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A discrete boundary value problem

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In the present note we are concerned with the question of the uniqueness and the existence of solutions of two-point boundary value problem for non-linear difference equations. In the proof of uniqueness we follow the idea of M. P. Colautti [2] and Z. Opial [7] to use inequalities of the Wirtinger's type ([1], p. 140). The proof of existence is based on the Brouwer theory of topological degree of continuous mappings in euclidean space. Finally, we prove that, as the mesh is refined to zero, the solutions of the finite difference problem converge to a solution of an ordinary differential equation. Similar results for equation with right-hand sides satisfying certain condition of monotonity were obtained be M. Lees [5].

Section 1 contains discrete analogs of Wirtinger's and Opial's inequalities. In section 2 "a priori" estimates for the solutions of the boundary value problem are given. The main result, uniqueness and existence theorems, is formulated and proved in section 3. The approximation theorem is given in section 4.

1. Consider the space R^{n+1} (R denotes the real line) of sequences $u = (u_0, \ldots, u_n)$ of real numbers, with the usual scalar product

$$(u, v) = \sum_{i=0}^{n} u_i v_i$$

and the norm $\|u\| = (u, u)^{1/2}$. The difference operators Δ: $R^{n+1} \to R^{n+1}$ and δ^2: $R^{n+1} \to R^{n+1}$ are defined by the formulae

$$\Delta u_i = \begin{cases} u_{i+1} - u_i, & i = 0, \ldots, n-1, \\ 0, & i = n; \end{cases}$$

$$\delta^2 u_i = \begin{cases} u_{i+1} - 2u_i + u_{i-1}, & i = 1, \ldots, n-1, \\ 0, & i = 0,\ i = n. \end{cases}$$

By $|u|$ ($u \in R^{n+1}$) we denote the sequence $(|u_0|, \ldots, |u_n|)$.

The following discrete analog of Wirtinger's inequality is due to Ky Fan, O. Taussky and J. Todd [3]:

THEOREM 1.1. *If $u \in R^{n+1}$ satisfies the homogeneous boundary value condition*

$$u_0 = 0, \quad u_n = 0, \tag{1.1}$$

then

$$\|u\| \leqslant \lambda_n \|\Delta u\|, \quad \lambda_n = \frac{1}{2\sin(\pi/2n)}, \tag{1.2}$$

and the strong inequality holds unless

$$u_i = C \sin\frac{\pi i}{n}, \quad i = 0, \ldots, n.$$

For the convenience of the reader we shall give a proof of Theorem 1.1 based on the theory of finite Fourier series. It is well known (see e.g. [9], p. 39) that the vectors $u^1, \ldots, u^{n-1}$ such that

$$u_i^k = \sqrt{\frac{2}{n}} \sin\frac{k\pi i}{n}, \quad i = 0, \ldots, n; \ k = 1, \ldots, n-1,$$

form an orthonormal base in the subspace of R^{n+1} determined by (1.1). A straightforward calculation gives also

$$(\Delta u^k, \Delta u^l) = 4\delta_{kl} \sin^2\frac{k\pi}{2n}, \quad k, l = 1, \ldots, n-1.$$

Setting

$$u = \sum_{k=1}^{n-1} a_k u^k$$

we obtain, by the Parseval equality,

$$\|u\|^2 = \sum_{k=1}^{n-1} a_k^2. \tag{1.3}$$

On other hand, we have

$$\begin{aligned} \|\Delta u\|^2 = (\Delta u, \Delta u) &= \Big(\sum_{k=1}^{n-1} a_k \Delta u^k, \sum_{l=1}^{n-1} a_l \Delta u^l\Big) \\ &= \sum_{k,l=1}^{n-1} a_k a_l (\Delta u^k, \Delta u^l) = 4 \sum_{k=1}^{n-1} a_k^2 \sin^2\frac{k\pi}{2n}. \end{aligned} \tag{1.4}$$

From (1.3) and (1.4) the conclusion of the theorem is immediate.

Combining Theorem 1.1 and the Cauchy inequality we obtain

$$(|u|, |\Delta u|) \leqslant \lambda_n \|\Delta u\|^2.$$

However, in this inequality the constant λ_n is not the best possible and we shall prove the following discrete analog of an Opial's inequality [8] (see also [6]):

THEOREM 1.2. *If* $u \in R^{n+1}$ *satisfies condition* (1.1), *then*

$$(1.5)\qquad (|u|, |\Delta u|) \leqslant \mu_n^2 \|\Delta u\|^2, \quad \mu_n^2 = \frac{1}{2}\left[\frac{n+1}{2}\right] (^1) .$$

For even n *the constant* μ_n *is the best possible.*

Proof. In view of the boundary value condition (1.1) we may write

$$u_i = \sum_{k=0}^{i-1} \Delta u_k, \quad u_i = -\sum_{k=i}^{n-1} \Delta u_k, \quad i = 1, \ldots, n-1 .$$

Hence

$$\begin{aligned}(|u|, |\Delta u|) &= \sum_{i=1}^{s} |u_i||\Delta u_i| + \sum_{i=s+1}^{n-1} |u_i||\Delta u_i| \\ &\leqslant \sum_{i=1}^{s} |\Delta u_i| \sum_{k=0}^{n-1} |\Delta u_k| + \sum_{i=s+1}^{n-1} |\Delta u_i| \sum_{k=i}^{n-1} |\Delta u_k| .\end{aligned}$$

Using the inequality $ab \leqslant \frac{1}{2}(a^2+b^2)$ and putting $s = \left[\frac{n+1}{2}\right]$ we have

$$\begin{aligned}(|u|, |\Delta u|) &\leqslant \frac{1}{2} \sum_{i=1}^{s} \sum_{k=0}^{i-1} (|\Delta u_i|^2 + |\Delta u_k|^2) + \frac{1}{2} \sum_{i=s+1}^{n-1} \sum_{k=i}^{n-1} (|\Delta u_i|^2 + |\Delta u_k|^2) \\ &= \frac{s}{2} \sum_{i=0}^{s} |\Delta u_i|^2 + \frac{n-s}{2} \sum_{i=s+1}^{n-1} |\Delta u_i|^2 \leqslant \frac{1}{2} \max(s, n-s) \sum_{i=0}^{n-1} |\Delta u_i|^2 \\ &= \mu_n^2 \|\Delta u\|^2 .\end{aligned}$$

To complete the proof it is sufficient to verify that $(|v|, |\Delta v|) = \frac{1}{4} n \|\Delta v\|^2$ for even n and $v_i = \frac{1}{2}n - |i - \frac{1}{2}n|$.

2. Theorems 1.1 and 1.2 permit us to deduce "a priori" estimates for the solutions of two-point boundary value problem.

THEOREM 2.1. *Suppose that* $u \in R^{n+1}$ *satisfies condition* (1.1) *and the inequality*

$$(2.1)\qquad |\delta^2 u_i| \leqslant A|u_i| + B|\Delta u_i| + C, \quad i = 1, \ldots, n-1 .$$

If the constants A, B, C *are non-negative and if*

$$(2.2)\qquad \varrho_n = \lambda_n^2 A + \mu_n^2 B < 1 ,$$

(1) $[x]$ denotes the whole part of x.

then

$$\|u\| \leqslant \sqrt{n-1}\,\frac{\lambda_n^2 C}{1-\varrho_n}, \quad \|\Delta u\| \leqslant \sqrt{n-1}\,\frac{\lambda_n C}{1-\varrho_n}. \tag{2.3}$$

Proof. Multiplying (2.1) by $|u_i|$ and summing over i we get

$$(|u|, |\delta^2 u|) \leqslant A\|u\|^2 + B(|u|, |\Delta u|) + C\sum_{i=1}^{n-1} |u_i|.$$

Hence using Theorems 1.1, 1.2 and the Cauchy inequality we may write

$$(|u|, |\delta^2 u|) \leqslant \varrho_n \|\Delta u\|^2 + \lambda_n C\sqrt{n-1}\,\|\Delta u\|. \tag{2.4}$$

On other hand, from (1.1) it follows

$$\|\Delta u\|^2 = (\Delta u, \Delta u) = -(u, \delta^2 u) \leqslant (|u|, |\delta^2 u|). \tag{2.5}$$

Comparing (2.4) and (2.5) we obtain the second of inequalities (2.3). The first one is the consequence of Theorem 1.1 and of the second one.

3. Now consider the difference equation

$$\delta^2 u_i = g_i(u_i, \Delta u_i), \quad i = 1, \ldots, n-1, \tag{3.1}$$

and the boundary value condition

$$u_0 = \alpha, \quad u_n = \beta. \tag{3.2}$$

Here α and β are given real numbers and $g_i(v_0, v_1)$ are given real-valued functions defined for $(v_0, v_1) \in R^2$.

THEOREM 3.1. *If the functions $g_i(v_0, v_1)$ are continuous and if they satisfy the inequalities*

$$|g_i(v_0, v_1)| \leqslant A|v_0| + B|v_1| + C, \tag{3.3}$$

where the constants A, B, C are non-negative and such that (2.2) *holds, then there exist at least one solution of* (3.1), (3.2).

Proof. By substitution

$$v_i = u_i - \alpha - \frac{i}{n}(\beta - \alpha)$$

we can reduce the difference problem (3.1), (3.2) to the same with homogeneous boundary value conditions. Thus, in the sequel we consider without loss of generality only the case $\alpha = \beta = 0$.

Write

$$\Gamma_{ij} = \begin{cases} \dfrac{ij}{n} - j, & i \geqslant j, \\[2mm] \dfrac{ij}{n} - i, & i < j \end{cases}$$

and define for each $t \in \langle 0, 1\rangle$ the mapping $v = F_t(u)$ of R^{n+1} into itself by the formula

$$v_i = u_i - t\sum_{j=1}^{n-1} \Gamma_{ij} g_j(u_j, \Delta u_j)\,, \qquad i = 0, \ldots, n\,.$$

It is easy to see that any vector $u \in R^{n+1}$ satisfying the condition $F_t(u) = 0$ is a solution of the difference problem

$$\delta^2 u_i = t g_i(u_i, \Delta u_i)\,, \qquad u_0 = 0\,, \qquad u_n = 0\,.$$

Write

$$S = \left\{u\colon\ \|u\| = \sqrt{n-1}\,\frac{\lambda_n^2 C}{1-\varrho_n} + 1\right\}.$$

From (3.3) and Theorem 2.1 it follows that $F_t(u) \neq 0$ for $u \in S$. Hence the degree of $F_t(u)$ at the origin is independent of t and, since $F_0(u) \equiv u$, it is equal to one. Thus there exist a solution of the equation $F_1(u) = 0$ which is the desired solution of the difference problem.

From Theorems 2.1 and 3.1 immediately follows

THEOREM 3.2. *If the functions $g_i(v_0, v_1)$ satisfy the Lipschitz conditions*

$$|g_i(w_0, w_i) - g_i(v_0, v_1)| \leqslant A\,|w_0 - v_0| + B\,|w_1 - v_1| \tag{3.4}$$

and (2.2) *holds, then there exists exactly one solution of* (3.1), (3.2).

In fact, the Lipschitz conditions (3.4) imply (3.3) and the existence of a solution is given by Theorem 3.1. Now suppose that $\bar{\bar{u}}$ and $\bar{u}$ are two solutions of (3.1), (3.2). Then $u = \bar{\bar{u}} - \bar{u}$ satisfies the homogeneous conditions (1.1) and inequality (2.1) with $C = 0$. Therefore, by Theorem 2.1, $\|u\| = 0$, which completes the proof.

4. The aim of this section is to obtain approximative solutions of the second order differential equation

$$x'' = f(t, x, x')\,, \qquad a \leqslant t \leqslant b, \tag{4.1}$$

satisfying the boundary value condition

$$x(a) = \alpha\,, \qquad x(b) = \beta, \tag{4.2}$$

where the real-valued function $f(t, v_0, v_1)$ is continuous in the region

$$D\colon\ a \leqslant t \leqslant b\,,\ -\infty < v_0, v_1 < +\infty\,.$$

It is known [4] that if the function $f(t, v_0, v_1)$ satisfies the Lipschitz condition

$$|f(t, w_0, w_1) - f(t, v_0, v_1)| \leqslant K\,|w_0 - v_0| + M\,|w_1 - v_1| \tag{4.3}$$

and the constants K, M are such that

$$\varrho = K\frac{(b-a)^2}{\pi^2} + M\frac{b-a}{4} < 1\,, \tag{4.4}$$

then there exists one and only one solution of (4.1), (4.2).

Side by side of the differential equation (4.1) consider the difference equation

$$\delta^2 u_i = h_n^2 f\left(t_i^n, u_i, \frac{\Delta u_i}{h_n}\right), \qquad i = 1, \ldots, n-1\,, \tag{4.5}$$

together with the boundary value condition

$$u_0 = \alpha\,, \qquad u_n = \beta\,. \tag{4.6}$$

The sequence $(t_0^n, \ldots, t_n^n)$ and the mesh h_n are defined by the formulae

$$t_0^n = a\,, \qquad \Delta t_i^n = h_n = \frac{b-a}{n}, \qquad i = 0, \ldots, n-1\,.$$

THEOREM 4.1. *If the function $f(t, v_0, v_1)$ is continuous in D and if it satisfies conditions* (4.3), (4.4), *then*

1° *for sufficiently great n there exists exactly one solution u^n of the difference problem* (4.5), (4.6) *and*

2° $\lim\limits_{n\to\infty} |u_i^n - x(t_i^n)| = 0$ *uniformly in i, where $x(t)$ denotes the solution of the differential problem* (4.1), (4.2).

Proof. From (4.3) if follows that the functions

$$g_i^n(v_0, v_1) = h_n^2 f(t_i, v_0, v_1/h_n)$$

satisfy the Lipschitz conditions (3.4) with the constants $A_n = Kh_n^2$ and $B_n = Mh_n$. Set

$$\varrho_n = \lambda_n^2 A_n + \mu_n^2 B_n = K(b-a)^2\frac{\lambda_n^2}{n^2} + M(b-a)\frac{\mu_n^2}{n}\,.$$

By the definition of λ_n and μ_n (see (1.2), (1.5)) we have

$$\lim_{n\to\infty}\frac{\lambda_n}{n} = \frac{1}{\pi}, \qquad \lim_{n\to\infty}\frac{\mu_n^2}{n} = \frac{1}{4}\,. \tag{4.7}$$

Hence

$$\lim_{n\to\infty}\varrho_n = K\frac{(b-a)^2}{\pi^2} + M\frac{b-a}{4} = \varrho < 1 \tag{4.8}$$

and obviously $\varrho_n < 1$ for sufficiently great n. Thus 1° immediately follows from Theorem 3.2.

Write $x_i^n = x(t_i^n)$. There exist points $s_i^n, c_i^n \in (t_{i-1}^n, t_{i+1}^n)$ such that

$$\delta^2 x_i^n = h_n^2 x''(s_i^n)\,, \qquad \Delta x_i^n = h_n x'(c_i^n)\,, \qquad i = 1, \ldots, n-1\,.$$

In view of (4.1), (4.2) we find

$$x_0^n = \alpha\,, \qquad x_n^n = \beta \tag{4.9}$$

and

$$\delta^2 x_i^n = h_n^2 f\big(s_i^n, x(s_i^n), x'(s_i^n)\big)\,, \qquad i = 1, \ldots, n-1\,.$$

This equation may be written in the form

$$\delta^2 x_i^n = h_n^2 f(t_i, x_i^n, \Delta x_i^n/h_n) + r_i^n\,, \qquad i = 1, \ldots, n-1 \tag{4.10}$$

where

$$r_i^n = h_n^2\Big(f\big(s_i^n, x(s_i^n), x'(s_i^n)\big) - f\big(t_i^n, x(t_i^n), x'(c_i^n)\big)\Big)\,.$$

From the continuity of the functions $f(t, v_0, v_1)$, $x(t)$, $x'(t)$ it follows that

$$\lim_{n\to\infty} \frac{1}{h_n^2} \max_j |r_j^n| = 0\,. \tag{4.11}$$

Putting $v_i^n = u_i^n - x_i^n$, subtracting (4.5), (4.10) and using (4.3) we obtain

$$|\delta^2 v_i^n| \leqslant A_n |v_i^n| + B_n |\Delta v_i^n| + \max_j |r_j^n|\,, \qquad i = 1, \ldots, n-1\,.$$

Besides the boundary value conditions (4.6), (4.9) imply $v_0^n = 0$, $v_n^n = 0$. Hence Theorem 2.1 yields

$$\|\Delta v^n\| \leqslant \sqrt{n-1}\,\frac{\lambda_n}{1-\varrho_n} \max_j |r_j^n|\,.$$

On other hand, we have

$$|v_i^n| = \frac{1}{2}\Big|\sum_{k=0}^{i-1} \Delta v_k^n - \sum_{k=i}^{n-1} \Delta v_k^n\Big| \leqslant \frac{1}{2}\sum_{i=0}^{n-1} |\Delta v_i^n| \leqslant \frac{\sqrt{n}}{2}\|\Delta v^n\|$$

and consequently

$$2|v_i^n| \leqslant \frac{n\lambda_n}{1-\varrho_n} \max_j |r_j^n| = \frac{(b-a)^2}{1-\varrho_n} \cdot \frac{\lambda_n}{n} \max_j \frac{|r_j^n|}{h_n^2}\,.$$

Thus, to complete the proof it is sufficient to use (4.7), (4.8) and (4.11).

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