Изложенная в статье методика позволяет исследовать вопрос о существовании производной по параметру t для решений более общих граничных задач, в которых коэффициенты оператора L и граничных операторов B_j являются достаточно гладкими функциями от t и x.

Для нас существенным являлось условие $\max m_j \leqslant 2m-2$. Как показывают простейшие примеры, при нарушении этого ограничения производная $D_t u_t$ может существовать лишь как обобщенная функция. Нам кажется интересным исследование до конца возникающей здесь ситуации.

Важным было ы также рассмотрение поведения решений эллиптических задач в условиях их нормальной разрешимости, а не единственности.

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A uniform boundedness theorem and mappings into spaces of operators

by

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Dedicated to Professor
Władysław Orlicz
on the occasion of the 40th anniversary
of his scientific research

In a recent paper [1] B. E. Johnson proved the remarkable fact that every strictly irreducible representation of a Banach algebra is continuous. In the present note we use a similar argument to prove a certain modification of the classical uniform boundedness principle; this simple result (theorem (2.2) of the present note) is interesting in its own right and turns out to be the basis of theorems concerning the behaviour of algebraic homomorphisms of Banach spaces into spaces of linear operators. Indeed, if A is a Banach space, Y and X two normed spaces and T an algebraic homomorphism of A into $L(Y, X)$, then the modification of the uniform boundedness theorem mentioned above may be used to show that T is continuous provided it satisfies some surprisingly weak conditions (theorem (2.5) of the present note). These conditions being automatically satisfied if A is a Banach algebra and T a strictly irreducible representation thereof, this result constitutes a slight generalization of Johnson's theorem. At the same time it puts into evidence the way in which use is made of the assumption that A is a Banach algebra.

1. Preliminaries. In this section we intend to collect some simple propositions which will be needed in the sequel. We begin by listing several simple facts concerning rare and meagre sets in topological spaces.

(1.1) *Let T be a topological space and H a subset of T. Then*

1° *if $A \subset H$ and A is rare (meagre) in H, then A is rare (meagre) in T as well;*

2° *if H is open in T, then $A \subset H$ is rare (meagre) in T if and only if it is rare (meagre) in H.*

Further, let us recall the definition of the operator D. Given a topological space T, we denote, for each $A \subset T$, by $D(A)$ the set of all points $x \in T$ with the following property: for each neighbourhood U of x the intersection $U \cap A$ is non-meagre in T.

(1.2) *Let T be a topological space. Then the following conditions are equivalent:*

1° *we have $G \subset D(G)$ for each non-void G open in T;*

2° *every non-void G open in T is non-meagre in T;*

3° *every non-void G open in T is non-meagre in itself*

Proof. We observe, first that 2° and 3° are equivalent by (1.1). Now suppose that 1° is satisfied and let G be an open set in T, G non-void. Since G is non-void, there exists a point $x \in G$. By 1° we have $x \in D(G)$ so that every neighbourhood of x is non-meagre in T. Hence G itself is non-meagre in T which proves 2°. If 3° is satisfied and if G is non-void and open in T, take an arbitrary $x \in G$ and an arbitrary open neighbourhood U of x. Then $H = U \cap G$ is non-void and open in T hence non-meagre in itself; it follows from (1.1) that H is non-meagre in T. Hence, $x \in D(G)$ so that $G \subset D(G)$. This proves 1° and completes the proof.

A space which satisfies one (and hence all) of the conditions of the preceding lemma will be called a *Baire space*. In the case of a linear space we have the following simple proposition:

(1.3) *Let E be a topological vector space. If E is non-meagre in itself then, E is a Baire space.*

Proof. Take an arbitrary non-void open set G and suppose that G is meagre in T. Choose a $g \in G$; then $G-g$ is again meagre in T and so is $n(G-g)$ for any natural number n. Since $E = \cup\, n(G-g)$, this is a contradiction.

We shall also need the following simple lemma:

(1.4) *Let E be a linear space and P, Q two subspaces of E such that their set theoretical union $P \cup Q = E$. Then either P or Q equals E.*

Proof. Suppose that P does not fill the whole of E and let us show that $P \subset Q$. Take a fixed q_0 outside P. If $p \in P$, both $p+q_0$ and $p-q_0$ are points outside P hence $p+q_0 \in Q$ and $p-q_0 \in Q$. It follows that $p = \frac{1}{2}\big((p+q_0)+(p-q_0)\big) \in Q$ so that $P \subset Q$, whence $Q = P \cap Q = E$.

If (P, p) and (Q, q) are two normed spaces, we denote by $L\big((P,p),(Q,q)\big)$ the space of all continuous linear transformations of (P,p) into (Q,q). If we drop the requirement of continuity, we obtain the space of all algebraic homomorphisms of P into Q which we denote by $L_a\big((P,p),(Q,q)\big)$ or, shortly, $L_a(P,Q)$.

If E is a linear space and F a set of homomorphisms of E into a linear space X, we denote by $N(F)$ the set of all $x \in E$ such that $f(x) = 0$ for all $f \in F$.

2. A uniform boundedness theorem. The classical uniform boundedness principle may be formulated as follows:

(2.1) *Let (B,p) be a normed Baire space and S a set of algebraic homomorphisms of B into a normed space (X,w). Assume further that*

1° *each $y \in S$ is continuous;*

2° *the set S is pointwise bounded for each $b \in B$.*

Then the set S is equicontinuous.

The basic result of the present paper consists in showing that this tesult remains almost true if we replace condition 1° by a weaker hyporhesis. Instead of requiring y to be continuous we take the weaker postulate that the kernel $N(y)$ be closed in (B,p).

(2.2) Theorem. *Let (B,p) be a Banach space, S a set of algebraic homomorphisms of B into a normed space (X,w). Assume further that*

1° *for each $y \in S$, the set $N(y)$ is closed in (B,p);*

2° *the set S is pointwise bounded for each $b \in B$.*

Then there exists a finite set $y_1, \ldots, y_n \in S$ such that the set S is equicontinuous on $N(y_1, \ldots, y_n)$ or, more precisely, there exists a $\sigma > 0$ such that $b \in N(y_1, \ldots, y_n)$ and $y \in S$ imply $w(by) \leqslant \sigma p(b)$.

Proof. For each $a \in B$, write $M(a) = \sup\, w(ay)$; $y \in S$. According to condition 2°, the constant $M(a)$ is finite for each $a \in B$. Suppose that the statement of the theorem is false.

It follows that, for each $\sigma > 0$ and each finite set $y_1, \ldots, y_n$ there exists an $a \in N(y_1, \ldots, y_n)$ and a $y \in S$ such that $p(a) \leqslant 1$ and $w(ay) > \sigma$. We shall proceed by induction. There exists an a_1 and a $y_1 \in S$ such that $p(a_1) \leqslant 1$ and $w(a_1 y_1) \geqslant 2$. Further, there exists an $a_2 \in N(y_1)$ and a $y_2 \in S$ such that $p(a_2) \leqslant 1$ and $w(a_2 y_2) \geqslant 2^2\big(2+\frac{1}{2}M(a_1)\big)$. Similarly, there exists an $a_3 \in N(y_1, y_2)$ and a $y_3 \in S$ such that $p(a_3) \leqslant 1$ and $w(a_3 y_3) \geqslant 2^3\big(3+ +\frac{1}{2}M(a_1)+(1/2)^2 M(a_2)\big)$. Proceeding by induction we construct two sequences $a_i \in B$, $y_j \in S$ such that

$$p(a_i) \leqslant 1, \quad a_i \in N(y_1, \ldots, y_{i-1}),$$

$$w(a_n y_n) \geqslant 2^n\Big(n + \sum_{j \leqslant n-1} (1/2)^j M(a_j)\Big).$$

Consider now the point $a = \sum_{j \geqslant 1} (1/2)^j a_j$; clearly $p(a) \leqslant 1$. Given a natural number n, we have

$$ay_n = \sum_{j \leqslant n} (1/2)^j a_j y_n + v y_n, \qquad \text{where } v = \sum_{j \geqslant n+1} (1/2)^j a_j.$$

For $j \geqslant n+1$ we have $a_j \in N(y_n)$ so that, $N(y_n)$ being closed, the vector v belongs to $N(y_n)$ as well. It follows that

$$w(ay_n) = w\Big((1/2)^n a_n y_n + \sum_{j\leqslant n-1} (1/2)^j a_j y_n\Big)$$
$$\geqslant (1/2)^n w(a_n y_n) - \sum_{i\leqslant n-1} (1/2)^j w(a_j y_n)$$
$$\geqslant (1/2)^n w(a_n y_n) - \sum_{j\leqslant n-1} (1/2)^j M(a_j) \geqslant n,$$

which contradicts condition 2°. The proof is complete.

It would be interesting to know whether the preceding result remains true under the weaker hypothesis that (B, p) is a Baire space.

The result just obtained may be considerably improved if we add another condition.

(2.3) *Let (B, p) be a Banach space, (X, w) a normed space. Let S be a subset of $L_a(B, X)$. Denote by Y the subspace of $L_a(B, X)$ generated by S and suppose that the following three conditions are satisfied:*

1° *for each $y \in Y$, the set $N(y)$ is closed in (B, p);*

2° *the set S is pointwise bounded for each $b \in B$;*

3° *given $y_1, \ldots, y_n \in Y$ and $x_1, \ldots, x_n \in X$ such that the y_i are linearly independent, there exists a $b \in B$ such that $by_i = x_i$ for $i = 1, 2, \ldots, n$.*

Then either Y is finite-dimensional or S is equicontinuous on the whole of B.

Proof. The proof will be divided into four steps.

I. According to (2.2) there exist $y_1, \ldots, y_n \in Y$ and a $\sigma > 0$ such that $b \in N(y_1, \ldots, y_n)$ and $y \in S$ imply $w(by) \leqslant \sigma p(b)$.

II. Let us prove now the following assertion: if $y \in Y$ is linearly independent of $y_1, \ldots, y_n$, then y is continuous on (B, p). We begin by showing that $B = N(y_1, \ldots, y_n) + N(y)$. Indeed, given $b \in B$ there exists, by assumption 3°, a $c \in B$ such that $cy_i = 0$ and $cy = by$. It follows that $c \in N(y_1, \ldots, y_n)$ and $b - c \in N(y)$. Since (B, p) is complete and the subspaces N are closed therein there exists a $\beta > 0$ such that every $b \in B$ may be written in the form $b = u + v$, $u \in N(y_1, \ldots, y_n)$, $v \in N(y)$ with $p(u) + p(v) \leqslant \beta p(b)$. Since y may be written in the form $y = \sum \lambda_j s_j$, $s_j \in S$, we have, for $h \in N(y_1, \ldots, y_n)$, the estimate

$$w(hy) = w\Big(\sum \lambda_j hs_j\Big) \leqslant \sum |\lambda_j|\, w(hs_j) \leqslant \Big(\sum |\lambda_j|\Big) \sigma p(h) = \gamma p(h),$$

where we set $\gamma = \sigma(\Sigma|\lambda_j|)$. If $b \in B$, we have thus

$$w(by) = w(uy + vy) = w(uy) \leqslant \gamma p(u) \leqslant \gamma\beta p(b)$$

which completes the proof of our assertion.

III. Denote by P the subspace of Y generated by $y_1, \ldots, y_n$ and by Q the subspace of those $y \in Y$ which are continuous on (B, p). According to the preceding part of the proof $P \cup Q = Y$. It follows that either $P = Y$ so that Y is finite-dimensional or $Q = Y$.

IV. If Y is not finite-dimensional, we have $Q = Y$; in particular, S is a set of continuous linear transformations of (B, p) into (X, w) which is pointwise bounded on (B, p), a complete space. It follows that S is equicontinous. The proof is complete.

The preceding results may also be formulated in the form of propositions concerning mappings of a Banach space into spaces of linear operators. Before considering the main theorem, let us state a simple result about mappings of a Banach space into spaces of linear functionals which helps to understand the general case.

(2.4) *Let A and E be two normed spaces and let T be a continuous linear mapping of A into $(E', \sigma(E', E))$. Suppose that A is a Baire space. Then T is continuous into (E', w), where w is the norm topology of E'.*

Proof. According to our assumption, for each $x \in E$, the mapping $a \to \langle T(a), x\rangle$ iscontinuous on A hence $\langle T(\cdot), x\rangle$ is a certain element $T'x \in A'$. Denote by B the set of all $T'x$, where $x \in E$ and $|x| \leqslant 1$ and let us show that B, a subset of A', is bounded in the norm. Since A is a Baire space it suffices to show that B is pointwise bounded on A. To see that, take a fixed $a \in A$ and an arbitrary element $b \in B$, $b = T'x$. Then

$$|\langle a, b\rangle| = |\langle a, T'x\rangle| = |\langle T(a), x\rangle| \leqslant |T(a)|\,|x| \leqslant |T(a)|;$$

this estimate being independent of $b \in B$, the proof is complete.

(2.5) Theorem. *Let (Y, q) and (X, w) be two normed spaces. Let (A, p) be a Banach space and T an algebraic homomorphism of A into $L((Y, q), (X, w))$. Suppose that the following two conditions are satisfied:*

1° *for each $y \in Y$ the set $N(y) = \{a \in A;\ T_a y = 0\}$ is closed in (A, p);*

2° *given $y_1, \ldots, y_n \in Y$ and $x_1, \ldots, x_n \in X$ such that the y_i are linearly independent, then there exists an $a \in A$ such that $T_a y_i = x_i$.*

Then either Y is finite-dimensional or the mapping T is continuous.

Proof. Let us define, for each $y \in Y$, an element $h(y) \in L_a(A, X)$ by the relation $a \cdot h(y) = T_a y$.

Take, in proposition (2.3), for (B, p), Y, S, the following objects respectively: (A, p), $h(Y)$, $h(U)$, where U is the unit ball of (Y, q). Let us prove that the assumptions of (2.3) are satisfied. In order to see that the set $h(U)$ is bounded for each $a \in A$ take a fixed $a \in A$ and an arbitrary $y \in Y$, $q(y) \leqslant 1$. It follows that $w(a \cdot h(y)) = w(T_a y) \leqslant |T_a|\, q(y) \leqslant |T_a|$ for all $y \in U$. The other two conditions being immediate, it follows that either Y is finite-dimensional or the set $h(U)$ is equi-continuous. In the second case there exists a $\beta > 0$ such that $|h(y)| \leqslant \beta$ for each y for which $q(y) \leqslant 1$. Hence $|h(y)| \leqslant \beta q(y)$ for all $y \in Y$ so that

$$w(T_a y) = w(a \cdot h(y)) \leqslant p(a)|h(y)| \leqslant \beta p(a) q(y).$$

3. Bilinear mappings. The preceding results admit interesting reformulations in the form of statements about bilinear mappings. We begin by stating a classical result about separately continuous bilinear mappings and then proceed to investigate what happens if we relax the requirement of continuity for one of the two variables.

(3.1) *Let (A, p), (Y, q) and (X, w) be three normed spaces, (A, p) a Baire space. Let F be a separately continuous bilinear mapping of $A \times Y$ into X. Then there exists a constant $a > 0$ such that*

$$w(F(x, y)) \leqslant ap(x) q(y).$$

Proof. For each $y \in Y$ denote by T_y the transformation $F(\cdot, y)$ so that T_y is a continuous linear mapping of (A, p) into (X, w). Let $M \subset L((A, p), (X, w))$ be the set of all T_y for $y \in Y$, $q(y) \leqslant 1$. For each fixed $x \in A$ it follows from the continuity of $F(x, \cdot)$ that the set $M(x)$ is bounded. Since (A, p) is Baire space and M is pointwise bounded, it follows that there exists a constant a such that $|T_y| \leqslant a$ for each $T_y \in M$. Hence $|T_y| \leqslant aq(y)$ so that $w(F(x, y)) = w(T_y x) \leqslant |T_y| p(x) \leqslant ap(x) q(y)$, which completes the proof.

(3.2) *Let (A, p), (Y, q) and (X, w) be three normed spaces and let f be a bilinear mapping of $A \times Y$ into X. Suppose that (A, p) is complete and*

1° *for each fixed $a \in A$ the value $f(a, \cdot)$ is continuous;*

2° *for each fixed $y \in Y$ the set $N(y) = \{a \in A, f(a, y) = 0\}$ is closed in (A, p).*

Then there exists a finite sequence $y_1, \ldots, y_n \in Y$ and an $a > 0$ such that $a \in N(y_1, \ldots, y_n)$ and $y \in Y$ imply

$$w(f(a, y)) \leqslant ap(a) q(y).$$

Proof. We are going to apply proposition (2.2). For each $y \in Y$ the value $f(\cdot, y)$ is an algebraic homomorphism $h(y)$ of A into X. Denote by S the set of all $h(y)$ for $q(y) \leqslant 1$. It follows from condition 1° of the present theorem that, given a fixed $a \in A$, $f(a, \cdot)$ is continuous so that $w(f(a, y)) \leqslant \beta q(y)$ for a suitable $\beta > 0$. If $s \in S$, we have $s = h(y)$ for some $q(y) \leqslant 1$, whence $w(as) = w(f(a, y)) \leqslant \beta q(y) \leqslant \beta$.

Condition 2° of (2.2) is thus satisfied. The rest follows immediately.

(3.3) *Let (A, p), (Y, q) and (X, w) be three normed spaces and let f be a bilinear mapping of $A \times Y$ into X. Suppose that (A, p) is complete and*

1° *for each fixed $y \in Y$ the set $N(y) = \{a \in A, f(a, y) = 0\}$ is closed in (A, p);*

2° *for each fixed $a \in A$ the value $f(a, \cdot)$ is continuous;*

3° *given $y_1, \ldots, y_n \in Y$ and $x_1, \ldots, x_n \in X$ such that the y_i are linearly independent, there exists an $a \in A$ such that $f(a, y_i) = x_i$.*

Then either Y is finite-dimensional or there exists a $\beta > 0$ such that $w(f(a, y)) \leqslant \beta p(a) q(y)$ for all $a \in A$ and $y \in Y$.

Proof. An immediate consequence of (2.3).

4. Representations of Banach algebras. Now we are ready to apply the results about mappings into spaces of linear operators to representations of Banach algebras. This immediately yields Johnson's theorem.

(4.1) THEOREM. *Let (A, p) be a Banach algebra and let T be a strictly irreducible representation of A on the normed space (X, w). Then T is continuous.*

Proof. The mapping T is an algebraic homomorphism of A into $L((X, w), (X, w))$. We shall use theorem (2.5) where we put $(Y, q) = (X, w)$. First of all, it follows from the general theory of Banach algebras (see theorem 2.4.6 of [3]) that condition 3° of theorem (2.5) is satisfied. Further, for each y, the set $N(y)$ is the set of all $a \in A$ for which $T_a y = 0$. It is a well known fact that this kernel is a maximal modular left ideal in A and, accordingly, closed in (A, p). This gives condition 1°. It follows that either X is finite-dimensional or T is continuous. If X is finite-dimensional, denote by H the intersection of all $N(x)$, $x \in X$, and observe that H is a closed two sided ideal in (A, p). It follows that A/H is isomorphic to a subalgebra of the finite-dimensional algebra $L((X, w), (X, w))$. Since all norms on a finite-dimensional vector space are equivalent, we have $|T_a| \leqslant \beta p(a)$ for some $\beta > 0$. The proof is complete.

A preliminary report about these results is contained in [2].

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