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DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY

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A note on a Szegö type properties of semi-spectral measures

by

W. MLAK (Kraków)

The present paper deals with some properties of semi-spectral measures of representations of subalgebras of $C(X)$, provided such measures exist. The properties in question were first proved for representations associated with completely non-unitary contractions in [7]. We try in the present paper to bring to light some prediction-free essential points of the reasonings involved in the proofs of similar properties within the frames of the general theory of representations of function algebras. We also present some examples of operator-theoretic interpretation of some simple features concerning function algebras in connection with the above-mentioned properties of semi-spectral measures.

Let H be a complex Hilbert space. The inner product of $f, g \in H$ is denoted by (f, g). $\|f\|$ stands for the norm of f induced by this product. We write $L(H)$ for the algebra of all linear bounded operators in H. $\|T\|$ stands for the norm of $T \in L(H)$, T^* for the adjoint of T. I is the identity operator in H.

Suppose we are given a compact Hausdorff space X. $C(X)$ ($C_R(X)$) is the Banach algebra of all complex (real) continuous functions on X with the norm

$$(*) \qquad \|u\| = \sup_X |u(x)|.$$

We say that $A \subset C(X)$ is the *algebra* (strictly: *subalgebra of* $C(X)$) if the following conditions are satisfied:

(1) A is a closed subspace of $C(X)$ which is closed under multiplication, i.e. $u, v \in A$ implies $uv \in A$.

(2) The function $u_0(x) \equiv 1$ belongs to A.

(3) The functions of A separate the points of X.

The mapping $\varphi : A \to L(H)$ is called the *representation* of A if the following holds true:

(4) φ is a homomorphism of A into $L(H)$ such that $\varphi(u_0) = I$.

(5) $\|\varphi(u)\| \leqslant \|u\|$ for every $u \in A$ ($\|u\|$ is defined by $(*)$).

In what follows the measures and functions are Baire ones. The totality of all Baire subsets of X is denoted by B. The mapping $F\colon B \to L(H)$ is called the *semi-spectral measure* if it satisfies the following conditions:

(6) $F(X) = I$.

(7) For every $f \in H$ the function $\mu_f(\sigma) = (F(\sigma)f, f)$ is a non-negative measure on X.

The semi-spectral measure F is called *spectral* if additionally $F(\sigma_1 \cap \sigma_2) = F(\sigma_1)F(\sigma_2)$ for $\sigma_1, \sigma_2 \in B$. Given the semi-spectral measure F and bounded u we write

$$T = \int u\,dF$$

for $T \in L(H)$ if

$$(**) \qquad (Tf, g) = \int u\,d\mu_{f,g}$$

for all $f, g \in H$: $\mu_{f,g}(\sigma) = (F(\sigma)f, g)$ by definition. $\int u\,dF$ is called the *semi-spectral integral* of u (*spectral* if F is spectral). In any case the right-hand side of $(**)$ determines a bounded bilinear form and there is exactly one $T \in L(H)$ such that $(**)$ holds true. It is not difficult to show that $u \to \int u\,dF$ is a linear map and

$$\left\|\int u\,dF\right\| \leqslant \|u\|_\infty,$$

where $\|u\|_\infty$ stands for the essential supremum of u with respect to F.

It is a classical result that for every φ which satisfies (4) for $A = C(X)$ and which is an involution-preserving mapping that is such that

$$(\overset{*}{**}) \qquad \varphi(\bar{u}) = \varphi(u)^* \quad \text{for } u \in C(X)$$

($\bar{u}$ = complex conjugate of u) condition (5) holds true. The representation φ of general algebra A which satifies $\varphi(\bar{u}) = \varphi(u)^*$ for $u, \bar{u} \in A$ is called a *$*$-representation*. It follows that any homomorphism $\varphi\colon C(X) \to L(H)$ $(\varphi(u_0) = I)$ which satisfies $(\overset{*}{**})$ is a $*$-representation. Moreover, there exists for such φ a unique spectral measure $F\colon B \to L(H)$ such that

$$\varphi(u) = \int u\,dF$$

for every $u \in C(X)$. This is the result of Foiaş (see for instance [1]) which extends the above statement; namely, the following holds true:

(i) *If φ is the representation $\varphi\colon A \to L(H)$ and A is the Dirichlet algebra* (¹), *then thereis a unique semi-spectral measure $F\colon B \to L(H)$ such that*

$$(\overset{**}{**}) \qquad \varphi(u) = \int u\,dF \quad \textit{for } u \in A$$

(¹) A is the *Dirichlet algebra* if the set $\operatorname{Re} A = \{v/v(x) = \operatorname{Re} u(x)$ for some $u \in A\}$ is uniformly dense in $C_R(X)$; see [5].

The above statement holds true if A is a log modular algebra (²) provided φ satisfies some additional conditions—see [2]. The uniqueness of F satisfying $(\overset{**}{**})$ for a given fixed φ is trivial if A is a Dirichlet algebra. It is a result of [2] that the log modularity of A implies that there is at most one F which satisfies $(\overset{**}{**})$ for a given fixed φ. In general, for any A and any φ the uniqueness of F does not necessarily hold true. A trivial example is at hand:

Consider the algebra A of functions u which are analytic in $D = \{z/|z| < 1\}$ and continuous in the closure D, and are such that $u(0) = u(1)$. The restrictions of u to the unit circle $\Gamma = \{z/|z| = 1\}$ form the subalgebra of $C(\Gamma)$. Let m stand for the normalized Lebesgue measure on Γ and m_0 for the unit point mass at $\{1\}$. Simply take H as the one-dimensional space and define the representation $\varphi\colon A \to L(H)$ by

$$\varphi(u)h = u(1)h, \quad h \in H.$$

Now F_1, F_2 are defined by the formulas

$$F_1(\sigma)h = m(\sigma)h, \quad F_2(\sigma)h = m_0(\sigma)h.$$

They are different semi-spectral measures on Γ and are such that

$$\left(\int u\,dF_1\right)h = u(1)h = \varphi(u)h = u(1)h = \left(\int u\,dF_2\right)h$$

for all $u \in A$.

The interplay between theorem (i) and that which concerns $*$-representations of $C(X)$ may be expressed in terms of dilation theory. We introduce the following definition:

The family $\{R_a\} \subset L(K)$ is the *dilation* of $\{T_a\} \subset L(H)$ if the following conditions are satisfied:

(8) H is a Hilbert subspace of K.

(9) $T_a f = PR_a f$ for every a and every $f \in H$.

P stands here (and in all that follows) for the orthogonal projection of K onto H. H is called the *initial space*, K the *dilation* space. $\{R_a\}$ is called a *minimal dilation* if $K = \bigvee R_a H$. For references in dilation theory see [9], [12].

Two dilations $\{R_a\} \subset L(K)$, $\{R'_a\} \subset L(K')$ of $\{T_a\} \subset L(H)$ are called *equivalent* if there is a unitary isomorphism $V\colon K \to K'$ such that $Vf = f$ for $f \in H$ and $R_a = V^{-1}R'_a V$ for each a. Now, one of the basic theorems of the dilation theory is the theorem of M. A. Naĭmark, which reads as follows (see [15]):

(²) A is called *log modular* if the set $\log|A^{-1}| = \{v/v = \log|u^{-1}|$ for some $u, u^{-1} \in A\}$ is uniformly dense in $C_R(X)$; see [5] for the general theory.

(ii) *Let $F: B \to L(H)$ be a semi-spectral measure. Then there is a spectral measure $E: B \to L(K)$ such that $\{E(\sigma)\}_{\sigma \in B}$ is a dilation of $\{F(\sigma)\}_{\sigma \in B}$. The minimality condition $K = \bigvee_{\sigma \in B} E(\sigma)H$ determines E up to equivalence.*

We say that *the representation $\varphi: A \to L(H)$ is X-dilatable to the $*$-representation $\varphi[\cdot]$ of $C(X)$* if $\{\varphi[u]\}_{u \in A}$ is the dilation of $\{\varphi(u)\}_{u \in A}$. $\varphi[\cdot]$ is called the *X-dilation* of $\varphi(\cdot)$.

It follows from (ii) that the representation φ is X-dilatable if and only if there is a semi-spectral measure F such that for $u \in A$ $\varphi(u) = \int u\,dF$. Indeed, to any such F there is a spectral dilation E which induces the $*$-representation of $C(X)$, which, when restricted to A, gives the desired dilation. *Vice versa*, if $\varphi[\cdot]$ is the X-dilation of $\varphi(\cdot)$ and E is the spectral measure corresponding to $\varphi[\cdot]$, then $F = PE_H$ (H — *initial space*) is the suitable semi-spectral measure. If F is uniquely determined by the formulas

$$\varphi(u) = \int u\,dF, \quad u \in A,$$

then, since we deal with regular measures, we have

$$\bigvee_{\sigma \in B} E(\sigma)H = \bigvee_{u \in C(X)} \varphi[u]H,$$

which implies that in this case $\varphi[\cdot]$ is determined uniquely up to equivalence by the minimality condition. This certainly happens if A is log modular and $\varphi(\cdot)$ is its X-dilatable representation, as follows from the uniqueness results of [2].

The preliminary discussion being over, we are able to formulate and prove some results announced at the beginning of the paper. Let us consider the algebra $A \subset C(X)$ and let m be a probability measure on X which is multiplicative on A. Write

$$A_m = \{u / u \in A \text{ and } \int u\,dm = 0\}.$$

We are interested in the application of the Szegö-Krein-Kolmogorov theorem (see [4] for its classical version) in its abstract form, which reads as follows:

(iii) *Let μ be a non-negative Baire measure on X. Then*

$$\inf_{u \in A_m} \int |1-u|^2 d\mu = \exp\left(\int \log \mu'\,dm\right);$$

μ' stands here for the Radon-Nikodym derivative $\dfrac{d\mu}{dm}$.

It follows from the results of [14] that the uniqueness of m implies (iii) and *vice versa*. This is the case where A is log modular as shown in [5]. This applies in particular to the algebra H^∞ considered below. We shall now prove the following theorem:

THEOREM. *Suppose* (iii) *holds true. Let $\varphi: A \to L(H)$ be a representation of A which is X-dilatable to $\varphi[\cdot]$. Suppose E is the spectral measure corresponding to $\varphi[\cdot]$. If $f \in H$, $f \neq 0$ and $\mathrm{Re}(\varphi(u)f, f) \leqslant k\|\varphi[u]f\|\,\|f\|$ for a certain $k \in [0,1)$ and all $u \in A_m$, then*

$$\log \mu_f' \in L^1(m), \quad \textit{where} \quad \mu_f' = \frac{d(E(\cdot)f, f)}{dm}.$$

Proof. We have to show that

$$\varrho = \inf_{u \in A_m} \int |1-u|^2 d\mu_f > 0$$

with $\mu_f(\sigma) = (E(\sigma)f, f)$. One can assume without loss of generality that $\|f\| = 1$. Note now that $\|f - \varphi[u]f\|^2 = \int |1-u|^2 d\mu_f$ for all $u \in C(X)$. Consequently,

$$\int |1-u|^2 d\mu_f = 1 - 2\,\mathrm{Re}(\varphi[u]f, f) + \|\varphi[u]f\|^2.$$

Since $f \in H$ and $\varphi[u]$ is the dilation of $\varphi(u)$, we get $(\varphi(u)f, f) = (\varphi[u]f, f)$. By our assumption $\mathrm{Re}(\varphi(u)f, f) \leqslant k\|\varphi[u]f\|$ for $u \in A_m$. It follows that

$$1 - 2k\|\varphi[u]f\| + \|\varphi[u]f\|^2 \leqslant \|f - \varphi[u]f\|^2$$

for $u \in A_m$. But

$$1 - k^2 = \min_{\xi}(1 - 2k\xi + \xi^2) \leqslant (1 - 2k\|\varphi[u]f\| + \|\varphi[u]f\|^2).$$

Since $0 \leqslant k < 1$, we get

$$0 < 1 - k^2 \leqslant \varrho = \inf_{u \in A_m} \int |1-u|^2 d\mu_f, \qquad \text{q.e.d.}$$

COROLLARY. *Under the assumptions of the theorem, since*

$$\varrho = \exp\left(\int \log \mu_f'\,dm\right),$$

we have

$$\log(1-k^2) \leqslant \int \log \mu_f'\,dm.$$

This gives us the estimation of the interplay between the contraction coefficient k and μ_f'.

We will now present some applications of the above theorem. Suppose we are given a discrete subgroup G of reals naturally ordered by its subsemigroup G_+ of non-negative elements. The contraction-valued function $T_\xi: G_+ \to L(H)$ is called a *semi-group of contractions* if

$$T_0 = I, \quad T_{\xi+\eta} = T_\xi T \quad \text{for} \quad \xi, \eta \in G_+.$$

It is proved in [10] that for every such semi-group T_ξ there is a unique (up to equivalence minimal) unitary dilation U_ξ (3). Write E for the spectral measure of U_ξ defined on $X = \hat{G}$ = the dual of G and $A(G_+)$ = the uniform algebra on X spanned by characters $\xi(x)$ such that $\xi \geqslant 0$ $(x \in \hat{G})$. One extends uniquely the mapping

$$\sum_{\xi_i \geqslant 0} a_i \xi(x) \to \sum_{\xi_i \geqslant 0} a_i T_{\xi_i} \quad (\xi_i \geqslant 0)$$

to the representation φ of $A(G_+)$. The unique $\hat{G}$-dilation of φ is the natural representation of $\hat{C}(G)$ induced by the unitary representation U_ξ, which is the unitary dilation of T_ξ. This representation, we shall call it $\varphi[\cdot]$, is unique up to equivalence. For details, see [8]. Let us take as m the Haar normalized measure on $\hat{G}$. Then A_m = the uniform span of polynomials of the form $\sum_{\xi > 0} a_\xi \xi(x)$.

Assume that for a certain $f \in H$, $\|f\| = 1$, and a certain $\xi_0 \geqslant 0$, $\xi_0 \in G_+$, we have

$$\|T^*_{\xi_0} f\| = \|f\| \quad \text{and} \quad \sup_{\eta > \xi_0} \|T^*_\eta f\| = k < 1. \tag{10}$$

Write

$$u(x) = \sum_{i|1}^{n} a_i \zeta_i(x), \quad \text{where} \quad \zeta_i > 0 \ (x \in \hat{G}).$$

It follows from (10) that $f = U_{\xi_0} T^*_{\xi_0} f$. Now, for our u

$$(\varphi(u)f, f) = \Big(\sum_i a_i U_{\zeta_i + \xi_0} T^*_{\xi_0} f, f\Big).$$

Let us take such an η that $\xi_0 < \eta \leqslant \xi_0 + \zeta_i$ for $i = 1, \dots n$ and put $\varrho_i = (\xi_0 + \zeta_i) - \eta$. Then we have

$$(\varphi(u)f, f) = \Big(\sum a_i U_{\eta + \varrho_i} T^*_{\xi_0} f, f\Big) = \Big(T_\eta \sum a_i T_{\varrho_i} T^*_{\xi_0} f, f\Big)$$
$$= \Big(\sum a_i T_{\varrho_i} T^*_{\xi_0} f,\, T^*_\eta f\Big).$$

Since $\|T^*_\eta f\|$, as a function of η, decreases, we have by (10)

$$|(\varphi(u)f, f)| \leqslant k \Big\| \sum a_i T_{\varrho_i} T^*_{\xi_0} f \Big\|.$$

But

$$\sum a_i T_{\varrho_i} T^*_{\xi_0} f = P \sum a_i U_{\varrho_i} U_{-\xi_0} f = P \sum a_i U_{\zeta_i - \eta} f = P U_{-\eta} \varphi[u],$$

where $\varphi[\cdot]$ is the previously mentioned $\hat{G}$-dilation of $\varphi(\cdot)$. It follows that

$$\Big\| \sum a_i T_{\varrho_i} T^*_{\xi_0} f \Big\| \leqslant \|\varphi[u] f\|,$$

(3) For references in unitary dilations, see [9] and [12]

which by the previous inequalities implies that

$$\operatorname{Re}(\varphi(u)f, f) \leqslant k \|\varphi[u] f\|.$$

Since the polynomials u span A_m, the last inequality holds true for all $u \in A_m$. Applying the above arguments to $\overline{A(G_+)}$ if necessary and using the Theorem we infer that the following property holds true:

(iv) *If there is a* $\xi_0 \in G_+$ *such that* $\|T^*_{\xi_0} f\| = \|f\| = 1$ *and*

$$\sup_{\eta > \xi_0} \|T^*_\eta f\| = k < 1 \quad \textit{or} \quad \|T_{\xi_0} f\| = \|f\| = 1 \quad \textit{and} \quad \sup_{\eta > \xi_0} \|T_\eta f\| = k < 1,$$

then

$$\log \frac{d(E(\cdot)f, f)}{dm} \in L^1(m).$$

The above statement generalizes a series of results proved by the author in [7], [9] and [11].

We will now illustrate the general considerations in the case of representations of the H^∞-algebra induced by a completely non-unitary contraction. This is in fact the case where G_+ is the semigroup of non-negative integers. Then the semi-group of contractions T_ξ reduces to the semi-group of powers of a fixed contraction T. We then write φ_T for the representation induced by the semi-group T^m as generally described before. The algebra $A(G_+)$ becomes here the algebra of functions analytic in D and continuous in $\bar{D}$. Theorem (iv) reduces in this case to Theorem 8.5 of [9]. Assume now that T is completely non-unitary. Then the spectral measure of the minimal unitary dilation U, say E, is concentrated on the unit circle Γ and is mutually absolutely continuous with the normalized Lebesgue measure m_L on Γ (see [12], Theorem 6.4, II). Consider the algebra $L^\infty(m_L)$ of m_L essentially bounded measurable functions on Γ and the algebra H^∞ of bounded functions analytic (in D) with the ess sup norm over Γ. The algebra $L^\infty(m_L)$ is isometrically $*$-isomorphic to its Gelfand image. Let X be the space of maximal ideals of $L^\infty(m_L)$, and identify $L^\infty(m_L)$ with $C(X)$ *via* the Gelfand representation $u \to \hat{u}$. Under this representation, H^∞ is mapped onto a subalgebra $\hat{H}^\infty$ of $C(X)$. $\hat{H}^\infty$ is log modular and its Shilov boundary is exactly X (see [4], [5]). Now, since E and m_L are mutually absolutely continuous, the formula

$$\varphi_T[u] = \int_\Gamma u(z)\, dE(z) \leftrightarrow u \leftrightarrow \hat{u}$$

$(u \in L^\infty(m_L))$ establishes an isometric $*$-isomorphism between the algebra formed by $\varphi_T[u]$ $(u \in L^\infty(m_L))$ and $C(X)$). It follows that there is a unique spectral measure $\hat{E}$ on X such that

$$\varphi_T[u] = \varphi_T[\hat{u}] = \int_X \hat{u}\, d\hat{E} \quad \text{for} \quad u \in L^\infty(m_L).$$

Suppose now that $u \in H^{\infty}$ and write $u_r(z) = u(rz)$ for $r \in [0,1)$. It is known (see [12], III) that the formula $\psi(\hat{u}) = \lim_{r\to 1-} \varphi_T(u_r)$ (strong limit) defines a representation of $\hat{H}^{\infty}$. Since $\varphi(u_r)f = P\varphi_T[u_r]f$ for $u \in H^{\infty}$ and f from the initial space and since $\varphi_T[u_r] \xrightarrow[r\to 1-]{} \varphi_T[u]$ strongly, we have

$$\psi(\hat{u}) = P\int_X \hat{u}\, d\hat{E},$$

which implies that

$$\psi(\hat{u}) = \int_X \hat{u}\, d\hat{F} \quad \text{for} \quad \hat{u} \in \hat{H}^{\infty},$$

where $\hat{F} = P\hat{E}_{|H}$ is a semi-spectral measure on X. It is not difficult to prove that $\hat{F}$ and the representing measure $\hat{m}_L$ for the evaluation functional at the origin are mutually absolutely continuous. Note that T is completely non-unitary if and only if

(11) $$\inf_n (\|T^{*n}f\|, \|T^n f\|) < 1 \text{ for each } f, \quad \|f\| = 1.$$

It follows that for such f

$$\mathrm{Re}\big(\varphi_T(u)f, f\big) \leqslant k_f \|\varphi_T[u]f\|$$

with suitable $k_f < 1$ and for $u \in A(D)$ for which $u(0) = 0$. Since $\varphi_T[\hat{u}]$ and $\psi(\hat{u})$ for $u \in H^{\infty}$ are strong limits of $\varphi[u_r]$ and $\varphi_T(u_r)$ respectively, $u_r \in A(D)$ and $\hat{H}^{\infty}_{\hat{m}_L} = \{\hat{u} / \int \hat{u}\, d\hat{m}_L = 0\}$ consists exactly of those $\hat{u}$ for which $u(0) = 0$, the above inequality yields

$$\mathrm{Re}\big(\psi(\hat{u})f, f\big) \leqslant k_f \|\varphi_T[\hat{u}]f\|$$

for $\hat{u} \in \hat{H}^{\infty}_{\hat{m}_L}$. It follows from the Theorem that $\log \dfrac{d(\hat{E}(\cdot)f, f)}{d\hat{m}_L}$ is summable over X, because (iii) holds true in this case (see [4]).

The last statement, however, can be obtained without making use of the Theorem. We proceed as follows:

Since (11) implies that (iv) holds true and $A(D) = A(G_+)$, $m = m_L$ in this case, we obtain

$$1 - k_f^2 \leqslant \|f - \varphi_T[u]f\|^2$$

for $u \in A_{m_L}(D)$. It follows by the limit passage $\varphi_T[u_r] \to \varphi_T[\hat{u}]$ that

$$1 - k_f^2 \leqslant \|f - \varphi_T[\hat{u}]f\|^2$$

for $\hat{u} \in \hat{H}^{\infty}_{m_L}$. Now we use (iii) by the same argument as before.

Consider now the representation $\varphi: A \to L(H)$. Call the subspace H_0 of H which reduces φ an *X-reducing subspace* if there is a $*$-representation $\psi: C(X) \to L(H)$ such that $\varphi(u)f = \psi(u)f$ for $u \in A$ and $f \in H_0$.

It is proved in [1] that every representation φ is a uniquely determined orthogonal sum of an X-reducing representation φ_0 and a representation φ_1 no part of which is a non-trivial X-reducing representation. φ_1 is called the *X-pure part* of φ. In particular, if $\varphi = \varphi_1$, φ is called *X-pure*.

Given a non-zero positive measure μ on X, we write $H^2(A, \mu)$ for the $L^2(\mu)$-span of A. If $K \subset L^2(\mu)$ and χ_σ is the characteristic function of the measurable set σ, $\chi_\sigma K$ stands for the set $\{f / f = \chi_\sigma g$ for a certain $g \in K\}$. We say that μ is a *Szegö measure* [1] if the inclusion $\chi_\sigma L^2(\mu) \subset H^2(A, \mu)$ implies $\mu(\sigma) = 0$. It is proved in [1] that:

(v) *If ν is an arbitrary non-zero complex measure orthogonal to A, then $|\nu|$ is a Szegö measure.*

(vi) *The elementary representation*

$$\varphi(u)h = uh, \quad u \in A, \; h \in H^2(A, \mu),$$

is X-pure if and only if μ is a Szegö measure.

The representation $\varphi_T: A(D) \to L(H)$ induced by the contraction T is *Γ-pure* if and only if T is completely non-unitary. It follows that in this case the purity of the representation allows us to use the Theorem or condition (iv). The suitable property concerning the summability of logarithm of the Radon-Nikodym derivative of elementary measures induced by semi-spectral measures for φ_T when extended for T c.n.u. to H^{∞} is, as we have seen, a simple strong limit transfer of the corresponding property previously proved for the algebra $A(D)$. This, however, is a very special situation. In general, the purity of the representation is in no general way related to the above-mentioned property of elementary measures created with the help of semi-spectral measures of representations. We will consider some examples.

It is proved in [4] that $\hat{H}^{\infty}$ is not a Dirichlet algebra when considered as a subalgebra of $\hat{L}^{\infty}(m_L)$. Using the previous notation, we infer therefore that there is a real non-zero measure μ such that $\int \hat{u}\, d\mu = 0$ for $\hat{u} \in A = \hat{H}^{\infty}$. It follows from Theorem 1 of [13] (valid for log modular algebras, as follows easily from [3]) that

$$\mu = \mu_s + \sum_i \mu_i,$$

where μ_s is completely singular, $\mu_s \perp A$ and $\mu_i \perp A$: μ_i are absolutely continuous parts of μ with respect to suitable multiplicative probability measures on A. Since μ_i are real, Theorem 6.7 of [5] implies that they are zero measures. It follows that μ is completely singular. Since $\mu \perp A$, $|\mu|$ is by (v) a Szegö measure and the elementary representation

$$\varphi(u)h = \hat{u}h, \quad \hat{u} \in A = \hat{H}^{\infty}, \quad h \in H^2(A, |\mu|),$$

is by (vi) $\mathfrak{X}$-pure. But since $|\mu|$ is completely singular, for every probability measure m multiplicative on A we have

$$\inf_{u \in A_m} \|f - \varphi[\hat{u}]f\|^2 = 0 \quad \text{for each } f \in H^2(A, |\mu|).$$

$\varphi[\hat{u}]$ stands here for the minimal $\mathfrak{X}$-dilation of $\varphi(\cdot)$ (necessarily unique up to equivalence).

The absolute continuity of the spectral measure of the dilation with respect to a multiplicative measure does not help in any way. Consider the additive group G of reals in the discrete topology. It is proved in [6] that there is a non-negative function w which is summable with respect to the Haar measure m_H of $\hat{G}$ and such that $\int \log w \, dm_H = -\infty$, but the stochastic process $\xi \to \xi(x)$ ($\xi \in G$, $x \in \hat{G}$) in $L^2(w\, dm_H)$ has a trivial remote past. It follows that the natural representation

$$\varphi(u)h = uh, \quad u \in A(G_+), \ h \in H^2\big(A(G_+), w\, dm_H\big) = H,$$

is $\hat{G}$-pure. The semi-spectral measure of φ, say F, is defined by the formula

$$\big(F(\sigma)f, g\big) = \int_\sigma fgw\, dm_H, \quad f, g \in H, \ \sigma \text{ a Baire subset of } \hat{G}.$$

Since $\log w$ is not m_H-summable, the same holds true for the Radon-Nikodym derivative of the measure $\big(F(\sigma)f, f\big)$ with respect to m_H for every f. But $F \ll m_H$.

Added in proof: The complete singularity of μ follows immediately from a corollary to Theorem 6.7 of [5]. Indeed, since μ is real and $\mu \perp \operatorname{Re} A$, we have $\mu \perp A + \bar{A}$ which implies that μ is singular with respect to every probability multiplicative measure on A.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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