STUDIA MATHEMATICA, T. XXXI. (1968)

Inverse limits of compact spaces and direct limits of spaces of continuous functions

by

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If X is a topological space, then $\mathscr{C}(X)$ denotes the space of bounded scalar-valued continuous functions on X (the scalar field is either $\boldsymbol{R}$ or $\boldsymbol{C}$); if X is empty, then $\mathscr{C}(X)$ consists of the single element 0. We recall that for any X there is exactly one function from $\emptyset$ into X, the empty function (the empty subset of $\emptyset \times X$), which is continuous. If $X \neq \emptyset$, then there is no map from X into $\emptyset$. If $\varphi\colon X \to Y$ is a continuous map, then

$$\mathscr{C}(\varphi) : \mathscr{C}(Y) \to \mathscr{C}(X)$$

denotes the induced linear operator defined as $\mathscr{C}(\varphi)g = g \circ \varphi$ for g in $\mathscr{C}(Y)$. **Top** denotes the category of topological spaces and continuous maps and **Comp** denotes the full subcategory of compact (= compact Hausdorff) spaces (the empty space being included). $\mathbf{Ban}_1$ denotes the category of Banach spaces and linear contractions (i.e., linear operators of norm $\leqslant 1$). It is clear that

$$\mathscr{C}\colon \mathbf{Comp} \to \mathbf{Ban}_1 \tag{1}$$

is a contravariant functor. The purpose of this paper [1] is to prove that the functor (1) and some related functors are *inversely continuous*, i.e., that they transform the inverse limits to direct limits.

Throughout this paper T denotes an upward filtering preordered set (i.e., it is assumed that (i) $t \leqslant t$, (ii) $t \leqslant s$ and $s \leqslant r$ imply $t \leqslant r$, (iii) $\forall s, t \in T \ \exists r \in T$ such that $r \geqslant s$ and $r \geqslant t$).

Let $\mathfrak{A}$ be a category. We recall (see, e.g., [2], p. 215, and [3], p. 48) that an *inverse system* (in $\mathfrak{A}$) is a family $(A_t)_{t\in T}$ of objects together with a family $(a_t^s)_{t\leqslant s}$ of morphisms $a_t^s : A_s \to A_t$ such that (i) a_t^t is the identity ι_{A_t} and (ii) if $t \leqslant s \leqslant r$, then $a_t^r = a_t^s a_s^r$. An *inverse* (or *projective*) *limit* of this system (in $\mathfrak{A}$) is an object A_∞ together with a family of morphisms $a_t : A_\infty \to A_t$ $(t \in T)$ satisfying the following conditions: (i) $a_t^s a_s = a_t$ for

(1) Theorem 1 and the Remarks were presented at the Symposium on Extension of Topological Structures in Berlin, 14-19 August 1967 (cf. [6]). Unexplained terminology concerning categories and functors can be found in [3] and [7].

$t \leqslant s$, (ii) for any object B and any family of morphisms $\beta_t : B \to A_t$ such that $\alpha_t^s \beta_s = \beta_t$ for $t \leqslant s$, there exists a unique morphism $\beta : B \to A_\infty$ such that $\alpha_t \beta = \beta_t$ for t in T.

A *direct* (or *inductive*) limit of a direct family $\alpha_t^s : A^t \to A^s$ $(t \leqslant s)$ in $\mathfrak{A}$ is defined dually; it is an object A^∞ together with a family of morphisms $\alpha_t : A^t \to A^\infty$ $(t \in T)$ satisfying the following conditions: (i) $\alpha_s \alpha_t^s = \alpha_t$ if $t \leqslant s$, (ii) for any object B and any family of morphisms $\beta_t : A_t \to B$ such that $\beta_s \alpha_t^s = \beta_t$ if $t \leqslant s$, there exists a unique morphism $\beta : A^\infty \to B$ such that $\beta \alpha_t = \beta_t$ for t in T.

It is well known that if an inverse [direct] limit exists, it is unique up to unique commuting isomorphism. If the morphisms α_t are taken for granted, we shall also say that A_∞ $[A^\infty]$ itself is an inverse [direct] limit of the family. A_∞ will also be denoted by $\lim^{\leftarrow} A_t$.

In the categories **Top** and **Comp**, an inverse limit of an inverse system $\big((X_t)_{t \in T}, (\varphi_t^s)_{t \leqslant s}\big)$ may be constructed as follows. Let a *thread* mean a family $x = (x_t)_{t \in T}$ in the Cartesian product $X = \mathop{\mathsf{P}}\limits_{t \in T} X_t$ such that $\varphi_t^s(x_s) = x_t$ whenever $t \leqslant s$. Let X_∞ be the set of all threads (with the topology induced by X) and let $\varphi_t(x) = x_t$; thus, φ_t is the restriction (to X_∞) of the projection from X onto the t-th axis X_t. It is clear that $\big(X_\infty, (\varphi_t)_{t \in T}\big)$ satisfies the above conditions; this pair (or simply X_∞) will be called *the* inverse limit of the system. We should point out that the notion of a direct limit in $\mathbf{Ban}_1$ does not coincide, in general, with that of a direct limit (of the same system) in the category of all locally convex topological vector spaces and continuous linear operators.

The inverse continuity of the functor (1) may be formulated as

THEOREM 1. *Let X_∞ be the inverse limit of an inverse family $\varphi_t^s : X_s \to X_t$ $(t, s \in T,\ t \leqslant s)$ of compact spaces and continuous maps. Then $\mathscr{C}(X_\infty)$ is a direct limit (in the category $\mathbf{Ban}_1$) of the direct system of linear operators $\mathscr{C}(\varphi_t^s) : \mathscr{C}(X_t) \to \mathscr{C}(X_s)$.*

In the special case where all maps φ_t^s are surjections, this theorem was recently proved by Pełczyński [4], p. 14.

Now, let $\mathbf{Comp}_{\mathscr{F},\subset}$ denote the category defined as follows: an object is a pair (X, A) where X is compact and A is a closed subset of X; a morphism from (X, A) to (X', A') is a continuous map $\varphi : X \to X'$ such that $\varphi(A) \subset A'$ (cf. [2], p. 3). If (X, A) is an object of $\mathbf{Comp}_{\mathscr{F},\subset}$, let

$$\mathscr{C}_0(X \| A) = \{f \in \mathscr{C}(X) : x \in A \Rightarrow f(x) = 0\}.$$

Any morphism $\varphi : (X, A) \to (X', A')$ in $\mathbf{Comp}_{\mathscr{F},\subset}$ determines a linear contraction

$$\mathscr{C}_0(\varphi) : \mathscr{C}_0(X' \| A') \to \mathscr{C}_0(X \| A)$$

which is the restriction of $\mathscr{C}(\varphi) : \mathscr{C}(X') \to \mathscr{C}(X)$. Thus, $\mathscr{C}_0(\varphi) g = g \circ \varphi$ for g in $\mathscr{C}(X' \| A')$; since $\varphi(A) \subset A'$ and g vanishes on A', the composite $g \circ \varphi$ vanishes on A. It is clear that

$$\mathscr{C}_0 : \mathbf{Comp}_{\mathscr{F},\subset} \to \mathbf{Ban}_1$$

is a contravariant functor. If

$$\varphi_t^s : (X_s, A_s) \to (X_t, A_t) \qquad (t \leqslant s) \tag{2}$$

is an inverse system in $\mathbf{Comp}_{\mathscr{F},\subset}$, then

$$\varphi_t^s : X_s \to X_t \quad \text{and} \quad \operatorname{rest}_{A_s} \varphi_t^s : A_s \to A_t,$$

where $\operatorname{rest}_A \varphi$ denotes the restriction of φ to the set A, are inverse systems in **Comp**. The pair (X_∞, A_∞), where X_∞ is the set of all threads with $x_t \in X_t$ and A_∞ is the set of all threads with $a_t \in A_t$, is an inverse limit in $\mathbf{Comp}_{\mathscr{F},\subset}$; we will refer to it as *the* inverse limit in $\mathbf{Comp}_{\mathscr{F},\subset}$.

The category **Comp** may be regarded as a full subcategory of $\mathbf{Comp}_{\mathscr{F},\subset}$ (identify X with $(X, \emptyset)$); moreover, $\mathscr{C}_0(X \| \emptyset) = \mathscr{C}(X)$. Thus, Theorem 1 is a special case of the following theorem:

THEOREM 2. *If* (2) *is an inverse system in $\mathbf{Comp}_{\mathscr{F},\subset}$, then $\mathscr{C}(X_\infty \| A_\infty)$ is a direct limit (in the category $\mathbf{Ban}_1$) of the direct system*

$$\mathscr{C}_0(\varphi_t^s) : \mathscr{C}_0(X_t \| A_t) \to \mathscr{C}_0(X_s \| A_s). \tag{3}$$

The proof is based on the following lemmas:

LEMMA 1 (see [2], p. 217, and [7], 11.8.5). *Let $\varphi_t^s : X_s \to X_t$ be an inverse system in* **Comp** *and let all X_t be non-empty. Then X_∞ is non-empty and*

$$\varphi_t(X_\infty) = \bigcap_{s \geqslant t} \varphi_t^s(X_s) \quad \text{for } t \text{ in } T.$$

LEMMA 2 (see [8] and [7], 11.8.3). *Suppose that for each t in T we are given a linear subset F^t of a vector space F and a norm $\| \ \|_t$ on F^t satisfying the following conditions: if $t \leqslant s$, then $F^t \subset F^s$ and $\|f\|_t \geqslant \|f\|_s$ for f in F^t. Write*

$$G = \bigcup F^t, \quad \|f\| = \lim \|f\|_t \quad \text{for } f \text{ in } G,$$

and $N = \{f \in G : \|f\| = 0\}$. Then G/N is a direct limit of the system $(F^t)_{t \in T}$ (with the embeddings $\varepsilon_t^s : F^t \to F^s$) in the category of normed vector spaces and linear contractions. If all $(F^t, \| \ \|_t)$ are complete, then the completion of G/N is a direct limit of this system in $\mathbf{Ban}_1$.

If Z is a locally compact space, $\mathscr{M}(Z)$ will denote the set of all finite Radon (= regular Borel scalar-valued) measures on Z.

LEMMA 3 (see [1] and [7], 18.8.1). *Let A and B be closed subsets of compact spaces X and Y, respectively. Then any bounded linear operator*

$$\Gamma : \mathscr{C}(Y\|B) \to \mathscr{C}(X\|A)$$

is of the form

$$\Gamma g(x) = \int_{Y\setminus B} g(y)\,d\mu_x(y) \quad \text{for } g \in \mathscr{C}(Y\|B),\ x\in X,$$

*where $\mu : X \to \mathscr{M}(Y\setminus B)$ is a *weakly continuous map, $\mu_x = 0$ for x in A, and $\|\Gamma\| = \sup_{x\in X}\|\mu_x\|$.*

Proof of Theorem 2. Let us write $Y_t = \varphi_t(X_\infty)$, $B_t = \varphi_t(A_\infty)$,

$$\psi_t^s = \operatorname{rest}_{Y_s}\varphi_t^s, \qquad a_t^s = \operatorname{rest}_{A_s}\varphi_t^s, \qquad \beta_t^s = \operatorname{rest}_{B_s}\varphi_t^s \qquad (t \leqslant s).$$

Since $\varphi_t^s\varphi_s = \varphi_t$, the maps $\psi_t^s : Y_s \to Y_t$ and $\beta_t^s : B_s \to B_t$ are surjections. The map $\varphi_t : X_\infty \to X_t$ is factored as $\varphi_t = \varepsilon_t\pi_t$, where $\pi_t : X_\infty \to Y_t$ is a surjection and $\varepsilon_t : Y_t \to X_t$ is the embedding. The resulting morphisms are shown in Fig. 1, where $\twoheadrightarrow$ stands for surjection and $\rightarrowtail$ stands for injection.

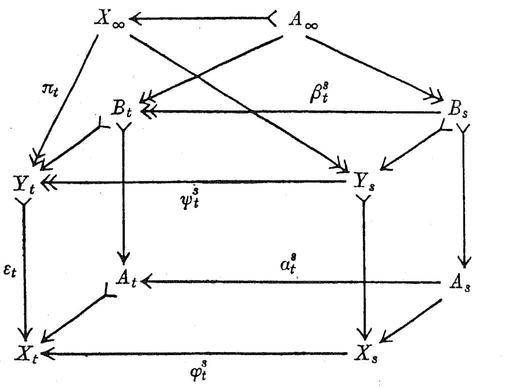

Fig. 1

The proof consists of two main steps.

(a) $\mathscr{C}(X_\infty\|A_\infty)$ is a direct limit (in $\mathbf{Ban}_1$) of the system

(4) $$\mathscr{C}_0(\psi_t^s) : \mathscr{C}_0(Y_t\|B_t) \to \mathscr{C}_0(Y_s\|B_s), \qquad t \leqslant s.$$

(b) The systems (3) and (4) have the same direct limits (see Fig. 2).

$$\begin{tikzcd}
 & \mathscr{C}_0(X_\infty\|A_\infty) & \\
\mathscr{C}_0(Y_t\|B_t) \arrow[ur, "\mathscr{C}_0(\pi_t)"] \arrow[rr, tail, "\mathscr{C}_0(\psi_t^s)"'] & & \mathscr{C}_0(Y_s\|B_s) \arrow[ul, "\mathscr{C}_0(\pi_s)"'] \\
\mathscr{C}_0(X_t\|A_t) \arrow[u, two heads, "\mathscr{C}_0(\varepsilon_t)"] \arrow[rr, "\mathscr{C}_0(\varphi_t^s)"'] & & \mathscr{C}_0(X_s\|A_s) \arrow[u, two heads, "\mathscr{C}_0(\varepsilon_s)"']
\end{tikzcd}$$

Fig. 2

(a). It will be shown that (X_∞, A_∞) coincides with the inverse limit (Y_∞, B_∞) of the system $\psi_t^s : (Y_s, B_s) \to (Y_t, B_t)$, $t \leqslant s$ (in $\mathrm{Comp}_{\mathscr{F},\subset}$). If at least one space X_t is empty, then each X_s ($s \geqslant t$) is empty and so is X_∞. Therefore we may assume that $X_t \neq \emptyset$ for all t in T. Let $x = (x_t)_{t\in T}$ be a thread in X_∞. If t is fixed, then $x_t = \varphi_t^s(x_s) \in \varphi_t^s(X_s)$ for $s \geqslant t$. Consequently, by Lemma 1,

$$x_t \in \bigcap_{s\geqslant t}\varphi_t^s(X_s) = \varphi_t(X_\infty) = Y_t.$$

Thus, $x \in Y_\infty$. We have shown that $X_\infty = Y_\infty$; similarly, any thread in A_∞ belongs to B_∞, i.e., $A_\infty = B_\infty$. Moreover, $\psi_t^s\pi_s = \pi_t$ for $s \geqslant t$.

Since $\pi_t : X_\infty \to Y_t$ is a surjection, $\mathscr{C}_0(\pi_t)$ is a linear isometrical injection; let H_t denote the range of $\mathscr{C}_0(\pi_t)$. Thus, H_t may be described as

$$\{g\circ\pi_t : g\in\mathscr{C}_0(Y_t\|B_t)\} = \{g\circ\pi_t : g\in\mathscr{C}(Y_t)\} \cap \mathscr{C}_0(X_\infty\|A_\infty);$$

in other words, H_t consists of all functions in $\mathscr{C}(X_\infty)$ which vanish on A_∞ and are constant on each inverse set $\pi_t^{\leftarrow}(y)$, $y \in Y_t$. If $s \geqslant t$, then the partition of X_∞ determined by π_s is a refinement of that determined by π_t; hence H_t is a subspace of H_s. Write $H = \bigcup H_t$. Each H_t and H are self-adjoint subalgebras of $\mathscr{C}_0(X_\infty\|A_\infty)$. If $x \in X_\infty\setminus A_\infty$, then there exists a t in T such that $x_t \notin A_t$ (hence $x_s \notin A_s$ for $s \geqslant t$). Consequently, there exists a g in $\mathscr{C}_0(X_t\|A_t)$ such that $g(x_t) \neq 0$; letting $f = \mathscr{C}(\pi_t)g$ we get $f \in H$ and $f(x) \neq 0$. It is also clear that H separates the points of $X_\infty\setminus A_\infty$. Consequently, by the Weierstrass-Stone theorem, H is dense in $\mathscr{C}(X_\infty\|A_\infty)$; by Lemma 2, $\mathscr{C}(X_\infty, A_\infty)$ is a direct limit of the family $(H_t)_{t\in T}$ (in $\mathbf{Ban}_1$). Since the square

$$\begin{tikzcd}
\mathscr{C}_0(Y_t\|B_t) \arrow[r, tail, "\mathscr{C}_0(\psi_t^s)"] \arrow[d, two heads, "\mathscr{C}_0(\pi_t)"'] & \mathscr{C}_0(Y_s\|B_s) \arrow[d, two heads, "\mathscr{C}_0(\pi_s)"] \\
H_t \arrow[r, tail] & H_s
\end{tikzcd}$$

Fig. 3

$(t \leqslant s)$ is commutative, $\mathscr{C}(X_\infty\|A_\infty)$ is a direct limit of the system (4) as well.

(b). Let F be any Banach space and let

$$\Gamma_t : \mathscr{C}_0(X_t\|A_t) \to F$$

be linear contractions such that $\Gamma_s\mathscr{C}_0(\varphi_t^s) = \Gamma_t$ for $s \geqslant t$. We have to find a linear contraction $\Gamma : \mathscr{C}_0(X_\infty\|A_\infty) \to F$ such that $\Gamma\mathscr{C}_0(\varphi_t) = \Gamma_t$ for t in T. Without loss of generality we may assume that F is a subspace of a space $\mathscr{C}(Z)$, where Z is compact. Write $U_t = X_t \setminus A_t$. By Lemma 3, for each t in T there is a *weakly continuous map $\mu^t : Z \to \mathscr{M}(X_t)$ such that $\|\mu_z^t\| \leqslant 1$, μ_z^t is concentrated on U_t (i.e., the variation of μ_z^t on A_t equals 0), and

$$\Gamma_t f(z) = \int_{U_t} f d\mu_z^t \quad \text{for } f \text{ in } \mathscr{C}(X_t\|A_t),\ z \text{ in } Z.$$

Let t and z be fixed. If $s \geqslant t$, then the condition $\Gamma_s\mathscr{C}_0(\varphi_t^s) = \Gamma_t$ may be written in the form

$$\int_{U_t} f d\mu_z^t = \int_{U_s} f\big(\varphi_t^s(u)\big)\, d\mu_z^s(u).$$

In particular, if $f \in \mathscr{C}_0(X_t\|A_t)$ and f vanishes on $\varphi_t^s(X_s)$, then the right-hand integral vanishes and so does the left-hand one. Since f is arbitrary, $\mu_z^t\big(U_t \setminus \varphi_t^s(X_s)\big)$ vanishes as well. In other words, for each $s \geqslant t$, μ_z^t is concentrated on $\varphi_t^s(X_s)$. Let W be any open subset of X_t containing Y_t. By Lemma 1, Y_t is the intersection of the downward filtering family of compact sets $\varphi_t^s(X_s)$, $s \geqslant t$. Hence, there exists an r in T such that $r \geqslant t$ and $\varphi_t^r(X_r) \subset W$. Since μ_z^t is regular, it must be concentrated on Y_t.

Now, if $g \in \mathscr{C}_0(Y_t\|B_t)$ and $z \in Z$, define $\Lambda_t g(z) = \int g d\mu_z^t$. It is obvious that $\Lambda_t : \mathscr{C}_0(Y_t\|B_t) \to \mathscr{C}(Z)$ is a linear contraction and $\Gamma_t = \Lambda_t\mathscr{C}_0(\varepsilon_t)$. By (a), there exists a linear contraction

$$\Gamma : \mathscr{C}_0(X_\infty\|A_\infty) \to \mathscr{C}(Z)$$

such that $\Gamma\mathscr{C}_0(\pi_t) = \Lambda_t$ for each t in T. Hence

$$\Gamma\mathscr{C}_0(\varphi_t) = \Gamma\mathscr{C}_0(\varepsilon_t\pi_t) = \Gamma\mathscr{C}_0(\pi_t)\mathscr{C}_0(\varepsilon_t) = \Lambda_t\mathscr{C}_0(\varepsilon_t) = \Gamma_t.$$

Moreover, Γ transforms each set H_t, the range of $\mathscr{C}_0(\varphi_t)$, into F. Since $\bigcup H_t$ is dense in $\mathscr{C}_0(X_\infty\|A_\infty)$, Γ is the unique linear contraction such that $\Gamma\mathscr{C}_0(\varphi_t) = \Gamma_t$ for t in T and the range of Γ is contained in F. This concludes the proof of Theorem 2.

Let $\mathbf{Comp}_\square$ denote the category of *pointed compact spaces*. In other words, a $\mathbf{Comp}_\square$-object is a pair $(X, x^\odot)$, where X is compact and $x^\odot$ is any element of X, called the *base point* of the object; a morphism from $(X, x^\odot)$ to $(Y, y^\odot)$ is a continuous map $\varphi : X \to Y$ such that $\varphi(x^\odot) = y^\odot$. The contravariant functor

(5) $$\mathscr{C}_0 : \mathbf{Comp}_\square \to \mathbf{Ban}_1$$

is defined in the obvious way: if $x^\odot \in X$, then

$$\mathscr{C}_0(X, x^\odot) = \{f \in \mathscr{C}(X) : f(x^\odot) = 0\} = \mathscr{C}_0(X\|x^\odot),$$

and if $\varphi : (X, x^\odot) \to (Y, y^\odot)$ is a morphism in $\mathbf{Comp}_\square$, then $\mathscr{C}_0(\varphi)$ is the restriction of $\mathscr{C}(\varphi)$ to the subspace $\mathscr{C}_0(Y\|y^\odot)$. If one thinks of $x^\odot$ as the one-point set $\{x^\odot\}$, then the category $\mathbf{Comp}_\square$ becomes a full subcategory of $\mathbf{Comp}_{\mathscr{F},\subset}$. Moreover, if $(X_t, x_t^\odot)$ are objects in $\mathbf{Comp}_\square$ and

(6) $$\varphi_t^s : X_s \to X_t, \quad \varphi_t^s(x_s^\odot) = x_t^\odot \quad (t \leqslant s)$$

form an inverse system, then the compact set X_∞ of all threads with the base point $x^\odot = \{x_t^\odot\}_{t\in T}$ is an inverse limit of this system (in $\mathbf{Comp}_\square$). Thus, the canonical functor $\mathbf{Comp}_\square \to \mathbf{Comp}_{\mathscr{F},\subset}$ preserves the inverse limits and from Theorem 2 it follows that the functor (5) is inversely continuous. More precisely:

THEOREM 3. *Let $(X_\infty, x^\odot)$ be the inverse limit of the system* (6) *in the category* $\mathbf{Comp}_\square$. *Then $\mathscr{C}_0(X_\infty\|x^\odot)$ is a direct limit* (*in the category* $\mathbf{Ban}_1$) *of the direct system*

$$\mathscr{C}_0(\varphi_t^s) : \mathscr{C}_0(X_t\|x_t^\odot) \to \mathscr{C}_0(X_s\|x_s^\odot).$$

Let us recall (see [5], p. 287, and [7], 18.3.2) that the Radon functor

(7) $$\mathscr{M} : \mathbf{Comp} \to \mathbf{Ban}_1$$

assigns to each compact space X the space $\mathscr{M}(X)$ of Radon measures on X, and to each continuous map $\varphi : X \to Y$ the induced transformation $\mathscr{M}(\varphi) : \mathscr{M}(X) \to \mathscr{M}(Y)$ defined as $\mathscr{M}(\varphi)\mu = \nu$, where $\nu(E) = \mu\big(\varphi^{\leftarrow}(E)\big)$ for any Borel subset E of Y. It is a covariant functor. Since (7) is naturally equivalent to the composition of (1) with the conjugate-space functor $J^* : \mathbf{Ban}_1 \to \mathbf{Ban}_1$ (see [5], p. 292, and [7], 18.4.3) and J^* transforms direct limits in $\mathbf{Ban}_1$ onto inverse limits in $\mathbf{Ban}_1$ (see [8]), we get the following corollary:

COROLLARY. *If $\varphi_t^s: X_s \to X_t$ $(t \leqslant s)$ is an inverse system in* $\mathbf{Comp}$, *then $\mathscr{M}(X_\infty)$ is an inverse limit (in the category $\mathbf{Ban}_1$) of the inverse system*

$$\mathscr{M}(\varphi_t^s): \mathscr{M}(X_s) \to \mathscr{M}(X_t), \quad t \leqslant s.$$

Similar statements are valid for categories $\mathbf{Comp}_\square$ and $\mathbf{Comp}_{\mathscr{F},\subset}$. The inverse limit in $\mathbf{Ban}_1$ may be represented, e.g., as the set of all threads $\mu = (\mu_t)_{t\in T}$ with $\mu_t \in \mathscr{M}(X_t)$ and $\sup \|\mu_t\| < \infty$.

Remarks. The functor $\mathscr{C}: \mathbf{Top} \to \mathbf{Ban}_1$ is not inversely continuous. Indeed, let $X_1 \leftarrow X_2 \leftarrow \ldots$ be an inverse system of non-empty topological spaces such that X_∞ is empty (e.g., $X_n = \{x \in \mathbf{R}: x \geqslant n\}$ with the embeddings). Applying the Stone-Čech functor

$$\beta: \mathbf{Top} \to \mathbf{Comp}$$

we get an inverse system $\beta X_1 \leftarrow \beta X_2 \leftarrow \ldots$ with non-empty inverse limit Y. Consequently, by Theorem 1, the space $\mathscr{C}(Y)$ is a direct limit (in $\mathbf{Ban}_1$) of the direct system

$$\mathscr{C}(\beta X_1) \to \mathscr{C}(\beta X_2) \to \ldots$$

This system is equivalent to the system

$$\mathscr{C}(X_1) \to \mathscr{C}(X_2) \to \ldots \tag{8}$$

Therefore $\mathscr{C}(Y)$ is also a direct limit of (8). Since $Y \neq \emptyset$, $\mathscr{C}(Y)$ contains non-zero functions and is not isomorphic to $\mathscr{C}(X_\infty) = \{0\}$. It is easy to modify this example (e.g., by adding an isolated point to each space X_n) as to show that $\mathscr{C}(X_\infty)$ need not be a direct limit of (8) even if $X_\infty \neq \emptyset$ and that the analogous functors

$$\mathscr{C}_0: \mathbf{Top}_\square \to \mathbf{Ban}_1 \quad \text{and} \quad \mathscr{C}_0: \mathbf{Top}_{\mathscr{F},\subset} \to \mathbf{Ban}_1$$

are not inversely continuous either.

Both the functors $\mathscr{C}: \mathbf{Top} \to \mathbf{Ban}_1$ and $\mathscr{C}: \mathbf{Comp} \to \mathbf{Ban}_1$ transform direct limits to inverse limits (indeed, they transform coproducts to products and coequalizers to equalizers; on the other hand, the [direct] inverse limits can be constructed as certain coequalizers [equalizers] of certain coproducts [products], see [3] and [7], 11.7.2 and 12.5.4 (A)).

The functor (1) does not transform products to coproducts neither does it transform equalizers to coequalizers (e.g., $\mathscr{C}(X \times Y)$ is not isomorphic to the l_1-product of $\mathscr{C}(X)$ and $\mathscr{C}(Y)$, cf. also [7], 11.5.8 (C)). Yet, this functor transforms inverse limits to direct limits.

Now, let $\mathbf{Top}_{\mathscr{F},\subset}$ denote the category defined similarly to $\mathbf{Top}_{\mathscr{F},\subset}$ but without requiring that the distinguished subsets be closed. The functor

$$Q: \mathbf{Top}_{\mathscr{F},\subset} \to \mathbf{Top}_\square \tag{9}$$

assigns to each pair (X, A), with $A \neq \emptyset$, the quotient space X/A obtained by pinching A to a base point, and to each pair $(X, \emptyset)$ — the space $X + \mathbf{1}$ obtained by adding a new isolated base point (the morphism transformation being obvious). It is clear that (9) is a left adjoint of the forgetful functor from $\mathbf{Top}_\square$ to $\mathbf{Top}_{\mathscr{F},\subset}$. Therefore the functor (9) preserves the coproducts, coequalizers, and direct limits.

Let us consider the diagram in Fig. 4, where γ_t is the unique map such that $\gamma_t \pi_\infty = \pi_t \alpha_t$ and β is the unique map such that $\beta_t \beta = \gamma_t$ for t in T. In this diagram, it is assumed for simplicity that $X/\emptyset = X + \mathbf{1}$.

$$\begin{tikzcd}
X_\infty \arrow[rr, "\varphi_t"] \arrow[d, "\pi_\infty"'] & & X_t \arrow[d, "\pi_t"] \\
X_\infty/A_\infty \arrow[rr, dashed, "\gamma_t"] \arrow[dr, dashed, "\beta"'] & & X_t/A_t \\
& \lim^{\leftarrow} X_s/A_s \arrow[ur, "\beta_t"'] &
\end{tikzcd}$$

Fig. 4

It is clear that the canonical map β is a continuous bijection; we shall show that it need not be a homeomorphism.

If $A_s \neq \emptyset$ for s in T, then the space $\lim^{\leftarrow} X_s/A_s$ consists of all threads $x = (x_s)_{s\in T}$ with $x_s \in X_s \setminus A_s$ for s sufficiently large and of one thread $a = (a_s)_{s\in T}$, where a_s is the base point of X_s/A_s; the thread a is the base point of $\lim^{\leftarrow} X_s/A_s$. If some A_r is empty, then $A_s = \emptyset$ for $s \geqslant r$, and β is a homeomorphism. It may happen, however, that $A_t \neq \emptyset$ for t in T and $A_\infty = \emptyset$. E.g., let

$$A_n = \{n, n+1, \ldots\}$$

and let X_n be $A_n \cup \{\omega\}$ with the interval topology (ω is the unique accumulation point of X_n). The spaces $X_1, X_2, \ldots$ and the embeddings form an inverse system. Each space X_n/A_n is a two-point topological space with three open subsets whereas the constant thread $\{\omega\}$ is the unique element of X_∞ and $Q(X_\infty, \emptyset)$ is a two-point discrete space. Thus, in this case the objects

$$\lim^{\leftarrow} Q(X_s, A_s) \quad \text{and} \quad Q\big(\lim^{\leftarrow}(X_s, A_s)\big)$$

are not isomorphic.

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Reçu par la Rédaction le 22. 3. 1968

STUDIA MATHEMATICA, T. XXXI. (1968)

О разложении унитарного представления комплексной полупростой группы Ли на ее неприводимые представления

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1. Введение. Пусть G — топологическая группа со счетной базой окрестностей. Как известно (см. напр. [13], гл. VIII) каждое непрерывное унитарное представление $g \to V_g$ группы в сепарабельном гильбертовом пространстве H разлагается в прямой интеграл ее неприводимых представлений. С другой стороны, фактическое разложение на неприводимые представления заданного унитарного представления может представить значительные трудности. Для связной комплексной полупростой группы Ли G Гельфанд и Граев [1] разработали весьма общий метод фактического получения подобного разложения, который в ряде интересных случаев (например, в случае тензорного произведения двух представлений основной невырожденной серии) довольно просто приводит к цели. С другой стороны, в ряде других интересных случаев (например, в случае тензорного произведения неприводимых представлений других серий (¹)) этот метод наталкивается пока на существенные трудности.

В настоящей статье мы предлагаем другой метод фактического разложения на неприводимые представления также для случая связной комплексной полупростой группы Ли G. В этом методе использована конструкция, предложенная ранее автором [11] для описания неприводимых унитарных представлений группы G и развитая далее Желобенко и автором [9] (см. также Желобенко [6]-[8]) для получения описания всех вполне неприводимых (унитарных и неунитарных) представлений этой группы. В следующих сообщениях излагаемый здесь метод будет перенесен на некоторые неунитарные представления, а также будут даны приложения этого метода к конкретным представлениям.

2. Некоторые вспомогательные сведения. Всюду в дальнейшем G обозначает связную комплексную полупростую группу Ли, r — ее ранг, U — ее максимальную компактную подгруппу, H — сепара-

(¹) Для $GL(2, C)$ этот и вообще все случаи разобраны в статьях автора [12].