ANNALES
POLONICI MATHEMATICI
XX (1968)

Several theorems about uniform convergence of the Fourier series

by L. KARADŽIĆ (Beograd)

Let $f(x)$ be an integrable and periodic function with period 2π, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \, . \tag{1}$$

THEOREM 1. *If there exists a closed interval $\langle a, b\rangle$ $(-\pi < a < b < \pi)$ at whose points the following condition is satisfied:*

$$0 < 2\pi\delta < \varphi_2 - \varphi_1 + x < \ldots < \varphi_n - \varphi_{n-1} + x < 2\pi(1-\delta) \qquad (0 < \delta < \tfrac{1}{2}\,,\ x \in \langle a, b\rangle)\,, \tag{2}$$

where

$$\varphi_n = \arg(a_n + ib_n)\,,$$

and if

$$\begin{aligned} &\binom{2n}{k} |a_k + ib_k| < \binom{2n}{k+1} |a_{k+1} + ib_{k+1}|\,, \quad k = 1, 2, \ldots, n\,, \quad \textit{for} \quad n > n_0\,, \\ &\binom{2n}{k} |a_{2n-k} + ib_{2n-k}| < \binom{2n}{k+1} |a_{2n-k-1} + ib_{2n-k-1}|\,, \qquad k = 0, 1, \ldots, n \quad \textit{for} \quad n > n_0\,, \end{aligned} \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{|a_n + ib_n|}{\sqrt{n}} < \infty\,, \tag{4}$$

then series (1) *converges uniformly for $x \in \langle a, b\rangle$.*

Proof. If the partial sum of series (1) is written in the form

$$\begin{aligned} s_n(x) &= \frac{a_0}{2} \sum_{k=1}^{n} + a_k \cos kx + b_k \sin kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} \frac{a_k - ib_k}{a^k} (ae^{xi})^k + \frac{a_k + ib_k}{\bar{a}^k} (\bar{a}e^{-xi})^k \end{aligned}$$

and if we effect on it the following sequence of the successive transformations:

$$(ae^{xi})^k = \frac{(ae^{xi})^k - (ae^{xi})^{k+1}}{1 - ae^{xi}}, \qquad k = 1, 2, \ldots,$$

after the procedure shown in paper [1] the following sequence results:

$$S_n(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{k=1}^{n}\left[\frac{\Delta_{k-1}}{(e^{-xi} - a)^k} + \frac{\bar{\Delta}_{k-1}}{(e^{xi} - \bar{a})^k}\right] - R_n \tag{5}$$

$$= \frac{a_0}{2} + \mathrm{re}\left\{\sum_{k=1}^{n}\frac{\Delta_{k-1}}{(e^{-xi} - a)^k}\right\} - 2\,\mathrm{re}\{R_n\},$$

where

$$\Delta_{k-1} = a_k - ib_k - \binom{k-1}{1}(a_{k-1} - ib_{k-1})\,a + \ldots + (-1)^{k-1}(a_1 - ib_1)\,a^{k-1},$$

$$R_n = \frac{1}{2}\sum_{k=1}^{n-1}\left[\frac{\Delta'_{k-1}}{(e^{-xi} - a)^k}\,e^{(n-k+1)xi} + \frac{\bar{\Delta}'_{k-1}}{(e^{xi} - \bar{a})^k}\,e^{-(n-k+1)xi}\right],$$

$$\Delta'_{k-1} = a_n - ib_n - \binom{k-1}{1}(a_{n-1} - ib_{n-1})\,a + \ldots + (-1)^{k-1}(a_{n-k+1} - ib_{n-k+1})\,a^{k-1}.$$

The sum of sequence (5), where we consider the relations

$$a_n - ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,e^{-nti}\,dt,$$

$$\Delta_{k-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,(e^{-ti} - a)^{k-1} e^{-ti}\,dt, \tag{6}$$

$$\Delta'_{k-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,(e^{-ti} - a)^{k-1} e^{-(n-k+1)ti}\,dt,$$

may be written in the form

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{\sin(n+\frac{1}{2})(t-x)}{\sin\big((t-x)/2\big)} + 2\,\mathrm{re}\left\{\frac{ae^{-(t-x)i}}{e^{-xi} - a}\left(\frac{e^{-ti} - a}{e^{-xi} - a}\right)^{n-1}\right\}\right] dt. \tag{7}$$

Sequence (7) for $a = -e^{xi}$ implies the sum

$$S_n(x) = s_n(x) - \frac{\mathrm{re}\{\Delta'_{n-1}(e^{xi})\}}{2^n}, \tag{8}$$

where

$$\mathrm{re}\{\Delta'_{n-1}(e^{xi})\} = a_n\cos nx + b_n\sin nx + \binom{n-1}{1}\big(a_{n-1}\cos(n-1)x + \\ + b_{n-1}\sin(n-1)x\big) + \ldots + a_1\cos x + b_1\sin x .$$

So, from relation (5) for $a = -e^{-xi}$ and on the basis of relation (8) we obtain the relation

$$(9)\qquad s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx + b_k\sin kx \\ = \frac{a_0}{2} + \mathrm{re}\left\{\sum_{k=1}^{n}\frac{\Delta_{k-1}(e^{xi})}{2^k} + \sum_{k=1}^{n}\frac{\Delta'_{k-1}(e^{xi})}{2^k}\,e^{(n-k)xi}\right\}.$$

If on the closed interval $\langle a, b\rangle$ condition (2) is satisfied, according to van der Corput [2] there exists an inequality

$$\left|\sum_{k=1}^{n} e^{(\varphi_k+x)i}\right| < M/\delta \qquad (0<\delta<\tfrac{1}{2}),$$

where M is a number which is independent of φ_k, δ and of $x \in \langle a, b\rangle$. If we consider this inequality and hypothesis (3), and if

$$\binom{2n}{n} = O\left(\frac{2^{2n}}{\sqrt{n}}\right),$$

then

$$|\Delta_{2n+r}(e^{xi})| = O\left(\frac{2^{2n}}{\sqrt{n}}\,|a_n+ib_n|\right), \qquad r = 0, \pm 1,\ x\in\langle a, b\rangle .$$

According to the above we may get the following results:

$$\left|\frac{\Delta_{k-1}(e^{xi})}{2^k}\right| = O\left(\frac{|a_p+ib_p|}{\sqrt{p}}\right), \qquad p\to\infty \qquad \left(p = \left[\frac{k-1}{2}\right]\right),$$

$$\left|\frac{\Delta'_{k-1}(e^{xi})}{2^k}\right| = O\left(\frac{|a_q+ib_q|}{\sqrt{q}}\right), \qquad q\to\infty \qquad \left(q = \left[\frac{n-k}{2}\right]\right).$$

From these results and on the basis of hypothesis (4) we conclude that sequence (9) converges uniformly on the closed interval $\langle a, b\rangle$.

COROLLARY. *If the sequence has positive terms*

$$(10)\qquad \left\{\frac{a_{n-1}b_n - a_nb_{n-1}}{a_{n-1}a_n + b_{n-1}b_n}\right\}$$

and is monotonically increasing and if conditions (3) *and* (4) *are satisfied, series* (1) *converges uniformly in the interval* $(2\pi\delta+\varphi_1-\varphi_2,\ 2\pi(1-\delta)-\varphi)$,

$\varphi = \lim\limits_{n\to\infty}(\varphi_n - \varphi_{n-1}) \leqslant \pi/2$, *but if sequence* (10) *has negative terms and is a monotonically increasing zero-sequence and if conditions* (3) *and* (4) *are satisfied, then series* (1) *converges uniformly in the interval* $\big(2\pi\delta + \varphi_1 - \varphi_2,\ 2\pi(1-\delta)\big)$.

Proof. If in relation (2) the following condition is satisfied

$$0 < \varphi_2 - \varphi_1 < \dots < \varphi_n - \varphi_{n-1} < \pi/2 ,$$

then sequence (10) has the positive terms and is monotonically increasing because

$$\operatorname{tg}(\varphi_n - \varphi_{n-1}) = \frac{a_{n-1}b_n - a_n b_{n-1}}{a_{n-1}a_n + b_{n-1}b_n},$$

but if

$$-\pi/2 < \varphi_2 - \varphi_1 < \dots < \varphi_n - \varphi_{n-1} < 0 \qquad (\varphi = 0),$$

then sequence (10) has negative terms and is a monotonically increasing zero-sequence. On the basis of Theorem 1 we obtain the following corollary:

THEOREM 2. *If*

$$\Delta_{k-1}(c_{n-k+1}) = O(1)\,, \quad n \to \infty\,, \quad \textit{for } k = 1, 2, \dots, n\,, \tag{11}$$

where

$$\Delta_{k-1}(c_{n-k+1}) = c_n - \binom{k-1}{1} c_{n-1} + \dots + (-1)^{k-1} c_{n-k+1}$$

$(c_n = a_n - ib_n,\ n = 1, 2, \dots)$, *then series* (1) *uniformly converges for* $(\pi/3 + \varepsilon,\ 5\pi/3 - \varepsilon)$, *where* ε *is a very small positive number.*

Proof. The sequence (7) for $a = 1$ gets this form

$$S_n(x) = s_n(x) + \operatorname{re}\left\{e^{xi} \frac{\Delta_{n-1}(c_1)}{(e^{-xi}-1)^n}\right\}.$$

From this relation and relation (5) we obtain the relation

$$s_n(x) = \frac{a_0}{2} + \operatorname{re}\left\{\sum_{k=1}^{n} \frac{\Delta_{k-1}(c_1)}{(e^{-xi}-1)^k} - \sum_{k=1}^{n} \frac{\Delta_{k-1}(c_{n-k+1})}{(e^{-xi}-1)^k} e^{(n-k+1)xi}\,.\right.$$

If condition (11) is satisfied, this sequence converges uniformly in the interval $(\pi/3 + \varepsilon,\ 5\pi/3 - \varepsilon)$.

References

[1] L. Karadžić, *Sur un procédé pour le prolongement analytique de la série de Taylor*, C. R. Acad. Sc. Paris, 259 (1965).

[2] J. G. van der Corput, *Zahlentheoretische Abschätzungen*, Math. Ann. 84 (1921).

Reçu par la Rédaction le 30. 10. 1965