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Boundary properties of sets relative to function algebras*

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Introduction. Let Ω be a compact Hausdorff space and $\mathfrak{A}$ a given algebra of complex-valued continuous functions on Ω. We always assume that $\mathfrak{A}$ separates the points of Ω and contains the constants. Note that $\mathfrak{A}$ is a normed algebra under the "sup norm" given by

$$|a|_\Omega = \sup\{|a(\omega)| : \omega \in \Omega\}.$$

Let $\varphi : a \to \varphi(a)$ be a homomorphism of $\mathfrak{A}$ onto the complex numbers C. Then φ will be continuous if and only if $|\varphi(a)| \leqslant |a|_\Omega$ for each $a \in \mathfrak{A}$. If every continuous homomorphism φ is of the form $\varphi(a) = a(\omega_\varphi)$, for some point $\omega_\varphi \in \Omega$, then $\mathfrak{A}$ is said to be *natural*. If $\mathfrak{A}$ is a Banach algebra (under any norm), then the condition that it be natural means that Ω is the space of maximal ideals of $\mathfrak{A}$. It is well-known that, for arbitrary $\mathfrak{A}$, the Šilov boundary $\partial_\mathfrak{A}\Omega$ of Ω relative to $\mathfrak{A}$ exists. Recall that $\partial_\mathfrak{A}\Omega$ is a uniquely determined closed set which is minimal among all closed sets $F \subseteq \Omega$ with the property that $|a|_F = |a|_\Omega$ for each $a \in \mathfrak{A}$.

Consider an arbitrary set $X \subseteq \Omega$ and let $\mathfrak{B}$ denote an algebra of bounded functions on X which contains the restriction of $\mathfrak{A}$ to the set X. In Section 1, we define a "Šilov type" boundary for X relative to $\mathfrak{B}$ which reduces to the ordinary Šilov boundary when X is closed and $\mathfrak{B}$ consists of continuous functions. The more general notion is useful in the study of $\mathfrak{A}$-holomorphic functions (see Section 2) defined on an open set. This special situation is considered in Section 2, where the main result concerns the behavior of $\mathfrak{A}$-holomorphic functions outside of an $\mathfrak{A}$-analytic subvariety of Ω, generalizing a result due to Glicksberg [1], Theorem 4.8.

1. A "Šilov boundary" for arbitrary sets. Let X be a given subset of Ω and let $\mathfrak{B}$ denote an arbitrary algebra of bounded functions defined on X and containing the restriction $\mathfrak{A}|X$ of the algebra $\mathfrak{A}$ to the set X. Elements of $\mathfrak{B}$ need not be continuous and X is not assumed to be compact, though its closure $\overline{X} \subseteq \Omega$ is, of course, compact. A set $B \subseteq \overline{X}$ is called

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a $\mathfrak{B}$-*boundary* for X if for every open set $H \supseteq B$ and every $b \in \mathfrak{B}$ it is true that $|b|_{H \cap X} = |b|_X$.

The proof of the following theorem parallels that for the existence of the Šilov boundary for compact sets ([3], 3. 3.1):

1.1. THEOREM. *There exists a unique minimal closed $\mathfrak{B}$-boundary for X.*

Proof. Consider first a family $\mathscr{E}$ of closed $\mathfrak{B}$-boundaries for X which is simply ordered by inclusion (i.e., if $B_1, B_2 \in \mathscr{E}$, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$). Then the intersection B_0 of all sets in $\mathscr{E}$ is also a $\mathfrak{B}$-boundary. This is immediate since the compactness of Ω implies that every open set which contains B_0 must also contain an element of $\mathscr{E}$. Therefore, by application of Zorn's lemma, there exists a minimal closed $\mathfrak{B}$-boundary for X which we denote by Γ. It remains to prove uniqueness.

Let δ be any point of Γ and consider an arbitrary neighborhood N of δ in Ω. Since $\mathfrak{A}$ separates the points of Ω and consists of continuous functions, we may choose a neighborhood of δ of the form

$$N(\varepsilon) = N_\delta(a_1, \ldots, a_n; \varepsilon) = \{\omega : |a_i(\omega) - a_i(\delta)| < \varepsilon \ (i = 1, \ldots, n)\},$$

where $a_1, \ldots, a_n \in \mathfrak{A}$ and $\varepsilon > 0$, such that $N(\varepsilon) \subseteq N$. Also choose ε' such that $0 < \varepsilon' < \varepsilon$ and set $N(\varepsilon') = N_\delta(a_1, \ldots, a_n; \varepsilon')$. Since $\Gamma - N(\varepsilon')$ is a proper closed subset of Γ, it cannot be a $\mathfrak{B}$-boundary for X. Hence there exists an open set $H \supseteq \Gamma - N(\varepsilon')$ and $b \in \mathfrak{B}$ such that

$$|b|_{H \cap X} < |b|_X.$$

Let $\varrho = |b|_{H \cap X} |b|_X^{-1}$. Choose an integer k such that

$$\varrho^k < \Big(1 + \sum_{i=1}^{n} |a_i|_\Omega\Big)^{-1} \varepsilon' = \varepsilon''$$

and set $f = b^k$. Then

$$|f|_{H \cap X} < \varrho^k |f|_X < \varepsilon'' |f|_X.$$

Then, for each i,

$$|a_i f|_{H \cap X} < |a_i|_\Omega \varepsilon'' |f|_X < \varepsilon' |f|_X$$

and also

$$|a_i f|_{N(\varepsilon') \cap X} \leqslant |a_i|_{N(\varepsilon')} |f|_X \leqslant \varepsilon' |f|_X,$$

so we have

$$|a_i f|_{(H \cup N(\varepsilon')) \cap X} \leqslant \varepsilon' |f|_X.$$

Since $\Gamma \subseteq H \cup N(\varepsilon')$ and Γ is a $\mathfrak{B}$-boundary, it follows that

$$|a_i f|_{(H \cup N(\varepsilon')) \cap X} = |a_i f|_X.$$

Therefore

$$|a_i f|_X \leqslant \varepsilon' |f|_X$$

for each i. Next consider any $\omega \in X - N$. Then $\omega \notin N(\varepsilon)$, so $|a_i(\omega)| \geqslant \varepsilon$ for some $i = i_0$, and hence

$$\varepsilon |f(\omega)| \leqslant |a_{i_0}(\omega) f(\omega)| \leqslant \varepsilon' |f|_X.$$

therefore $\varepsilon |f|_{X-N} \leqslant \varepsilon' |f|_X$, and since $\varepsilon' < \varepsilon$, we conclude that

$$|f|_{X-N} < |f|_X.$$

Now let F be an arbitrary closed subset of $\overline{X}$ with $\Gamma \nsubseteq F$. Then there exists $\delta \in \Gamma - F$ and a neighborhood N of δ with $\overline{N} \cap F = \emptyset$. By the above result, there exists $f \in \mathfrak{B}$ such that $|f|_{X-N} < |f|_X$. Consider the open set $H = \Omega - \overline{N}$. Then $F \subset H$. However

$$|f|_{H \cap X} = |f|_{X - \overline{N}} < |f|_X,$$

so F cannot be a $\mathfrak{B}$-boundary for X. In other words, every closed $\mathfrak{B}$-boundary for X must contain Γ, thus proving the uniqueness.

When X is compact and $\mathfrak{B}$ consists of continuous functions, the minimal closed $\mathfrak{B}$-boundary given by the above theorem obviously reduces to the ordinary Šilov boundary. Therefore we denote it also by $\partial_{\mathfrak{B}} X$. As a consequence of the proof, we have the following characterization of points of $\partial_{\mathfrak{B}} X$:

1.2. COROLLARY. *A point δ belongs to $\partial_{\mathfrak{B}} X$ if and only if, for every neighborhood N of δ, there exists $b \in \mathfrak{B}$ with $|b|_{X-N} < |b|_X$.*

If $\mathfrak{A}$ is natural and $\mathfrak{B}$ consists of continuous functions, then $\partial_{\mathfrak{B}} X$ may be identified in another way which relates it to an ordinary Šilov boundary. Observe first that if $\overline{\mathfrak{B}}$ denotes the closure of $\mathfrak{B}$ under uniform convergence on X, then $\partial_{\overline{\mathfrak{B}}} X = \partial_{\mathfrak{B}} X$. Therefore we may assume that $\mathfrak{B}$ is already closed and is accordingly a Banach algebra with norm $|b|_X$. Denote by $\Phi_{\mathfrak{B}}$ the space of maximal ideals of $\mathfrak{B}$ and by $b \to \hat{b}$ the Gelfand representation of $\mathfrak{B}$ as an algebra $\hat{\mathfrak{B}}$ of continuous functions on $\Phi_{\mathfrak{B}}$. Let $\iota : \xi \to \iota(\xi)$ be the natural embedding of X in $\Phi_{\mathfrak{B}}$, where $\hat{b}(\iota(\xi)) = b(\xi)$, $b \in \mathfrak{B}$. Then ι is a homeomorphism ([3], 3.2.1). Denote the image of X in $\Phi_{\mathfrak{B}}$ by Y and let $\overline{Y}$ be the closure of Y in $\Phi_{\mathfrak{B}}$. Next, because $\mathfrak{A} | X \subseteq \mathfrak{B}$, every $\varphi \in \Phi_{\mathfrak{B}}$ defines a homomorphism $a \to \widehat{a|X}(\varphi)$ of $\mathfrak{A}$ onto C. Therefore, since $\mathfrak{A}$ is natural, we obtain a mapping $\pi : \Phi_{\mathfrak{B}} \to \Omega$ such that $\widehat{a|X}(\varphi) = a(\pi(\varphi))$, $a \in \mathfrak{A}$. The mapping π is obviously continuous and, on the set Y, is one-to-one and inverse to the natural embedding ι. In order to simplify notation, we shall write $\hat{a} = \widehat{a|X}$ and denote the Šilov boundary of $\overline{Y}$ relative to $\hat{\mathfrak{B}} | \overline{Y}$ by $\partial_{\mathfrak{B}} \overline{Y}$.

1.3. PROPOSITION. (i) $\pi(\overline{Y}) = \overline{X}$ *and* $\pi(\overline{Y}-Y) = \overline{X}-X$.

(ii) $\pi(\partial_{\mathfrak{B}}\overline{Y}) = \partial_{\mathfrak{B}}X$.

Proof. Consider a point $\psi\in\overline{Y}$ along with its image $\pi(\psi)\in\Omega$ and let $N = N_{\pi(\psi)}(a_1, \ldots, a_n; \varepsilon)$ be an arbitrary basic neighborhood of $\pi(\psi)$ in Ω. Denote by $U = N_\psi(\hat{a}_1, \ldots, \hat{a}_n; \varepsilon)$ the corresponding neighborhood of ψ in $\Phi_{\mathfrak{B}}$. Then $\pi(U\cap Y) = N\cap X$. It immediately follows that $\pi(\overline{Y}) \subseteq \overline{X}$. On the other hand, since $\overline{Y}$ is compact and $\pi(\overline{Y}) \supseteq X$, the continuity of π implies $\pi(\overline{Y}) \supseteq \overline{X}$ so $\pi(\overline{Y}) = \overline{X}$.

Suppose now that $\pi(\psi)\in X$ and let $b\in\mathfrak{B}$. Since b is continuous, we may, for arbitrary $\varepsilon' > 0$, choose the neighborhood N such that $\xi\in N\cap X$ implies

$$\left|b(\xi)-b\big(\pi(\psi)\big)\right| < \varepsilon'.$$

Again, since $\hat{b}$ is continuous on $\Phi_{\mathfrak{B}}$ and $\psi\in\overline{Y}$, there exists $\xi_0\in X$ with $\iota(\xi_0)\in U\cap Y$ such that

$$\left|\hat{b}(\psi)-\hat{b}\big(\iota(\xi_0)\big)\right| < \varepsilon'.$$

Note that $\xi_0\in N\cap X$ so $\left|b(\xi_0)-b\big(\pi(\psi)\big)\right| < \varepsilon'$. Also $b(\xi_0) = \hat{b}\big(\iota(\xi_0)\big)$ and $b\big(\pi(\psi)\big) = \hat{b}\Big(\iota\big(\pi(\psi)\big)\Big)$, so it follows that

$$\left|\hat{b}(\psi)-\hat{b}\Big(\iota\big(\pi(\psi)\big)\Big)\right| < 2\varepsilon'.$$

Since ε' is arbitrary, we have $\hat{b}(\psi) = \hat{b}\Big(\iota\big(\pi(\psi)\big)\Big)$ for every $b\in\mathfrak{B}$. Therefore $\psi = \iota\big(\pi(\psi)\big)\in Y$. This proves that $\pi(\overline{Y}-Y) = \overline{X}-X$.

Finally, assume that $\psi\in\partial_{\mathfrak{B}}\overline{Y}$. Then there exists $b\in\mathfrak{B}$ such that

$$|b|_{X-N} = |\hat{b}|_{Y-U} < |\hat{b}|_{\overline{Y}} = |b|_X.$$

Therefore, by Corollary 1.2, $\pi(\psi)\in\partial_{\mathfrak{B}}X$, so we conclude that $\pi(\partial_{\mathfrak{B}}\overline{Y}) \subseteq \partial_{\mathfrak{B}}X$. On the other hand, if H is any open set that contains $\pi(\partial_{\mathfrak{B}}\overline{Y})$, then, by the continuity of π, there exists an open set G in $\Phi_{\mathfrak{B}}$ containing $\partial_{\mathfrak{B}}\overline{Y}$ such that $\pi(G)\subseteq H$ and hence $\pi(G\cap Y)\subseteq H\cap X$. For arbitrary $b\in\mathfrak{B}$, we have

$$|b|_{H\cap X} \geqslant |\hat{b}|_{G\cap Y} = |\hat{b}|_{G\cap\overline{Y}} = |\hat{b}|_{\partial_{\mathfrak{B}}\overline{Y}} = |\hat{b}|_{\overline{Y}} = |b|_X,$$

so $|b|_{H\cap X} = |b|_X$ and it follows that $\pi(\partial_{\mathfrak{B}}\overline{Y})$ is a $\mathfrak{B}$-boundary for X. Since $\pi(\partial_{\mathfrak{B}}\overline{Y})$ is closed, it must therefore contain $\partial_{\mathfrak{B}}X$ which means that $\pi(\partial_{\mathfrak{B}}\overline{Y}) = \partial_{\mathfrak{B}}X$.

2. $\mathfrak{A}$-holomorphic functions on an open set. Let $\mathscr{F}$ be an arbitrary family of complex-valued functions defined on subsets of Ω and let g be a given function defined on a set $D\subseteq\Omega$. Then g is said to be *locally approximable* by elements of $\mathscr{F}$ if for each $\delta\in D$ there exists a neighborhood N of δ such that, on $N\cap D$, g is a uniform limit of elements of $\mathscr{F}$. If $\mathscr{F}$ contains every function which is locally approximable by its elements, then $\mathscr{F}$ is said to be *locally closed*. We denote by $\mathfrak{H}_{\mathfrak{A}}$ the smallest locally closed family that contains the given algebra $\mathfrak{A}$. Elements of $\mathfrak{H}_{\mathfrak{A}}$ are called *$\mathfrak{A}$-holomorphic functions* ([2], Definition 2.2). It is not difficult to see that $\mathfrak{H}_{\mathfrak{A}}$ does exist and consists of continuous functions. Also, $\mathfrak{H}_{\mathfrak{A}}$ is closed under uniform convergence and under the algebraic operations whenever they are defined. If $\mathfrak{A}$ is natural, then a deeper result is that elements of $\mathfrak{H}_{\mathfrak{A}}$ satisfy an extension of the Rossi local maximum modulus principle for Banach algebras [4]. This may be stated as follows: Let U be an open subset of $\Omega-\partial_{\mathfrak{A}}\Omega$ and let h be a function continuous on $\overline{U}$ and $\mathfrak{A}$-holomorphic on U. Then $|h|_{\mathrm{bd}\,U} = |h|_{\overline{U}}$, where $\mathrm{bd}\,U$ is the topological boundary of U in the space Ω ([2], Lemma 2.5). One can easily verify that it is sufficient here to require that h be only *almost* $\mathfrak{A}$-holomorphic on U. This means that it is continuous on U and $\mathfrak{A}$-holomorphic on that portion of U where it is non-zero.

Consider an open set G in Ω and denote by $\mathfrak{B}_G$ the algebra of all bounded continuous functions on G which are $\mathfrak{A}$-holomorphic on $G-\partial_{\mathfrak{A}}\Omega$. Note that $\mathfrak{A}|G\subseteq\mathfrak{B}_G$. In this case, we call any $\mathfrak{B}_G$-boundary for G an *$\mathfrak{A}$-holomorphic boundary* for G. Also, we shall denote by $\partial_{\mathfrak{A}\text{-hol}}G$ the minimal closed $\mathfrak{A}$-holomorphic boundary for G given by Theorem 1.1. Define

$$\mathrm{bd}_\partial G = (\mathrm{bd}\,G)\cup(G\cap\partial_{\mathfrak{A}}\Omega).$$

Then $\mathrm{bd}_\partial G$ is a closed subset of $\overline{G}$ and it follows from the local maximum modulus principle and Corollary 1.2 that $\partial_{\mathfrak{A}\text{-hol}}G\subseteq\mathrm{bd}_\partial G$. Observe also that the Šilov boundary of $\overline{G}$ relative to functions continuous on $\overline{G}$ and $\mathfrak{A}$-holomorphic on $G-\partial_{\mathfrak{A}}\Omega$ is contained in $\partial_{\mathfrak{A}\text{-hol}}G$.

If $\Theta\subseteq\Delta\subseteq\Omega$, then Θ is called an $\mathfrak{A}$-analytic subvariety of Δ if for each $\delta\in\Delta$ there exists a neighborhood N of δ such that $N\cap\Theta$ consists of the common zeros of functions that are almost $\mathfrak{A}$-holomorphic on $N\cap\Delta$ ([2], Definition 2.10). Note that Θ is automatically a relatively closed subset of Δ. Also, one could clearly replace N by $\overline{N}$ in the definition. The following theorem generalizes a result of Glicksberg ([1], Theorem 4.8) who considered, instead of a general $\mathfrak{A}$-analytic subvariety, the zero set of a single function belonging to $\mathfrak{A}$:

2.1. THEOREM. *Let G be an open set in Ω and let Θ be an $\mathfrak{A}$-analytic subvariety of Ω. Then $\mathrm{bd}_\partial G-\Theta$ is an $\mathfrak{A}$-holomorphic boundary for $G-\Theta$.*

Proof. Observe first that $G-\Theta$ is an open set in Ω and $\mathrm{bd}_\partial(G-\Theta)-\Theta = \mathrm{bd}_\partial G-\Theta$. Therefore we could replace G by $G-\Theta$. In other words, it may be assumed, without loss of generality, that $G\cap\Theta = \emptyset$. Next let $G_0 = G-\partial_{\mathfrak{A}}\Omega$ and suppose that the theorem has been proved for the open set G_0. Consider any open set $H\supseteq\mathrm{bd}_\partial G-\Theta$. Then

$$\mathrm{bd}_\partial G_0-\Theta\subseteq\mathrm{bd}_\partial G-\Theta\subseteq H.$$

Moreover, if $h \in \mathfrak{B}_G$, then $h|G_0 \in \mathfrak{B}_{G_0}$ so, by hypothesis, $|h|_{H \cap G_0} = |h|_{G_0}$. Also, since $H \supseteq G \cap \partial_{\mathfrak{A}} \Omega$, we have

$$|h|_{H \cap G} = \max(|h|_{H \cap G_0}, |h|_{G \cap \partial_{\mathfrak{A}} \Omega}) = \max(|h|_{G_0}, |h|_{G \cap \partial_{\mathfrak{A}} \Omega}) = |h|_G.$$

Therefore, if the theorem is true for G_0, then it is also true for G. Thus, we may also assume, without loss of generality, that $G \subseteq \Omega - \partial_{\mathfrak{A}} \Omega$. Note that in this case $\mathrm{bd}_\partial G = \mathrm{bd}\, G$.

Now suppose that the theorem were false for some open set $G \subseteq \Omega - \partial_{\mathfrak{A}} \Omega$ with $G \cap \Theta = \emptyset$. Then there exists $h \in \mathfrak{B}_G$ and an open set $H \supseteq \mathrm{bd}\, G - \Theta$ such that $|h|_{H \cap G} < \frac{1}{3}$, $|h|_G = 1$.

Let

$$O_n = \left\{\omega : \omega \in G,\ |h(\omega)| > 1 - \frac{1}{n}\right\},$$

and set

$$F = \bigcap_{n=1}^{\infty} \bar{O}_n.$$

Then F is a non-vacuous closed set and

$$F \cap (\mathrm{bd}\, G - \Theta) \subseteq F \cap H = \emptyset,$$

so $F \cap \mathrm{bd}\, G \subseteq \Theta$.

Let δ be a strong boundary point of F with respect to $\overline{\mathfrak{A}|F}$ ([3], 3.3.9) and let N be a neighborhood of δ such that $\bar{N} \cap \Theta$ is the set of common zeros of functions almost $\mathfrak{A}$-holomorphic on $\bar{N}$. Since δ is a strong boundary point, there exists $a \in \mathfrak{A}$ such that

$$|a|_{F-N} < \tfrac{1}{3}, \qquad |a(\delta)| = 1.$$

Define

$$W = \{\omega : \omega \in \Omega,\ |a(\omega)| < \tfrac{1}{3}\}.$$

Then $F \subset N \cup W$. Hence $O_n \subseteq N \cup W$ for large n. Therefore we may choose ϱ with $\frac{1}{3} < \varrho < 1$ such that

$$|h|_{G-(N \cup W)} \leqslant \varrho.$$

Choose m such that

$$\left(\frac{1}{3}\right)^m |a|_\Omega < \varrho^m |a|_\Omega < \frac{1}{3}.$$

Then, for $\omega \in G - (N \cup W)$,

$$|(h^m a)(\omega)| \leqslant |h(\omega)|^m |a|_\Omega \leqslant \varrho^m |a|_\Omega < \tfrac{1}{3}.$$

Since $G - N \subseteq [G - (N \cup W)] \cup [G \cap W]$, it follows that

$$|h^m a|_{G-N} \leqslant \tfrac{1}{3}.$$

Furthermore, since

$$|h|^m_{\bar{H} \cap G} = |h|^m_{H \cap G} \leqslant \left(\frac{1}{3}\right) < \varrho^m,$$

we also have

$$|h^m a|_{\bar{H} \cap G} \leqslant \tfrac{1}{3}.$$

Consider next the open set $U = (N - \bar{H}) \cap G$, and determine its topological boundary. Since $U \cap H = \emptyset$, we have $\bar{U} \cap H = \emptyset$ so

$$\bar{U} \cap (\mathrm{bd}\, G - \Theta) = \emptyset.$$

Since $\bar{U} \subseteq \bar{G}$, this implies that $\bar{U} - G \subseteq \Theta$ and hence

$$(\mathrm{bd}\, U) \cap (\mathrm{bd}\, G) \subseteq \Theta.$$

Also

$$(\mathrm{bd}\, U) \cap G \subseteq (G - N) \cup (\bar{H} \cap G).$$

Therefore we may write

$$\mathrm{bd}\, U = B_\Theta \cup B_G,$$

where $B_\Theta \subseteq \Theta$ and $B_G \subseteq (G - N) \cup (\bar{H} \cap G)$. From the second inclusion and the preceding inequalities, we have

$$|h^m a|_{B_G} \leqslant \tfrac{1}{3}.$$

Since $\delta \in F$ and $|a(\delta)| = 1$, there exists $\omega_0 \in N \cap G$ such that

$$|h(\omega_0)| > \left(\frac{2}{3}\right)^{1/2m}, \qquad |a(\omega_0)| > \left(\frac{2}{3}\right)^{1/2}.$$

Then

$$|(h^m a)(\omega_0)| > \tfrac{2}{3}.$$

Also

$$|h|_{H \cap G} \leqslant \tfrac{1}{3} < |h(\omega_0)|,$$

so $\omega_0 \notin \bar{H}$ and therefore $\omega_0 \in U$. Furthermore, since $\omega_0 \in \bar{N} - \Theta$, there exists g, almost $\mathfrak{A}$-holomorphic on $\bar{N}$, such that $g(\omega_0) \neq 0$ while $g(\omega) = 0$ for $\omega \in \bar{N} \cap \Theta$.

Finally choose k such that $(\frac{1}{2})^k |g|_{\bar{N}} < |g(\omega_0)|$ and define

$$f(\omega) = \begin{cases} ((h^m a)^k g)(\omega), & \omega \in \bar{U} \cap G, \\ 0, & \omega \in \bar{U} - G. \end{cases}$$

Since $\bar{U} - G \subseteq \Theta$ and h is bounded while g is continuous, it follows that f is almost $\mathfrak{A}$-holomorphic on $\bar{U}$. Moreover

$$|f(\omega_0)| = |(h^m a)(\omega_0)|^k |g(\omega_0)| > \left(\frac{2}{3}\right)^k |g(\omega_0)|.$$

Since $f(\omega)=0$ for $\omega\in B_\Theta$, we have $|f|_{\mathrm{bd}U}=|f|_{B_G}$. Also, since $|h^m a|_{B_G}\leqslant\frac{1}{3}$, we have

$$|f|_{B_G}\leqslant|h^m a|_{B_G}^k|g|_{B_G}\leqslant\left(\frac{1}{3}\right)^k|g|_{\bar{N}}=\left(\frac{2}{3}\right)^k\left(\frac{1}{2}\right)^k|g|_{\bar{N}}<\left(\frac{2}{3}\right)^k|g(\omega_0)|.$$

Therefore

$$|f|_{\mathrm{bd}U}<\left(\frac{2}{3}\right)^k|g(\omega_0)|<|f(\omega_0)|.$$

This contradicts the local maximum modulus principle for almost $\mathfrak{A}$-holomorphic functions and completes the proof.

The method of proof used in the above theorem enables us to obtain another related boundary property of G. First we define a point $\delta\in\mathrm{bd}G$ to be an $\mathfrak{A}$-*analytic boundary point* if there exists a neighborhood N of δ such that the set $N\cap\mathrm{bd}G$ consists of the common zeros of functions almost $\mathfrak{A}$-holomorphic on $N\cap\bar{G}$. Denote the set of all $\mathfrak{A}$-analytic boundary points of G by $(\mathrm{bd}G)_0$. It is obvious that $(\mathrm{bd}G)_0$ is an open set relative to $\mathrm{bd}G$. Also, if H is an open set in Ω such that $H\cap(\mathrm{bd}G)=(\mathrm{bd}G)_0$, then $(\mathrm{bd}G)_0$ is an $\mathfrak{A}$-analytic subvariety of the set $H\cap\bar{G}$.

2.2. THEOREM. *Let G be an open subset of $\Omega-\partial_{\mathfrak{A}}\Omega$. Then $\partial_{\mathfrak{A}-\mathrm{hol}}G\subseteq(\mathrm{bd}G)-(\mathrm{bd}G)_0$.*

Proof. Since $G\subseteq\Omega-\partial_{\mathfrak{A}}\Omega$, it follows that $\partial_{\mathfrak{A}\text{-hol}}G\subseteq\mathrm{bd}G$. Also, $(\mathrm{bd}G)-(\mathrm{bd}G)_0$ is a closed set. Therefore, if the theorem were false, there would exist a point $\delta\in\partial_{\mathfrak{A}\text{-hol}}G\cap(\mathrm{bd}G)_0$. Choose a neighborhood N of δ such that $N\cap\mathrm{bd}G$ is the set of common zeros of functions almost $\mathfrak{A}$-holomorphic on $N\cap\bar{G}$. Then choose a neighborhood V of δ such that $\bar{V}\subset N$. By Corollary 1.2, there exists $h\in\mathfrak{B}_G$ such that

$$|h|_{G-V}<\frac{1}{3},\qquad |h|_G=1.$$

Set $U=V\cap G$ and choose $\omega_0\in U$ such that $|h(\omega_0)|>\frac{2}{3}$. Since $\omega_0\in(N\cap\bar{G})-(N\cap\mathrm{bd}G)$, there exists g, almost $\mathfrak{A}$-holomorphic on $N\cap\bar{G}$, such that $g(\omega_0)\neq0$ while $g(\omega)=0$ for $\omega\in N\cap\mathrm{bd}G$. Next choose k such that

$$\left(\frac{1}{2}\right)^k|g|_{\bar{U}}<|g(\omega_0)|$$

and define

$$f(\omega)=\begin{cases}(h^kg)(\omega), & \omega\in\bar{U}\cap G,\\ 0, & \omega\in\bar{U}-G.\end{cases}$$

Note that $\bar{U}\subset N\cap\bar{G}$ and $\bar{U}-G\subseteq N\cap\mathrm{bd}G$ so it follows that f is almost $\mathfrak{A}$-holomorphic on $\bar{U}$ and is zero on $(\mathrm{bd}\,U)\cap(\mathrm{bd}G)$. Moreover,

$$|f(\omega_0)|>\left(\frac{2}{3}\right)^k|g(\omega_0)|$$

and

$$|f|_{\mathrm{bd}U}=|f|_{(\mathrm{bd}U)\cap G}\leqslant|h|_{(\mathrm{bd}U)\cap G}^k|g|_{\bar{U}}\leqslant|h|_{G-V}^k|g|_{\bar{U}}<\left(\frac{1}{3}\right)^k|g|_{\bar{U}}$$
$$=\left(\frac{2}{3}\right)^k\left(\frac{1}{2}\right)^k|g|_{\bar{U}}<\left(\frac{2}{3}\right)^k|g(\omega_0)|.$$

Thus $|f|_{\mathrm{bd}U}<|f(\omega_0)|$ and we have a contradiction of the local maximum modulus principle.

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