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A generalization of the Mazur-Ulam theorem

by

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Let two F-spaces X and Y be given. Since it does not lead to a misunderstanding, we shall denote the F-norms in both spaces by the same symbol $\| \ \|$.

An operator U (non-necessarily linear) transforming X into Y is called an *isometry* if

$$\|U(x)-U(y)\| = \|x-y\|.$$

An isometry is called a *rotation* if

$$U(0) = 0.$$

Mazur and Ulam [4] have proved that if X and Y are Banach spaces over reals and the norms in both spaces are homogeneous, then each rotation is a linear transformation.

The same question for general F-spaces over reals is still unsolved. Charzyński [2] has proved that if X and Y are finite-dimensional real F-spaces, then the Mazur-Ulam theorem is also valid.

In this note we generalize the Mazur-Ulam theorem to locally bounded spaces. The norms in question ought to be so called *concave norms* (1). Let us remark that the result of this paper contains the result of Charzyński only partially.

The method of the proof is similar to the proof of the Mazur and Ulam theorem.

THEOREM 1. *Let X and Y be locally bounded* (2) *spaces over reals. Let the norms in X and Y be concave. Then each rotation is a linear transformation.*

The proof of theorem 1 is based on some conceptions and lemmas.

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(1) We say that a norm $\| \ \|$ is *concave* if the function $\|tx\|$ is concave for all x and positive t, i.e.

$$\|(\alpha t+\beta t')\, x\| \geqslant \alpha\|tx\|+\beta\|t' x\|,$$

$\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$ (see [1]).

(2) A linear metric space X is called *locally bounded* if there is in X a bounded neighbourhood of zero (see [3] and [5]).

Let
$$n(r) = \sup_{\|2x\|\leqslant r} \|x\|.$$

Obviously, if the norm $\| \ \|$ is concave, the function $\|tx\|$ is non-decreasing, whence $n(r) \leqslant r$.

Lemma 1. *If the space X is locally bounded and the norm $\| \ \|$ is concave, then there is an $r_0 > 0$ such that for $0 < r < r_0$ we have $n(r) < r$.*

Proof. Let r_0 be a positive number such that the set
$$K_{2r_0} = \{x \in X\colon \|x\| < 2r_0\}$$
is bounded. Such an r_0 exists since the space X is locally bounded.

Obviously, for all r, $0 < r \leqslant r_0$, the set
$$K_{2r} = \{x \in X\colon \|x\| < 2r\}$$
is also bounded.

Let us suppose that $n(r) = r$. Then there is a sequence of elements $x_n \in X$ such that $\|x_n\| = r$ and $\|x_n/2\|$ tends to r. This implies
$$a_n = \frac{r}{2(r - \|x_n/2\|)} \to \infty.$$

Since the norm is concave, we have on the other hand
$$r = \|x_n\| = \left\| \frac{\frac{1}{2}}{a_n - \frac{1}{2}} a_n x_n + \frac{a_n - 1}{a_n - \frac{1}{2}} \frac{x_n}{2} \right\| \geqslant \frac{\frac{1}{2}}{a_n - \frac{1}{2}} \|a_n x_n\| + \frac{a_n - 1}{a_n - \frac{1}{2}} \left\| \frac{x_n}{2} \right\|.$$
Then
$$\|a_n x_n\| \leqslant 2\left[\left(a_n - \frac{1}{2}\right) r - (a_n - 1)\left\|\frac{x_n}{2}\right\|\right]$$
$$= 2a_n\left(r - \left\|\frac{x_n}{2}\right\|\right) - r + 2\left\|\frac{x_n}{2}\right\| = r - r + 2\left\|\frac{x_n}{2}\right\| < 2r.$$

This implies that $a_n x_n \in K_{2r}$, which leads to a contradiction since K_{2r} is bounded and $a_n^{-1} a_n x_n = x_n$ does not tend to 0, q. e. d.

Let us define r_n by induction: $r_n = n(r_{n-1})$. From lemma 1 it follows that $r_0 > r_1 > r_2 > \ldots$

Lemma 2. $\lim\limits_{n\to\infty} r_n = 0$.

Proof. Let us suppose that $\lim\limits_{n\to\infty} r_n = r' \neq 0$. Obviously, $n(r') < r'$. The function $n(r)$ is trivially continuous; therefore there is an $\bar{r} > r'$ such that $n(\bar{r}) < r'$. But from the definition of r_n it trivially follows that there is an n such that $r_n < \bar{r}$. Therefore
$$r_{n+1} = n(r_n) < n(\bar{r}) < r'$$
and we obtain a contradiction, q. e. d.

Proof of theorem 1. Let x_0 and y_0 be two arbitrary elements of X such that $\|x_0 - y_0\| < r_0/2$. Let
$$H_0 = \left\{z \in X\colon \|z - x_0\| \leqslant n\left(\frac{r_0}{2}\right),\ \|z - y_0\| \leqslant n\left(\frac{r_0}{2}\right)\right\}.$$

Obviously, the diameter of the set H_0 is
$$\delta(H_0) = \sup_{x,y \in H_0} \|x - y\| \leqslant 2n\left(\frac{r_0}{2}\right) \leqslant r_0.$$

We define H_n by induction:
$$H_n = \{x \in H_{n-1}\colon \|x - y\| < r_n \text{ for all } y \in H_{n-1}\}.$$

This definition trivially implies that the diameter of the set H_n, $\delta(H_n)$, is not greater than r_n. We shall show by induction that the point $(x_0 + y_0)/2$ belongs to all the sets H_n and, moreover, that the sets H_n are symmetric with respect to this point, i.e. if $x \in H_n$, then the point $\bar{x} = x_0 + y_0 - x$ also belongs to H_n.

For H_0 this is trivial. In fact, the definition of $n(r)$ implies that $(x_0 + y_0)/2 \in H_0$ and the second part of the statement follows from the identities $\bar{x} - x_0 = y_0 - x$ and $\bar{x} - y_0 = x_0 - x$.

Suppose that H_{n-1} satisfies our induction assumptions. Then the definition of r_n implies that $(x_0 + y_0)/2 \in H_n$. Let z and x be arbitrary elements of H_{n-1}. Then $\bar{x} - z = \bar{z} - x$ and this implies that if $x \in H_n$, then also $\bar{x} \in H_n$.

Since r_n tends to 0, the intersection of all sets H_n contains only one point $(x_0 + y_0)/2$. This is a metric characterization of the centre.

Using the same method, we can give a metric characterization of a centre of two points v and w belonging to Y provided the distance between them is small enough.

Therefore the distance between two points $x, y \in X$ is small enough, the centre of images is equal to the image of the centre. In other words, there is a positive number a such that if $\|x - y\| < a$, then
$$U\left(\frac{x + y}{2}\right) = \frac{U(x) + U(y)}{2}.$$

This implies that if $\|x\| < a/2$, then

(i) $$2U(kx) = U\big((k+1)x\big) + U\big((k-1)x\big)$$

for all positive integers k.

Basing ourselves on formula (i) we shall show by induction that

(ii) $$U(nx) = nU(x) \quad \text{for } n = 1, 2, \ldots$$

Putting $k = 1$ in formula (i) we find that (ii) holds for $n = 2$. Let us suppose that (ii) holds for $n = m$. Let us put $k = m$ in (i). Then the induction hypothesis implies

$$U(2mx) = 2mU(x) = U\big((m+1)x\big)+U\big((m-1)x\big)$$
$$= U\big((m+1)x\big)+(m-1)U(x).$$

Hence $U\big((m+1)x\big) = (m+1)U(x)$ and we have proved (ii). Hence $U(kx) = k\,U(x)$.

Let x and y be arbitrary elements of X. Obviously, there is a positive integer n such that $\|x/n\| \leqslant a/2$ and $\|y/n\| \leqslant a/2$. Therefore

$$U(x+y) = U\left(\frac{2n(x+y)}{2n}\right)$$
$$= 2nU\left(\frac{x+y}{2n}\right) = \frac{2n}{2}U\left(\frac{x}{n}\right)+U\left(\frac{y}{n}\right) = U(x)+U(y).$$

Hence the operator U is additive. Thus the continuity of U implies that U is continuous linear operator, q. e. d.

We do not know whether theorem 1 is true for arbitrary norms in locally bounded spaces. We can only prove

THEOREM 2. *Let X and Y be locally bounded spaces over reals with arbitrary F-norms $\|\ \|$. If a rotation U mapping X onto Y satisfies the identity*

$$\|tU(x)-tU(y)\| = \|tx-ty\|$$

for all positive t, then U is a linear operator.

Proof. In the same way as in [1] we define new norms in X and Y:

$$\|x\|' = \sup_{\substack{0<t,s\\ t+s<1}} (\|tx\|+\|sx\|).$$

The norms $\|\ \|'$ are equivalent to the given ones and concave. Moreover,

$$\|Ux-Uy\|' = \sup_{\substack{0<t,s\\ t+s<1}} \big(\|t(Ux-Uy)\|+\|s(Ux-Uy)\|\big)$$
$$= \sup_{\substack{0<t,s\\ t+s<1}} \big(\|t(x-y)\|+\|s(x-y)\|\big) = \|x-y\|'.$$

This means that U is an isometry with respect to the norms $\|\ \|'$. Therefore theorem 1 implies that U is a linear operator, q. e. d.

References

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