smooth functions f_n convergent to f. Since $(\varphi * f_n) E_p = \varphi E_p \dot{*} f_n E_p$, we obtain (6) on letting $n \to \infty$ (see [1], Theorem 5).

If f is a periodic distribution, then $(\varphi * f)_{(p)} = \varphi * f_{(p)} = \varphi \dot{*} f$, thus $g = \varphi * f$ is a periodic smooth function. Applying (6), we can calculate the Fourier coefficients of g, namely:

$$gE_{-p} \dot{*} \Pi = (\varphi * f) E_{-p} \dot{*} \Pi = (\varphi E_{-p} \dot{*} f E_{-p}) \dot{*} \Pi = \varphi E_{-p} * (f E_{-p} * \Pi)$$
$$= \varphi E_{-p} * c_p = \int \varphi E_{-p} \cdot c_p.$$

Letting $a_p = \int \varphi E_{-p}$, we see that the Fourier coefficients of $\varphi * f$ are $a_p c_p$.

Hence it follows that if all Fourier coefficients c_p of f vanish, then so do the Fourier coefficients of $\varphi * f$. This implies $\varphi * f = 0$, because $\varphi * f$ is a smooth function. If δ_n is a delta sequence of smooth functions, then $\delta_n * f \to f$. Since $\delta_n * f = 0$, this implies $f = 0$. Thus we have proved that if all the Fourier coefficients of a periodic distribution vanish, then $f = 0$.

This implies that periodic distributions are defined uniquely by their Fourier coefficients. In fact, if $f E_p \dot{*} \Pi = g E_p * \Pi$ for $p \in P$, then $(f - g) E_p * \Pi = 0$. Thus the Fourier coefficients of $f - g$ vanish. Hence $f - g = 0$.

Let, as before, c_p be Fourier coefficients of f and let

$$(7) \qquad g = \sum_{p \in P} c_p E_p;$$

this series converges distributionally, which is ensured by inequality (5). Hence

$$gE_{-q} * \Pi = \sum_{p \in P} (c_p E_{p-q} * \Pi)$$

(see e.g., [1], Theorem 5). But $E_{p-q} * \Pi = 0$ except for the case $p = q$, in which we have $E_0 * \Pi = 1 * \Pi = 1$. Thus $gE_{-q} * \Pi = c_p$. This means that the Fourier coefficients of g are c_p. Hence $f = g$, and by (7),

$$(8) \qquad f = \sum_{p \in P} c_p E_p.$$

Thus, we have proved that every periodic distribution equals its Fourier expansion. This is the well known theorem of Schwartz.

We shall prove that *every periodic distribution is a tempered distribution*, i.e., is the derivative of some order of a continuous function of polynomial growth.

We say that a periodic distribution f belongs to class (r), where r is an order (i.e., an integer point satisfying $r \geqslant 0$), iff, for (8),

$$\sum_{p \in P} \frac{|c_p|}{(1 + 2\pi \overline{p})^r} < \infty.$$

By (5), every periodic distribution belongs to some class (r). Evidently, distributions of class (0) are continuous functions, thus they are tempered distributions. Assume that all distributions of some class (r) are tempered and let f be a distribution of class $(r + e_j)$, where j is fixed. Letting (8), the distribution

$$g = \sum_{p \in P} d_p E_p$$

with

$$d_p = \frac{c_p}{(1 + 2\pi \overline{p})^{e_j}} = \frac{c_p}{1 + 2\pi |\pi_j|}$$

belongs to (r) and therefore is tempered. We have

$$f = \sum_{p \in P} (1 + 2\pi |\pi_j|) d_p E_p = g - i h^{(e_j)},$$

where

$$h = \sum_{p \in P} \operatorname{sgn} \pi_j \cdot d_p E_p.$$

Evidently, $h \in (r)$; thus h is tempered, and so is $h^{(e_j)}$. Thus f is tempered. Since j may be chosen arbitrarily from $1, \dots, q$, it follows by induction that distributions of all classes (r) are tempered. Thus, every periodic distribution is tempered.

3. We say that a sequence of distributions f_n in $\mathbf{R}^q$ converges tempered *strongly* to f, iff there are continuous functions F_n, F and ω, the last being positive and of polynomial growth in $\mathbf{R}^q$, such that $F_n^{(k)} = f_n$, $F^{(k)} = f$ for some order k, and the sequence F_n/ω converges to F/ω uniformly in $\mathbf{R}^q$.

We say that a sequence of distributions f_n in $\boldsymbol{R}^q$ converges *strongly* to f, iff, given any open bounded interval $I \subset \boldsymbol{R}^q$, there is an order k and there are continuous functions F_n, F such that $F_n^{(k)} = f_n$ and $F^{(k)} = f$ in I, and F_n converges to F uniformly in I. Such a convergence is called just *convergence*, in [3], or *distributional convergence*.

The following theorem is trivially true:

$\mathrm{I} \to \mathrm{II}$. *If a sequence of distributions converges tempered strongly to f, then it converges strongly to f.*

4. We say that a sequence of distributions f_n in $\boldsymbol{R}^q$ converges *weakly* to f, iff the sequence of integrals $\int f_n \varphi$ converges to $\int f \varphi$ for every real-valued smooth function of bounded carrier φ. For such a function φ, the integral $\int f \varphi$ is a regular operation, it is therefore defined uniquely for every distribution f as the limit of $\int f_n \varphi$, where f_n is a fundamental sequence for f (see [3]).

$\mathrm{II} \to \mathrm{III}$. *If a sequence of distributions converges strongly to f, then it converges weakly to f.*

Proof. If f_n converges strongly to f and the carrier of φ is in a bounded interval I, then there are an order k and continuous functions F_n, F such that $F_n^{(k)} = f_n$ and $F^{(k)} = f$ in I, and F_n converges to F uniformly in I. We have therefore

$$\int f_n \varphi = \int F_n^{(k)} \varphi = (-1)^k \int F_n \varphi^{(k)} \to (-1)^k \int F \varphi^{(k)} = \int F^{(k)} \varphi = \int f \varphi,$$

where $(-1)^k = (-1)^{\varkappa_1 + \dots + \varkappa_q}$ with $k = (\varkappa_1, \dots, \varkappa_q)$. This proves the assertion.

5. The class of all real-valued smooth functions of bounded carrier is denoted by $\mathscr{D}$. The class of all rapidly decreasing real-valued functions is denoted by $\mathscr{S}$.

In the sequel, we shall need the following

THEOREM 1. *Given any function $\psi \in \mathscr{S}$, there is a function $\varphi \in \mathscr{D}$ such that $f * \psi = f * \varphi$ holds for every periodic distribution f.*

Proof. Let $\varLambda \in \mathscr{D}$, $\sum_{p \in P} \varLambda_{(p)} = 1$. We shall show that the function

$$\varphi = \varLambda \sum_{p \in P} \psi_{(p)}$$

has the required properties. Let

$$\varLambda = \sum_{0 \leqslant m \leqslant k} |\varLambda^{(m)}| \quad \text{and} \quad \varPsi = \sum_{0 \leqslant m \leqslant k} |\psi^{(m)}|.$$

Evidently, $\varLambda$ is a continuous function of bounded carrier, and $\varPsi$ is a continuous function whose product with any polynomial is bounded.

We have

$$|(\varLambda_{(-p)} \psi)^{(k)}| \leqslant \sum_{0 \leqslant m \leqslant k} \binom{k}{m} |\varLambda_{(-p)}^{(m)}| \, |\psi^{(k-m)}| \leqslant \varLambda_{(-p)} \varPsi \sum_{0 \leqslant m \leqslant k} \binom{k}{m} = \varLambda_{(-p)} \varPsi \cdot 2^k.$$

It is easily seen that the series $\sum_{p \in P} \varLambda_{(-p)}$ is bounded by a constant function (for the carrier of $\varLambda$ is bounded); thus

$$(9) \qquad \sum_{p \in P} |(\varLambda_{(-p)} \psi)^{(k)}| \leqslant 2^k \cdot M \cdot \varPsi.$$

Now, let F be a continuous function of polynomial growth such that $F^{(k)} = f$. Then

$$f * \psi = f * \sum \varLambda_{(-p)} \psi = F * \sum (\varLambda_{(-p)} \psi)^{(k)},$$

where (and everywhere in the remaining part of this proof) the sum is stretched on all $p \in P$. In the last convolution, inequality (9) allows to interchange the signs $*$ and $\sum$ so that we obtain

$$f * \psi = \sum F * (\varLambda_{(-p)} \psi)^{(k)} = \sum f * \varLambda_{(-p)} \psi = \sum f_{(-p)} * \varLambda \psi_{(p)} = \sum f * \varLambda \psi_{(p)}.$$

But the carriers of $\varLambda \psi_{(p)}$ are included in the carrier of $\varLambda$, thus we may interchange again the signs $\sum$ and $*$ in the last convolution, provided the series $\sum \psi_{(p)}$ is distributionally convergent (see [1], Theorem 5). But this series converges even almost uniformly which follows from the fact that $\psi \in \mathscr{S}$. Similarly, the series $\sum \psi_{(p)}^{(k)}$ converges uniformly for every fixed order k, which implies that $\sum \psi_{(p)}$ is a smooth function. Subsequently, $\varphi = \varLambda \sum \psi_{(p)} \in \mathscr{D}$ and

$$f * \psi = f * \sum \varLambda \psi_{(p)} = f * \varphi.$$

6. If f is any tempered distribution and $\psi \in \mathscr{S}$, then the convolution $g = f * \psi$ is a smooth function. In fact, there is a continuous function F such that $F^{(k)} = f$ and $|F(x)| \leqslant M + |x|^\mu$ for some order k and positive numbers M and μ. Hence, for $h \in \boldsymbol{R}^q$, $|h| < 1$, we obtain

$$|F(x+h)| \leqslant M + (1 + |x|)^\mu$$

and subsequently

$$|F(x+h) - F(x)| \leqslant 2M + |x|^\mu + (1 + |x|)^\mu = G(x),$$

where G is, evidently, a continuous function of polynomial growth. Now,

$$|g(x+h) - g(x)| \leqslant \int |F(x+h-t) - F(x-t)| \cdot |\psi(t)| \, dt \quad \text{for } |h| < 1.$$

If x is fixed and $h \to 0$, then the integrand converges to 0 at every point t and is bounded by the product $G(x-t) |\psi(t)|$ which is an integrable function of t. Hence $g(x+h) - g(x) \to 0$, as $h \to 0$. Since x may be chosen

arbitrarily, g is a continuous function. The same argument can be used for the derivatives $g^{(k)} = f * \psi^{(k)}$ of any order k, thus the convolution $g = f * \psi$, as we said, is a smooth function.

Similarly, the convolution $f * \bar{\psi}$, where $\bar{\psi}(x) = \psi(-x)$, is a smooth function. If in particular, f is a continuous function of polynomial growth, then

$$(10) \qquad \int f \psi = (f * \bar{\psi})(0),$$

i.e., the integral $\int f \psi$ equals the value of the convolution $f * \bar{\psi}$ at the origin. If f is any tempered distribution and $\psi \epsilon \mathscr{S}$, then we shall use equality (10) as the definition of the integral $\int f \psi$. If, in particular, $\psi \epsilon \mathscr{D}$, then this definition coincides with the definition in section 4.

From Theorem 1, it follows the

COROLLARY. *Given any function $\psi \epsilon \mathscr{S}$, there is a function $\varphi \epsilon \mathscr{D}$ such that $\int f \psi = \int f \varphi$ holds for every periodic distribution f.*

We shall say that a sequence of tempered distributions f_n is *tempered weakly convergent to f*, iff the sequence of integrals $\int f_n \psi$ converges to $\int f \psi$ for every fixed $\psi \epsilon \mathscr{S}$.

Comparing the definitions of the weak convergence and of the tempered weak convergence, we immediately obtain, in view of the preceding Corollary,

III → IV. *If a sequence of periodic distributions converges weakly to f, then it converges tempered weakly to f.*

THEOREM 2. *If f is a periodic distribution, $\psi \epsilon \mathscr{S}$ and $g = f * \psi$, then g is a periodic smooth function whose Fourier coefficients are $b_p = c_p a_p$, where c_p are Fourier coefficients of f and $a_p = \int \psi E_{-p}$.*

Proof. As we already proved, g is a smooth function. Since f is periodic, we have $g_{(p)} = f_{(p)} * \psi = f * \psi = g$, thus g is periodic. By Theorem 1, there is a function $\varphi \epsilon \mathscr{D}$ such that $h * \psi = h * \varphi$ for every periodic distribution h. Thus we have

$$b_p = \pi * [(f * \psi) E_{-p}] = \pi * [(f * \varphi) E_{-p}] = \pi * (f E_{-p} * \varphi E_{-p}).$$

In the last expression there are two functions, π and φE_{-p}, which are of bounded carrier, thus the commutativity of the convolution holds and we have

$$b_n = (\pi * f E_{-p}) * \varphi E_{-p} = c_p * \varphi E_{-p} = c_p \int \varphi E_{-p}.$$

Since $\int \varphi E_{-p}$ is the value of $\varphi * E_p$ at the origin and, similarly, $\int \psi E_{-p}$ is the value of $\psi * E_p$ at the origin, we obtain $b_p = c_p a_p$.

7. So far we considered four kinds of convergence. We can denote them by I, II, III and IV, respectively. In each of those kinds, there is indicated a limit to which the sequence converges. However, we can formulate similar definitions of convergence, but without speaking explicitly of the limit of the sequence. The corresponding kinds of convergence will be denoted by I°, II°, III° and IV°, namely:

I° A sequence f_n *converges tempered strongly*, iff there are continuous functions F_n and ω, ω being of polynomial growth, such that $F_n^{(k)} = f_n$ for some fixed order k, and F_n/ω converges uniformly in $\boldsymbol{R}^q$.

II° A sequence f_n *converges strongly*, iff, given any open bounded interval $I \epsilon \boldsymbol{R}^q$, there is an order k and there are continuous functions F_n such that $F_n^{(k)} = f_n$ in I, and F_n converges uniformly in I.

III° A sequence f_n *converges weakly*, iff $\int f_n \varphi$ converges for every $\varphi \epsilon \mathscr{D}$.

IV° A sequence f_n *converges tempered weakly*, iff $\int f_n \psi$ converges for every $\psi \epsilon \mathscr{S}$.

The following implications are trivial: I → I°, II → II°, III → III° and IV → IV°. It is also evident that I° → I and II° → II, i.e., if a sequence converges, then it converges to some limit. We can also state the implication III° → IV° which immediately follows from Corollary in section 6. So we have already proved the following implications:

$$
\begin{array}{cccc}
\text{I}° & \text{II}° & \text{III}° \rightarrow \text{IV}° \\
\updownarrow & \updownarrow & \uparrow \quad\quad \uparrow \\
\text{I} \rightarrow & \text{II} \rightarrow & \text{III} \rightarrow & \text{IV}
\end{array}
$$

8. We shall say that a set (a_p) of real numbers a_p with $p \epsilon P$ is *rapidly decreasing*, iff, for every positive number μ, we have $|p|^\mu |a_p| \to 0$, as $|p| \to \infty$.

Given any $\mu > 0$, we have $|p|^\mu \leqslant (1 + 2\pi\bar{p})^k$, where the vector k has all its coordinates equal to $q^2\mu$. Similarly, given any order k, we have $(1 + 2\pi\bar{p})^k \leqslant |p|^q$ with $\mu = 2q|k|$, provided $|p|$ is large enough ($\geqslant 7$). From these inequalities it follows that (a_p) is rapidly decreasing, iff, for every order k, we have $(1 + 2\pi\bar{p})^k |a_p| \to 0$, as $|p| \to \infty$.

If a_p is rapidly decreasing, then given any order k, we have,

$$(1 + 2\pi\bar{p})^k |a_p| \leqslant (1 + 2\pi\bar{p})^{-2} M$$

for some positive number M. This implies

$$(11) \qquad \sum_{p \epsilon P} (1 + 2\pi\bar{p})^k |a_p| < \infty.$$

Conversely, from the last inequality it follows that

$$(1 + 2\pi\bar{p})^k |a_p| \to 0, \quad \text{as } |p| \to \infty.$$

Thus, a set (a_p) is rapidly decreasing, iff inequality (11) holds for every order k.

If c_p are Fourier coefficients of a periodic function, then from inequality (11) it follows that

$$(12) \qquad \sum_{p \epsilon P} |a_p c_p| < \infty .$$

If a set (c_p) of elements of a Banach space satisfies, for some order k, the inequality $|c_p| \leqslant (1 + 2\pi \overline{p})^k$, then we shall say that (c_p) is *tempered*. Thus, the set of Fourier coefficients of a periodic distribution is tempered.

Inequality (12) holds for any rapidly decreasing set of real numbers (a_p) and any tempered set (c_p) of elements of the given Banach space.

We shall say that a sequence of periodic distributions

$$f_n = \sum_{p \epsilon P} c_{np} E_p$$

converges weakly in Fourier coefficients, iff the sequence

$$b_n = \sum_{p \epsilon P} a_p c_{np}$$

is convergent for every rapidly decreasing set (a_p). We shall say that f_n *converges weakly in Fourier coefficients to the limit*

$$f = \sum_{p \epsilon P} c_p E_p ,$$

if $b_n \to b = \sum_{p \epsilon P} a_p c_p$. These kinds of convergence will be denoted by V° and V respectively. Evidently $V \epsilon V^\circ$. We shall prove the following implication:

$IV^\circ \to V^\circ$. *If a sequence of periodic distributions converges tempered weakly, then it converges weakly in Fourier coefficients.*

Proof. Let (a_p) be any given rapidly decreasing set and let φ be a smooth function such that $\omega(0) = 1$ and $\omega(x) = 0$ for every x not belonging to the interval $-1 < x < 1$. Then

$$\sigma = \sum_{p \epsilon P} a_p \omega_{(-p)} \epsilon \mathscr{S} \qquad \text{and} \qquad \sigma_{(p)} = a_p .$$

Let ψ be the Fourier transform of σ, i.e.,

$$\psi(x) = \int \sigma(t) e^{2\pi i t x} dt .$$

Then $\psi \epsilon \mathscr{S}$ and

$$\sigma(x) = \int \psi(t) e^{-2\pi i t x} dt .$$

Hence

$$\int \psi E_{-p} = \sigma(p) = a_p .$$

Let

$$f_n = \sum_{p \epsilon P} c_{np} E_p$$

be a sequence of periodic distributions which converges tempered weakly. Then, by Theorem 2, we have

$$(f_n * \psi)(x) = \sum_{p \epsilon P} a_p c_{np} E_p(x)$$

and hence, letting $x = 0$,

$$\int f_n \overline{\psi} = \sum_{p \epsilon P} a_p c_{np} ,$$

where $\overline{\psi}(x) = \psi(-x)$. But $\overline{\psi} \epsilon \mathscr{S}$, thus the sequence $\int f_n \overline{\psi}$ is convergent, by assumption that f_n converges tempered weakly. Therefore, the sequence $\sum_{p \epsilon P} a_p c_{np}$ is convergent for $n \to \infty$. Since (a_p) is an arbitrary rapidly decreasing set, this proves that f_n converges weakly in Fourier coefficients.

9. We shall say that a sequence of periodic distributions

$$f_n = \sum_{p \epsilon P} c_{np} E_p$$

converges strongly in Fourier coefficients, iff there is an integer $i \geqslant 0$ such that $n_{np}(1 + |p|)^{-i}$ converges uniformly in $p \epsilon P$, as $n \to \infty$. We say that f_n *converges strongly in Fourier coefficients to*

$$f = \sum_{p \epsilon P} c_p E_p ,$$

iff $c_{np}(1 + |p|)^{-i}$ converges to $c_p(1 + |p|)^{-i}$ uniformly in $p \epsilon P$, as $n \to \infty$. These kinds of convergence will be denoted by VI° and VI, respectively. The implication $VI \to VI^\circ$ is obvious. But also from VI° there follows VI, provided we take

$$f = \sum_{p \epsilon P} c_p E_p \qquad \text{with} \quad c_p = \lim_{n \to \infty} c_{np} .$$

We shall still prove the following implications:

$V^\circ \to VI^\circ$. *If a sequence of periodic distributions converges weakly in Fourier coefficients, then it converges strongly in Fourier coefficients.*

$VI \to V$ *and* $VI \to I$. *If a sequence of periodic distributions converges strongly in Fourier coefficients to f, then it converges to f weakly in Fourier coefficients and converges tempered strongly.*

Proof of V°-VI°. We order all $p \epsilon P$ into a sequence $p_1, p_2, \ldots$ Let

$$t_{ij} = i + |p_j|^i \qquad (i, j = 1, 2, \ldots).$$

The matrix $T = (t_{ij})$ has the following properties: (i) $t_{ij} \geqslant i$, (ii) for every i_1 there is an i_2 with $\sum_j t_{i_1 j} / t_{i_2 j} \to \infty$. We say that a sequence b_n of elements of a Banach space is T-*tempered*, iff there is an index i such that the sequence $t_{i1}^{-1} b_1, t_{i2}^{-1} b_2, \ldots$ is bounded.

If c_p are Fourier coefficients of a periodic distribution, then the sequence $b_j = c_{p_j}$ $(j = 1, 2, \ldots)$ is tempered, which follows from inequality (5).

Assume that a sequence of distributions

$$f_n = \sum_{p \in P} c_{np} E_p$$

converges weakly in Fourier coefficients. Let $B_n = (c_{np_1}, c_{np_2}, \ldots)$. We shall show that the sequence of scalar products

$$RB_n = \sum_{j=1}^{\infty} r_j c_{np_j}$$

is convergent for every sequence of real numbers $R = (r_1, r_2, \ldots)$ such that

(13)
$$\sum_{j=1}^{\infty} t_{ij} |r_j| < \infty.$$

Let $a_{p_j} = r_j$ $(j = 1, 2, \ldots)$. Then

$$RB_n = \sum_{p \in P} a_p c_{np},$$

and it suffices to show that the set (a_p) is rapidly decreasing. In fact, we have for every index $i = 1, 2, \ldots$,

$$\sum_{p \in P} |p|^i |a_p| = \sum_{j=1}^{\infty} |p_j|^i |r_j| \leqslant \sum_{i=1}^{\infty} t_{ij} |r_j| < \infty,$$

which implies that $|p|^i |a_p| \to 0$, as $|p| \to \infty$. Thus the set (a_p) is rapidly decreasing.

Now we apply a known theorem which says that, if for tempered sequences $B_n = (b_{n1}, b_{n2}, \ldots)$ the sequence of scalar products RB_n converges for every $R = (r_1, r_2, \ldots)$ satisfying (13), then there is an index i such that the sequence $A_n = (t_{i1}^{-1} b_{n1}, t_{i2}^{-1} b_2, \ldots)$ converges uniformly, as $n \to \infty$. Thus, in our case, there is an index i such that $t_{ij}^{-1} c_{np_j}$ converges uniformly in j, as $n \to \infty$. I.e., if $t_{ij}^{-1} c_{p_j}$ denotes the limit of $t_{ij}^{-1} c_{np_j}$, there is a sequence of positive numbers ε_n, tending to 0, such that

$$\frac{|c_{np_j} - c_{p_j}|}{i + |p_j|^i} < \varepsilon_n \quad \text{for } j, n = 1, 2, \ldots$$

Hence

$$\frac{|c_{np} - c_p|}{i + |p|^i} < \varepsilon_n \quad \text{for } p \in P, \; n = 1, 2, \ldots,$$

and subsequently

$$\frac{|c_{np} - c_p|}{(1 + |p|)^i} < (i+1)\varepsilon_n \quad \text{for } p \in P, \; n = 1, 2, \ldots$$

This proves that $c_{np}(1 + |p|)^{-i}$ converges uniformly in $p \in P$, as $n \to \infty$.

Proof of VI → V. Let $f_n = \sum c_{np} E_p$ be a sequence of periodic distributions which is VI-convergent to $f = \sum c_p E_p$; the signs of sum $\sum$ are stretched, here and everywhere in the remaining part of this paper, over all $p \in P$. Thus, there is an integer $\mu \geqslant 0$ such that $c_{np}(1 + |p|)^{-\mu}$ converges to $c_p(1 + |p|)^{-\mu}$ uniformly in $p \in P$, as $n \to \infty$. Let (a_p) be any rapidly decreasing set of real numbers. Then

$$\sum |c_{np} - c_p| \, |a_p| = \sum |c_{np} - c_p| (1 + |p|)^{-\mu} \cdot (1 + |p|)^{\mu} a_p$$
$$\leqslant \varepsilon_n \sum (1 + |p|)^{\mu} a_p,$$

where ε_n is a sequence of positive numbers tending to 0. This implies that f_n is V-convergent.

Proof of VI → I. As in the preceding proof, there is a positive integer μ such that $c_{np}(1 + |p|)^{-\mu}$ converges to $c_p(1 + |p|)^{-\mu}$ uniformly in $p \in P$, as $n \to \infty$. If r denotes the vector in $\boldsymbol{R}^q$ whose all the coordinates are $\mu + 2$, then the sequence

(14)
$$\sum \frac{c_{np}}{(1 + 2\pi \overline{p})^r} E_p$$

converges uniformly in $\boldsymbol{R}^q$ to a continuous periodic function, as $n \to \infty$. The scheme of the following proof is like by the end of section 2, where we proved that every periodic distribution is tempered. We say that a sequence of periodic distributions f_n belongs to the class (r), iff the corresponding sequence (14) is uniformly convergent. If a sequence f_n belongs to (0), then, evidently, it is I-convergent. Assume that all sequences of periodic distributions which belong to some class (r) are I-convergent, and let f_n be a sequence which belongs to $(r + e_j)$, where j is fixed arbitrary. If $f_n = \sum c_{np} E_p$, then the sequence $g_n = \sum d_{np} E_p$ with

$$d_{np} = \frac{c_{np}}{1 + 2\pi |\pi_j|}$$

belongs to (r) and, therefore, is I-convergent. We have

$$f_n = \sum (1 + 2\pi |\pi_j|) d_{np} E_p = g_n - i h_n^{(e_j)},$$

where i is the imaginary unit and $h_n = \sum \operatorname{sgn} \pi_j \cdot d_{np} E_p$. Evidently, the sequence h_n belongs to (r), thus is I-convergent and so is its derivative

$h_n^{(e_j)}$. Thus f_n is I-convergent. Since j may be chosen arbitrary, it follows by induction that all sequences belonging to some (r) are I-convergent. Thus, every VI-convergent sequence is I-convergent.

10. The following diagram shows which implications between the considered kinds of convergence have been stated, so far:
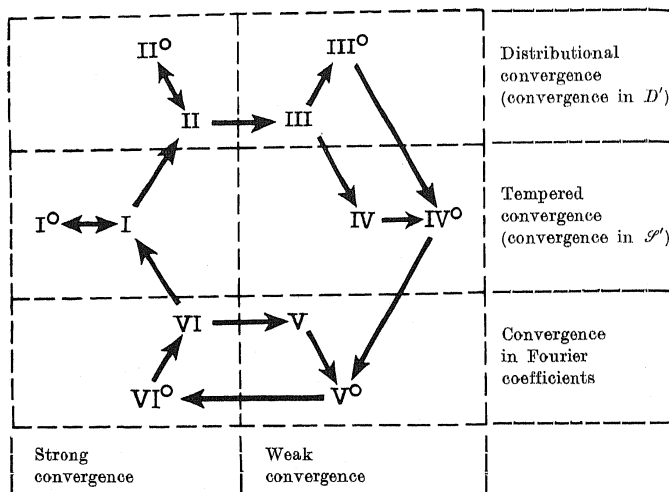
From this diagram we can immediately read that all possible implications hold between the 12 kinds of convergence, i.e., that all the 12 kinds of convergence are, for sequences of periodic distributions, equivalent.

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On symplectic mappings of contraction operators*

by

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Dedicated to
Stanislaw Mazur and Wladyslaw Orlicz

One of the more familiar theorems in function theory states that every conformal mapping of the unit disk onto itself is a fractional linear transformation. In 1943, Siegel [3] proved that this result holds as well for symmetric complex matrices. Our purpose is to generalize this theorem still further and show that it holds both for contraction operators and for symmetric (as distinguished from Hermitian symmetric) contraction operators.

More precisely, let $\mathscr{I}_1$ denote the set of all strictly contractive linear operators on a Hilbert space H,

$$\mathscr{I}_1 = [J; |J| < 1],$$

and let $\mathscr{Z}_1$ denote the set of all strictly contractive symmetric linear operators on H,

$$\mathscr{Z}_1 = [Z; |Z| < 1 \text{ and } Z = Z'],$$

where for a given conjugation $\mathscr{C}$,

$$Z' = \mathscr{C}Z^*\mathscr{C}.$$

We shall consider the group $\mathscr{G}\,[\mathscr{S}]$ of one-to-one bianalytic mappings φ of $\mathscr{I}_1\,[\mathscr{Z}_1]$ onto itself with the metric

$$|\varphi_1 - \varphi_2| = \sup |\varphi_1(J) - \varphi_2(J)| \text{ over } \mathscr{I}_1 \text{ [or } \mathscr{Z}_1].$$

Let $\mathscr{G}_1\,[\mathscr{S}_1]$ denote the principal component of $\mathscr{G}\,[\mathscr{S}]$. It will turn out that $\mathscr{S}_1 = \mathscr{S}$. The analogous assertion does not hold for $\mathscr{G}$ even in the case of matrices; for example $\varphi(J) = J'$ belongs to $\mathscr{G}$ but not to $\mathscr{G}_1$.

The transformation

(1) $$J \to (AJ + B)(CJ + D)^{-1}$$