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Reçu par la Rédaction le 17. 7. 1967

Strongly cellular cells in E^3 are tame

by

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1. Introduction. Bing and Kirkor [4] have shown that a 1-cell in E^3 is tame if and only if it is strongly cellular. The purpose of this paper is to extend this result to 2-cells and 3-cells, and to use the concept to characterize tame 2-spheres. The main results are these.

THEOREM I. *If Z is a k-cell in E^3, $k = 1, 2$, or 3, then Z is tame if and only if it is strongly cellular.*

THEOREM II. *A 2-sphere in S^3 is tame if and only if each of its complementary domains has a strongly cellular closure.*

Numerous characterizations of tame cells are known. Cells of dimension one are treated in [9]. The 3-cell case reduces to the question of whether or not the boundary 2-sphere is tame. Useful characterizations of tame 2-cells and 2-spheres have been given by Bing [3], Burgess [6], Harrold [8], Hemple [10], and others. The criteria to be used here are, in the 3-cell case, the result due to Bing [2] that *tame 2-spheres* are those which can be approximated in each complementary domain, and in the 2-cell case, the result given in [7] that *tame 2-cells* are those which have both the strong enclosure and the hereditary disk properties.

2. Definitions and notation. The real interval $[0,1]$ will be denoted by I. A *homotopy* of S in T is a continuous function $h\colon S\times I\to T$ such that $h(x,0) = x$ for all x in S, and h_t will then denote the function given by $h_t(x) = h(x,t)$. If C is a cell, then $C^\bullet$ and C° will denote the *combinatorial boundary* and *combinatorial interior* of C. The set of all points in E^n lying within ε of some point of A is denoted by $B(A,\varepsilon)$.

A set Z in E^n is strongly cellular (Bing and Kirkor [4]) *if there is an n-cell C in E^n and a homotopy $H\colon C\times I\to C$ such that, if $S = C^\bullet$, then*

(1) *H_0 is the identity map, and $H_t|Z$ is the identity for all t,*

(2) *$H_t|S$ is a homeomorphism and $Z\cap H_t(S) = \emptyset$ for $t<1$,*

(3) *$H_t(S)\cap H_u(S) = \emptyset$ for $t\neq u$.*

(4) *$H_1(C) = Z$.*

* Work on this paper was supported by the National Science Foundation under NSF G-5458.

In the paper cited condition (2) appeared as (2′) *$H_t|S$ is a homeomorphism for $t<1$*. It can be shown, however, that every arc in E^3 has a corresponding homotopy satisfying (1), (2′), (3) and (4). In a private communication Bing writes "the definition of strongly cellular should include the condition $Z \cap H_t(S) = \emptyset$ for $t<1$".

In the next section the following will be proved.

2.1. THEOREM. *If Z is a compact subset of E^n with a connected complement, then Z is strongly cellular if and only if there is an $(n-1)$-sphere S in $E^n\setminus Z$ and a homotopy $h\colon S\times I \to E^n$ such that*

(1) *h_0 is the identity,*

(2) *h_t is an imbedding for $t<1$,*

(3) *$h_t(S) \cap h_u(S) = \emptyset$ for $t \neq u$, and*

(4) *$h_1(S) = \operatorname{Bdry} Z$.*

A set Z for which there is an S and h satisfying these conditions will be said to *have a cocoon*, and henceforth when the hypothesis that Z is strongly cellular or Z has a cocoon is made, the symbols S and h will be used as in this theorem.

It should be noted that Z may have a cocoon without being compact or having a connected complement. In this notation theorems I and II become: *Tame cells in E^3 are those having cocoons*, and, *a 2-sphere in E^3 is tame if and only if each of its complementary domains has a cocoon.*

3. Preliminaries.

A strongly cellular set Z clearly has a cocoon, and the remainder of the proof of Theorem 2.1 will only be sketched. First it is verified that if $U = S\times[0,1)$, then $h|U$ is an embedding. Next, since Z does not separate E^n, any point of the component of $E^n\setminus S$ containing the connected set Z is either in Z or can be joined to a point of $h(S\times 1/2)$ by an arc missing both Z and S, so $Q = Z\cup h(U)$ is the closure the bounded component of $E^n\setminus S$. Since the bicollared sphere $h(S\times 1/2)$ could be used for S, Q may be assumed to be an n-cell (Brown [5]). The inverse of $h|U$ throws each $x\in h(U)$ to a point $(y[x], \tau[x])$ of U. The map $H\colon Q\times I\to Q$ defined by

$$H(x,t) = \begin{cases} x, & (x,t)\in Z\times I,\\ x, & (x,t)\in h(U)\times I \text{ and } t\leqslant \tau[x],\\ h(y[x],t), & (x,t)\in h(U)\times I \text{ and } t\geqslant \tau[x]\end{cases}$$

can be shown to be the desired homotopy.

Two straight forward consequences of the Zoretti Theorem ([11], p. 109) will be used.

3.1. LEMMA. *If A and B are distinct components of a compact subset K of the 2-sphere S and $\varepsilon>0$, then there is a simple closed curve J in $B(A,\varepsilon)\setminus K$ which separates A and B on S.*

3.2. LEMMA. *If A is a compact connected proper subset of the 2-sphere S with arcwise connected complement in S and $\varepsilon>0$ there is a disk D in S with $A\subset D^\circ$ and $D\subset B(A,\varepsilon)$.*

4. Cocooned sets in E^3.

Throughout this section Z will denote a subset of E^3 with a cocoon and with boundary W.

4.1. LEMMA. *If A is an arcwise connected closed subset of W and $W\setminus A$ is simply connected, then $h_1^{-1}(A)$ is connected.*

Proof. Suppose that p and q are points of distinct components of $h_1^{-1}(A)$. Then by Lemma 3.1 there is a simple closed curve J in $S\setminus h_1^{-1}(A)$ separating p and q on S. Let B be an arc from p to q meeting $h(S\times I)$ only at p and q, and let C be an arc in A from $h_1(p)$ to $h_1(q)$. Then $K = B\cup h(p\times I)\cup C\cup h(q\times I)$ is a simple closed curve and J links K. But J is homotopic in $h(J\times I)$ to a loop in $h_1(J)$ and hence is null-homotopic in $h(J\times I)\cup(W\setminus A)$. This set does not meet K, which is a contradiction.

4.2. LEMMA. *Let K be a finite or locally finite 1-complex in W with vertices $\{a_m|\ m\in M\}$ and 1-simplexes $\{e_n|\ n\in N\}$. Let U be an open connected subset of S such that if $f = h_1|U$, then $|K|\subset f(U)$. Let $\{\varepsilon_n|\ n\in N\}$ be a collection of positive numbers. Suppose that each $f^{-1}(e_n)$ is connected and each $f^{-1}(a_m)$ is connected and has arcwise connected complement in S.*

Then there is a 1-complex K' in U which is isomorphic to K and a homeomorphism $g\colon |K|\to|K'|$ which throws each simplex of K onto its isomorph in K' and is such that $h_1 g$ leaves vertices fixed and moves points of e_n less than $\operatorname{dia}(e_n)+\varepsilon_n$ for each $n\in N$.

Proof. Using 3.2 a pairwise disjoint collection $\{D_m\colon m\in M\}$ of disks can be chosen so that, for each $m\in M$,

$$f^{-1}(a_m)\subset D_m^\circ,\qquad D_m\subset U,\qquad f(D_m)\subset B(a_m,\varepsilon_n)$$

for each e_n having a_m as a vertex, and $f(D_m)\cap e_n = \emptyset$ otherwise. Then choose a pairwise disjoint collection $\{V_n\colon n\in N\}$ of open subsets of W such that, for each $n\in N$, $e_n^\circ\subset V_n\subset B(e_n,\varepsilon_n)$, no a_m is in V_n, and $V_n\cap f(D_m) = \emptyset$ when a_m is not a vertex of e_n.

Choose for each $m\in M$ a point a'_m of $f^{-1}(a_m)$ and a homeomorphism of a round 2-ball onto D_m throwing the center of the round ball onto a'_m. The images of the radii of the round ball will be called the *radii* of D_m.

For each $n\in N$, if e_n has vertices a_m and a_k, then the open set $D_m^\circ\cup f^{-1}(V_n)\cup D_k^\circ$ contains the connected set $f^{-1}(e_n)$ and hence contains an arc t_n from a point p_{nm} of $D_m^\bullet$ to a point p_{nk} of $D_k^\bullet$ which is otherwise disjoint from $D_m\cup D_k$. Choose one such t_n for each n.

For each $m\in M$ and $n\in N$ such that a_m is a vertex of e_n let v_{nm} be the radius of D_m to the point p_{nm} of $D_m^\bullet$. For each $n\in N$, let $e'_n = v_{nm}\cup t_n\cup v_{nk}$, where a_m and a_k are the vertices of e_n.

The collection K' of all a'_m and all e'_n is a 1-complex in U isomorphic to K, and $g\colon |K|\to|K'|$ can be defined by letting $g|e_n$ be any homeomorphism of e_n onto e'_n throwing vertices onto their isomorphs. Clearly $h_1 g(a_m) = h_1(a'_m) = a_m$, for $a'_m \in h_1^{-1}(a_m)$. Let $x \in e_n$. Then

$$g(x) \in e'_n \subset D^\circ_m \cup f^{-1}(v_n) \cup D^\circ_k$$

and f throws this set into $B(e_n, \varepsilon_n)$ by choice. Hence $fg = h_1 g$ moves x less than $\operatorname{dia} e_n + \varepsilon_n$.

5. Two spheres and three cells. Throughout this section Z is assumed to be a subset of E^3 with a cocoon, and having as boundary a 2-sphere W.

5.1. Lemma. *If A is a point or arc in W, then $h_1^{-1}(A)$ is connected and has arcwise connected complement in S.*

Proof. Lemma 4.1 assures that $h_1^{-1}(A)$ is connected and since it is also closed, it suffices to show that $S\setminus h_1(A)$ is connected. But A is the intersection of a countable family $\{U_n\}$ of open disks with $U_1 \supset U_2 \supset \ldots$ For each n, if $D_n = W\setminus U_n$, then $h_1^{-1}(D_n)$ is connected. Since $S\setminus h_1^{-1}(A)$ is the union of all $h_1^{-1}(D_n)$, it is also connected.

5.2. Lemma. *If $\varepsilon > 0$, there is a homeomorphism $g\colon W\to S$ such that $h_1 g$ moves points less than ε.*

Proof. The set $\mathfrak{D}$ of all D° such that D is a disk in W of diameter less than ε is an open cover of the compact set W, so if T is any finite triangulation of W of sufficiently small mesh, one can associate with each 2-simplex θ of T a sub-disk $\Delta(\theta)$ of W such that $B(\theta, 2\operatorname{dia}\theta) \subset \Delta(\theta)$ and $\operatorname{dia}\Delta(\theta) < \varepsilon$.

Lemma 5.1 is used to verify that the hypotheses of Lemma 4.2 are fulfilled for $K = T^1$, $U = S$ and $\operatorname{dia}(e_n) = \varepsilon_n$ for each 1-simplex e_n of T. Thus there is a 1-complex K' in S and a homeomorphism $g_0\colon |T^1|\to|K'|$ such that $h_1 g_0$ leaves vertices fixed and moves points of each e_n less than $2\operatorname{dia} e_n$.

Let $\sigma_0 = \emptyset$ and let $\sigma_1, \ldots, \sigma_m$ be the 2-simplexes of T, and make the inductive assumption that g_0 has been extended to an imbedding g_{n-1} of $X_{n-1} = |T^1| \cup \sigma_0 \cup \ldots \cup \sigma_{n-1}$. The image of $X_{n-1}\setminus\sigma_n^\bullet$ may be assumed connected and therefore disjoint from one component C_n of $S\setminus g_{n-1}(\sigma_n^\bullet)$, so g_{n-1} can be extended to an imbedding g_n of X_n by mapping $\sigma_n^\bullet$ onto C_n. Thus $g = g_m\colon W\to S$ is a homeomorphism and the lemma is established if it can be shown that $h_1 g(\sigma) \subset \Delta(\sigma)$ for each 2-simplex σ of T. But suppose that $x \in \sigma$ and $h_1 g(x) \in W\setminus\Delta(\sigma)$; then there is an arc A in $W\setminus\Delta(\sigma)$ joining $h_1 g(x)$ to a vertex a of T. Since $h_1^{-1}(A)$ is connected, $(h_1 g)^{-1}(A)$ is connected and contains both x and $(h_1 g)^{-1}(a) = a$, so some point p of $(h_1 g)^{-1}(A)$ lies in $\sigma^\bullet$. But then $h_1 g(p)$ is in both A and $\Delta(\sigma)$, and these were taken disjoint.

5.3. Proof of Theorem II and the 3-cell case of Theorem I. Let W be a 2-sphere in S^3 and let Z and X be the components of $S^3\setminus W$. Let $\varepsilon > 0$ be assigned. Then if Z has a cocoon, since $h\colon S\times I\to S^3$ must be uniformly continuous, there is a $t \in (0,1)$ such that $d\big(h_t(y), h_1(y)\big) < \varepsilon/2$ for all y in S. By Lemma 5.2, there is a homeomorphism $g\colon W\to S$ such that $d\big(x, h_1 g(x)\big) < \varepsilon/2$ for all $x \in S$. Hence $h_t g\colon W\to X$ is a homeomorphism which moves points of W less than ε. Theorem II and the 3-cell case of Theorem I follow from the results of Bing [2].

6. The 2-cell case. Throughout this section Z will denote a 2-cell in E^3 with a cocoon.

6.1. Lemma. *Let A be a point or arc in $Z^\bullet$, or an arc or disk in Z meeting $Z^\bullet$ in a single point. Then $h_1^{-1}(A)$ is connected and $S\setminus h_1^{-1}(A)$ is arcwise connected.*

Proof. That $h_1^{-1}(A)$ is connected in each case is an immediate consequence of Lemma 4.1. In each case $A = \bigcap_n U_n$, where $\{U_n\}$ is a countable family of open disks with $Z\setminus U_n$ a closed disk for each n and $U_1 \supset U_2 \supset \ldots$ Then $h_1^{-1}(Z\setminus U_n)$ is connected for each n by Lemma 4.1, so $S\setminus h_1^{-1}(A) = \bigcup_n h_1^{-1}(Z\setminus U_n)$ is connected. It is also open, and hence is arcwise connected.

6.2. Lemma. *If p is a point or arc of Z°, then $h_1^{-1}(p)$ has at most 2 components.*

Proof. Suppose that A, B, and C are distinct components of $h_1^{-1}(p)$. Then if $a \in A$, $b \in B$, and $c \in C$,

$$V = h(a\times I) \cup h(b\times I) \cup h(c\times I) \cup p$$

is a connected set meeting Z in p. There is a polyhedral 2-sphere Σ separating p from $Z^\bullet$ in E^3, and a subdisk D of Z which is a neighborhood of p in Z and does not meet Σ. By Bing's approximation theorem [1], there is a disk W_0 such that $D \subset W_0$, W_0 is locally polyhedral at points of $W_0\setminus D$, and $W_0^\bullet$ is separated from p by Σ. Using standard techniques a disk W containing D and meeting Σ in $W^\bullet$ can be found. Then W divides the 3-cell which Σ bounds into two crumpled cubes E and F such that $E \cap F = W$ and $E \cup F$ is a neighborhood of p.

Thus there is an open interval $K = (t, 1)$ such that $h(a\times K)$, $h(b\times K)$ and $h(c\times K)$ lie in $E^\circ \cup F^\circ$. Two of these three, say $h(a\times K)$ and $h(b\times K)$ lie in one of the two sets, say E°. By Lemma 3.1 there is a simple closed curve J in $S\setminus h_1^{-1}(p)$ separating A and B on S. The distance 2ε between $h(J\times I)$ and V is positive, and E° is u.l.c. ([12], p. 66), so there is a $\delta > 0$ such that any two points of E° within δ of each other are the end points of an arc in E° of diameter less than ε. Choose points $x_0 = h_1(a), x_1, \ldots, x_m$

$= h_1(b)$ in p, each within $\delta/3$ of its successor, and corresponding points $y_0 = h_s(a)$ for some s in $(t, 1)$, $y_1, \ldots, y_m = h_s(b)$ for the same s, such that $y_i \in E^\circ \cap B(x_i, \delta/3)$ for each i. Let α_i be an arc in E° from y_{i-1} to y_i having diameter less than ε, $i = 1, \ldots, m$. Since δ may be taken less than ε, each point of α_i lies within 2ε of x_i, so the union of the α_i's does not meet $h(J \times I)$. Therefore this union contains an arc α from $h_s(a)$ to $h_s(b)$ lying in $h(S \times I^\circ)$ and missing $h(J \times I)$. This is impossible for $h(J \times I)$ separates $h(a \times I)$ and $h(b \times I)$ in $h(S \times I^\circ)$.

6.3. Lemma. *For each $p \in Z^\circ$ there is an $\varepsilon > 0$ such that $p \in f(Z)$ for every continuous function $f\colon Z \to Z$ which moves points of $Z^\bullet$ less than ε.*

Proof. It suffices to consider the case where Z is a round ball in the plane with p as center. Let ν be the radial projection from p of $Z \setminus \{p\}$ onto $Z^\bullet$ and note that if $f\colon Z \to Z$ misses p, then νf provides a null homotopy of $\nu f|Z^\bullet$. But for ε sufficiently small, whenever f moves points of $Z^\bullet$ less than ε, $\nu f|Z^\bullet$ has Brouwer degree one.

6.4. Lemma. *If p is a point of Z°, then $h_1^{-1}(p)$ has exactly two components, and they lie in distinct components of $h_1^{-1}(Z^\circ)$.*

Proof. Suppose not. Then by Lemma 6.2 there is a point p of Z° such that either case 1: $h_1^{-1}(p)$ has two components not separated by $h_1^{-1}(Z^\bullet)$ or case 2: $h_1^{-1}(p)$ is connected. In case 1 let $A = h_1^{-1}(p) \cup \alpha$, where α is an arc in $h_1^{-1}(Z^\circ)$ joining the components of $h_1^{-1}(p)$, and in case 2 let $A = h_1^{-1}(p)$. In either case, A is a compact connected subset of $h_1^{-1}(Z^\circ)$ containing $h_1^{-1}(p)$. Now $h_1^{-1}(Z^\bullet)$ is connected by Lemma 4.1 so by Lemma 3.1 there is a simple closed curve J in S separating A and $h_1^{-1}(Z^\bullet)$. This J bounds a disk E in S containing $h_1^{-1}(p)$, and $h_1(E)$ has positive distance η from $Z^\bullet$. Choose ε in $(0, \eta)$ corresponding to p as in Lemma 6.3. Let K be a triangulation of $Z^\bullet$ of mesh less than $\varepsilon/2$. By Lemma 6.1, K satisfies the hypotheses of Lemma 4.2 with $U = S$ and $\varepsilon_n = \varepsilon/2$ for each 1-simplex e_n of K. Thus there is a homeomorphism $g\colon Z^\bullet \to S$ such that $h_1 g$ moves points less than ε.

Now $g(Z^\bullet)$ is the common boundary of two disks D_1 and D_2 in S. For $i = 1, 2$ extend g to a homeomorphism $g_i\colon Z \to D_i$. By choice of ε, p is in both $h_1 g_1(Z) = h_1(D_1)$ and $h_1 g_2(Z) = h_1(D_2)$. Thus, since E contains $h_1^{-1}(p)$, it meets both D_1 and D_2, and hence meets their common boundary $g(Z^\bullet)$. But the existence of $y \in Z^\bullet$ with $g(y) \in E$ yields the contradiction

$$\eta = d[Z^\bullet, h_1(E)] \leqslant d\big(y, h_1 g(y)\big) < \varepsilon < \eta .$$

6.5. Lemma. *$h_1^{-1}(Z^\circ)$ has exactly two components, each of which is thrown onto Z° by h_1.*

Proof. By Lemma 6.4, it has at least two. Suppose that a, b, and c lie in distinct components of $h_1^{-1}(Z^\circ)$. Then there is an arc A in Z° containing the h_1 images of a, b, and c, so $h_1^{-1}(A)$ contains the three points. But by Lemma 6.2, $h_1^{-1}(A)$ has at most two components, so one of its components contains at least two of the points. This contradicts the choice of a, b, and c.

6.6. Lemma. *If C is a component of $h_1^{-1}(Z^\circ)$ and A is an arc or a point of Z°, then $C \cap h_1^{-1}(A)$ is connected and has an arcwise connected complement in S.*

Proof. That $C \cap h_1^{-1}(A)$ is connected is clear, for $h_1^{-1}(A)$ has but two components and one of them lies in the other component of $h_1^{-1}(Z^\circ)$. As a component of $h_1^{-1}(A)$, $C \cap h_1^{-1}(A)$ is compact, so it suffices to show that $S \setminus C \cap h_1^{-1}(A)$ is connected. Let p and q be points of this set and choose a disk D in $Z \setminus A$ containing both $h_1(p)$ and $h_1(q)$, and meeting $Z^\bullet$ in a single point. Then $h_1^{-1}(D)$ is connected, contains p and q, and misses $h_1^{-1}(A)$.

6.7. Lemma. *If the arc A spans the boundary of Z there is a 2-sphere Σ in E^3 meeting Z in A and separating the two components of $Z \setminus A$.*

Proof. Let P_0 and Q_0 be the components of $Z \setminus A$ and let $Q = \bar{Q}_0$. It is not difficult to see there is a locally finite triangulation T of Q_0 such that (i) to each 2-simplex σ of T there corresponds a sub-disk $\Delta(\sigma)$ of Q_0 such that $Q_0 \setminus \Delta(\sigma)$ is connected and $B(\sigma, 2\operatorname{dia}\sigma) \subset \Delta(\sigma)$, and (ii) if $\{\sigma_n\}$ is a sequence of 2-simplexes of T with $\lim_n \operatorname{dia}\Delta(\sigma_n) = 0$, then $\lim_n \operatorname{dia}\Delta(\sigma_n) = 0$. Let N be an open neighborhood in Z of $Q^\bullet \setminus A$ meeting neither A nor any simplex σ of T having no vertex in $Z^\bullet$. Let C_1 and C_2 be the two components of $h_1^{-1}(Z^\circ)$, and let $U_k = C_k \cup h_1^{-1}(N)$, $k = 1, 2$.

If σ is a simplex of T^1, then either Lemma 6.1 or Lemma 6.6 applies, so it can be verified that both the pair U_1, T^1 and the pair U_2, T^1 satisfy the hypotheses on the pair U, K in Lemma 4.2. Using $\operatorname{dia} e_n$ as ε_n in this lemma provides two complexes, K_1' in U_1 and K_2' in U_2, each isomorphic to T^1, and two homeomorphisms $g_1\colon |T^1| \to |K_1'|$ and $g_2\colon |T^1| \to |K_2'|$. It is claimed that by making proper choices in the construction of K_1' and K_2', $L' = K_1' \cup K_2'$ will be a complex and the common subcomplex $K_1' \cap K_2'$ will be the isomorph of the sub-complex $T^\bullet$ of T consisting of all simplexes of T lying in $Z^\bullet$.

To see that this can be done, let D_{1m}, V_{1n}, t_{1n}, etc., be the sets chosen in the Lemma 4.2 construction of K_1' and let D_{2m}, V_{2n}, t_{2n}, etc. be those chosen for K_2'. If $a_m \in Z^\circ$, choose D_{km} to be a subset of C_k, $k = 1, 2$. If $a_m \in Z^\bullet$, take $D_{1m} = D_{2m}$ and $a_{1m}' = a_{2m}'$. Since a_{km}' must be taken in $C_k \cap h_1^{-1}(a_m)$ when $a_m \in Z^\circ$, the vertices common to K_1' and K_2' are precisely those corresponding to vertices of $T^\bullet$, as desired. If e_n is a 1-simplex of T with $e_n^\circ \subset Z^\circ$, take $V_{1n} = V_{2n}$ to be a subset of Z°, forcing $t_{km} \subset C_k$, $k = 1, 2$. If $e_n \subset Z^\bullet$, take $V_{1n} = V_{2n}$ and $t_{1n} = t_{2n}$.

These choices clearly force the isomorphs of simplexes of $T^\bullet$ to be common to K_1' and K_2', while if $e_n \subset Z^\circ$, then $e_{1n}' \subset C_1$ and $e_{2n}' \subset C_2$ so $e_{1n}' \cap e_{2n}' = \emptyset$. In the remaining case where e_n has one vertex a_i in Z° and the other a_m in $Z^\bullet$, the inclusions $t_{kn} \subset C_{kn}$ for $k = 1, 2$ still hold, so $e_{1n}' \cap e_{2n}'$ is a subset of $D_{1m} = D_{2m}$. But the portion of e_{kn}' inside D_{km} is a radius from a_{km}' to a point of $D_{km}^\bullet$ lying in C_k, $k = 1, 2$, so $e_{1n}' \cap e_{2n}' = a_{1m}' = a_{2m}'$ as desired.

It is assumed that g_1 and g_2 agree on $|T^\bullet|$, and the two maps are now extended inductively and simultaneously to $|T| = Q_0$. The typical step, assuming that $\{\sigma_i\}$ is an ordering of the 2-simplexes of T and that both g_1 and g_2 have been extended to $|T^1| \cup \sigma_1 \cup \ldots \cup \sigma_{n-1} = X_{n-1}$ is to note that $g_1(X_{n-1}) \cup g_2(X_{n-1}) \backslash g_1(\sigma_n^\bullet) = Y_n$ is connected and misses $g_1(\sigma_n^\bullet)$, so g_1 can be extended to throw σ_n° onto that component of $S \backslash g_1(\sigma_n^\bullet)$ not meeting Y_n. Then $g_1(X_{n-1}) \cup g_2(X_{n-1}) \cup g_1(\sigma_n) \backslash g_2(\sigma_n^\bullet)$ is connected, so g_2 can be extended.

The proof that each $h_1 g_k$ throws each σ into $\Delta(\sigma)$ is the same as the corresponding part of Lemma 5.2.

Now for $k = 1, 2$ a map $f_k\colon Q \to h(S \times I)$ is defined by taking $f_k(x) = x$ on A and

$$f_k[\varphi(x, t)] = h_t g_k[\varphi(x, t)] \text{ on } Q_0,$$

where φ is a homeomorphism of $I \times I$ onto Q throwing $I \times 1$ onto A. To show each f_k is an imbedding, it suffices to show that if $\{y_n\} \subset Q_0$ converges to $y \in A$ then $\{f_k(y_n)\}$ does also. Let $y_n = \varphi(x_n, t_n)$ so that $f_k(y_n) = h_{t_n} g_k(y_n)$ and note $\lim\limits_n t_n = 1$. For each n choose a 2-simplex σ_n of T with $y_n \in \sigma_n$ and a vertex a_n of σ_n. Note that $\{a_n\}$ converges to y and $\lim\limits_n \operatorname{dia} \Delta(\sigma_n) = 0$. Then

$$d[f_k(y_n), y] \leqslant d[h_{t_n} g_k(y_n), h_1 g_k(y_n)] + d[h_1 g_k(y_n), h_1 g_k(a_n)] + d[h_1 g_k(a_n), y].$$

Since h must be uniformly continuous on $S \times I$, the first term on the right is arbitrarily small for n sufficiently large. Since $h_1 g_k(a_n) = a_n$, the second term is less than $\operatorname{dia} \Delta(\sigma_n)$ and the third is $d(a_n, y)$, so $\lim\limits_n f_k(y_n) = y$.

Thus $f_1(Q)$ and $f_2(Q)$ are two disks meeting in their common boundary, so their union is a sphere Σ meeting Z in A. To see that Σ separates P_0 and Q_0, note first that h_1 throws both $g_1(Q_0)$ and $g_2(Q_0)$ into Q_0, so if $b \in h_1^{-1}(P_0)$, then $h(b \times I)$ is an arc missing Σ. This arc can be extended to a ray missing Σ so P_0 is exterior to Σ. On the other hand, if $a = \varphi(x, t)$ is a vertex of T, then $h[g_1(a) \times I]$ is seen to pierce Σ at $h_t g_1(a)$ so Q_0 is interior to Σ.

6.8. Proof of the 2-cell case of Theorem I. By Theorem 5.1 of [7], it suffices to show that a 2-cell with a cocoon has both the strong enclosure property and the hereditary disk property. A 2-cell Z has the strong enclosure property if to each pair of points a, b of $Z^\bullet$ there correspond rays A and B meeting Z in their initial points a and b and such that in each neighborhood of Z there is a 2-sphere enclosing Z and meeting both A and B in single points. A 2-cell with a cocoon clearly has this property.

A 2-cell Z has the hereditary disk property if given any sub-disk C of Z, any arc T which spans the boundary of C, and any $\varepsilon > 0$ there is a disk D such that

(i) $D^\bullet \cap C = \emptyset$,

(ii) $D \cap C$ spans the boundary of C and is within ε of T in the sense of a certain metric on arcs, and

(iii) If the two components of $C \backslash D$ are C_1 and C_2, then there is an $\eta > 0$ such that each connected set M with $\operatorname{dia}(M) < \eta$ which meets both C_1 and C_2, also meets D.

To see that a 2 cell Z with a cocoon has this property let T be an arc spanning the boundary of a sub-disk C of Z. Then T can be extended to an arc A spanning the boundary of Z and meeting C in T. A 2-sphere Σ meeting Z in A and separating the two components of $Z \backslash A$ can be chosen, by Lemma 6.7. Let D be a sub-disk of Σ with $A \subset D^\circ$. Then $D \cap C = T$, so the second condition above is satisfied for each $\varepsilon > 0$, no matter how the metric was defined. The first condition is evident, and that the third is also satisfied follows from the fact that Σ separates the components of $Z \backslash A$. Hence Z is tame.

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Reçu par la Rédaction 17. 8. 1967

Finitely additive measures and the first digit problem*

by

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1. Introduction. In his paper [4], R. S. Pinkham attempts to give a theoretical justification of the remarkable empirical conjecture that

(1) *the proportion of physical constants whose first significant digit lies between* 1 *and* n, *where* $1 \leqslant n \leqslant 9$, *is* $\log_{10}(n+1)$.

His approach consists of imposing 'reasonable' conditions on the distribution $F(x)$ of physical constants so as to yield (1) as a result. Two separate such considerations are given. In the first he argues that if every physical constant were multiplied by some real number $c > 0$, the resulting distribution $F(x/c)$ should agree with $F(x)$ regarding all data concerning first significant digits. This property of $F(x)$ is called scale invariance. Then in ([4], th. 1) it is shown that if the distribution $F(x)$ of physical constants is scale invariant and continuous, then (1) holds. His second argument consists of showing that (1) approximately holds independent of the specific nature of $F(x)$ and depending only on well-known statistical parameters associated with $F(x)$. In ([4], th. 2) bounds on this approximation are estimated in terms of the variation of the density function $f(x)$ associated with $F(x)$.

Our interest in this problem stems from the fact that several investigators have raised such questions as 'what is the probability that a natural number has property φ?' In this context we ask what is the probability that a natural number has a first significant digit which lies between 1 and n, where $1 \leqslant n \leqslant 9$. Our first task consists of giving a 'reasonable' definition of probability for natural numbers. This definition will then be tested against various sets of numbers to see whether it gives results which are in accord with our intuition. Finally, using a modified notion of scale invariance, we compute the probabilities of various sets associated with (1).

We use the following notation. Let $N = \{1, 2, 3, \ldots\}$ be the set of natural numbers, R the real numbers, and R^+ the non-negative real

* This paper is a report on some moonlighting performed by the authors while receiving support from the National Science Foundation via contracts GP 4368 and GP 5786, respectively.