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Tame singular integrals *

by

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Introduction. Let H be a real separable Hilbert space, B be a one-one Hilbert-Schmidt operator on H, and $y \to T_y$ be the regular representation of the additive group of H acting in $L^p(H)$, $1 < p < \infty$.

In [1] we studied singular integral operators

$$Z_p(f) = \lim_{\substack{\delta \downarrow 0 \\ \sigma \uparrow \infty}} \int_{\delta}^{\sigma} \Big[\int_H T_y f a(y/t)\, dn_{t^2} \circ B^{-1}(y) \Big] dt/t$$

acting on $L^p(H)$, where $\int_H a(y)\, dn \circ B^{-1}(y) = 0$ and $a(y)$ satisfies an integrability condition with respect to the Gaussian measure $n \circ B^{-1}$. In this note we shall restrict $a(y)$ to be either an absolutely integrable odd function or an r-power integrable even tame function for some $r > 1$. Under these conditions Z_p is a bounded operator on $L^p(H)$ as was shown in [1].

Extension of the results of the present note to the more general functions $a(y)$ used in [1] is a simple matter.

Singular integral operators Z_p generally map tame functions f in $L^p(H)$ to non-tame functions $Z_p(f)$. In this note we shall consider the tame singular integrals (introduced in [1]) which map tame functions to tame functions. Corresponding to each singular integral Z_p there is a net $\{(Z \circ Q^{-1}) \mid Q \in \mathscr{F}\}$ of tame singular integrals determined by the finite-dimensional orthogonal projections $Q \in \mathscr{F}$ on H and this net converges strongly to Z_p as Q tends strongly to the identity through the directed set $\mathscr{F}$. We shall prove this result in this note.

Preliminaries. We refer the reader to papers [3] and [4] of Gross and [5] of Segal for the measure theoretic preliminaries.

Definition (Segal). A *weak distribution* on a real Hilbert space H is an equivalence class F of linear maps from the conjugate space H^*

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of H to real-valued measurable functions on a probability space (depending on F). Two such maps, F and F', are *equivalent* if for any finite set of vectors $y_1, y_2, \ldots, y_k$ in H^* the two sets of measurable functions, $F(y_1), F(y_2), \ldots, F(y_k)$ and $F'(y_1), F'(y_2), \ldots, F'(y_k)$, have the same joint distribution in k-space. A weak distribution is *continuous* if a representative is continuous linear map (the range space has the topology of convergence in measure).

In what follows we shall be most interested in the normal distribution with variance parameter c, $c > 0$. This distribution is uniquely determined by the following properties:

(1) for any y in H^*, $n(y)$ is normally distributed with mean zero and variance $c\|y\|^2$;

(2) n takes orthogonal vectors to independent random variables.

The normal distribution is continuous. There is an essentially unique (up to expectation preserving isomorphism) probability space (S, Σ, μ) and a continuous linear map F from H^* to the real-valued measurable functions on (S, Σ, μ) such that F is a representative of the normal distribution. Σ has no proper sub-σ-field with respect to which all of the $F(y)$, $y \in H^*$, are measurable. The measurable functions on H are the measurable functions on (S, Σ, μ). $L^p(H, n_c) = L^p(S, \Sigma, \mu)$ and when $c = 1$ we set $n = n_1$ and $L^p(H, n) = L^p(H)$. The expectation $E(f)$ of a function f in $L^1(N, n_c)$ is $E(f) = \int_S f\,d\mu$.

A function $f(x)$ on the points of H is a *tame function* if there is a Baire function g on a finite-dimensional Euclidean space E_k and orthonormal vectors $h_1, h_2, \ldots, h_k$ in H^* such that $f(x) = g\big((x, h_1), \ldots, (x, h_k)\big)$. The span of the h_i, $i = 1, \ldots, k$, in H is called the *base space* of f. If F is a representative of the normal distribution and $f(x)$ is a tame function as above, $\tilde{f}(s) = g\big(F(h_1)(s), \ldots, F(h_k)(s)\big)$ is a measurable function on H. The expectation of f is

$$E(\tilde{f}) = (2\pi c)^{-k/2} \int_{E_k} g(t) \exp\left[\frac{-\|t\|^2}{2c}\right] dt,$$

where k is the dimension of the base space of f. This equality holds in the sense that if either side exists and is finite, then so does the other and the two are equal.

Several very useful representatives of the normal distribution are known. Of these the one in which we shall be most interested is the mapping studied by Gross (in [4]) from H^* to Borel measurable functions on an abstract Wiener space. We adopt the notation and terminology of [4]. Let B be a one-one Hilbert-Schmidt operator on a real separable Hilbert space H. Then $|x|_1 = \|Bx\|$ is a measurable norm on H. Let H_B denote the completion of H in this norm. Let $\mathscr{S}$ denote the σ-field generated by the closed sets in H_B. n_c induces a Borel probability measure N_c on H_B such that the extension of the identity map on H_B^* $(\subset H^*)$, regarded as a densely defined map on H^* to measurable functions on $(H_B, \mathscr{S}, N_c)$, to H^* is a representative of the normal distribution on H.

Continuous functions f on H_B are measurable functions on H and if g denotes the restriction of f to H and if $\mathscr{F}$ denotes the directed set (ordered by inclusion of the ranges) of finite-dimensional orthogonal projections on H, then the net $\{g(\widetilde{Qx}) \mid Q \in \mathscr{F}\}$ of measurable tame functions converges in measure to f as Q tends strongly to the identity through $\mathscr{F}$.

Let N_c be as above. We may regard B as an isometry from H_B to H. Hence $N_c \circ B^{-1}$ is a Borel measure on H. This measure is usually denoted by $n_c \circ B^{-1}$. If f is a bounded continuous function from H to a Banach space E, then

$$\int_H f(x)\, dn_c \circ B^{-1}(x) = \int_{H_B} f(By)\, dN_c(y) = E(f \circ \tilde{B}).$$

Tame singular integrals. Let B be a one-one Hilbert Schmidt operator on H. We may now rewrite the singular integral operator Z_p as

$$Z_p(f) = \lim_{\substack{\delta \downarrow 0 \\ \varrho \uparrow \infty}} \int_\delta^\varrho \Big[\int_{H_B} T_{tBy} f A(y)\, dn(y) \Big] dt/t,$$

where $A(y) = a(By)$. Let Q be a finite-dimensional orthogonal projection on H and let $\mathscr{F}$ denote the directed set (ordered by inclusion of the ranges) of finite-dimensional orthogonal projections on H. The tame singular integral operators corresponding to Z_p are

$$(Z \circ Q^{-1})_p(f) = \lim_{\substack{\delta \downarrow 0 \\ \varrho \uparrow \infty}} \int_\delta^\varrho \Big[\int_{H_B} T_{tQBy} f A(y)\, dn(y) \Big] dt/t.$$

The approximate tame singular integral operators are the

$$(Z \circ Q^{-1})_p^{\delta\varrho}(f) = \int_\delta^\varrho \Big[\int_{H_B} T_{tQBy} f A(y)\, dn(y) \Big] dt/t.$$

Tame operators have the advantage that they map tame functions to tame functions. The following result has been applied in [2]:

THEOREM. *Let Z_p be a bounded singular integral operator on $L^p(H)$ as described above. Let Q be a finite-dimensional orthogonal projection on H and $(Z \circ Q^{-1})_p$ be the tame singular integral operator corresponding to Z_p which is determined by Q. Z_p is the strong limit of the net $\{(Z \circ Q^{-1})_p \mid Q \in \mathscr{F}\}$ as Q tends strongly to the identity through $\mathscr{F}$.*

Proof. We shall assume that $A(y)$ is either an absolutely integrable odd function or an even tame function with $E(A) = 0$; initially we shall assume also that A is bounded.

One can see from modifications of the proofs of the main theorems of [1] that the tame singular integral operators $(Z\circ Q^{-1})_p$ are uniformly bounded in Q and that the approximate tame singular integral operators $(Z\circ Q^{-1})_p^{\delta\varrho}$ are uniformly bounded in Q, δ, and ϱ. We shall begin our proof by showing that if f is a boundedly differentiable tame function, then $(Z\circ Q^{-1})_p^{\delta\varrho}(f)$ converges uniformly in Q for sufficiently large Q as $\delta\downarrow 0$ and $\varrho\uparrow\infty$.

$$\|(Z\circ Q^{-1})_p^{\delta\varrho}(f)-(Z\circ Q^{-1})_p^{rR}(f)\| \leqslant \|(Z\circ Q^{-1})_p^{\delta r}(f)\|_p+\|(Z\circ Q^{-1})_p^{\varrho R}(f)\|_p .$$

Since $\int_{H_B} A(y)\,dn(y) = 0$,

$$(Z\circ Q^{-1})_p^{\delta r}(f) = \int_\delta^r \Big[\int_{H_B} (T_{tQBy}f-f)A(y)\,dn(y)\Big]dt/t .$$

Since f is boundedly differentiable, $\|T_{tQBy}f-f\|_p$ is dominated by a constant multiple of $t\|By\|$ and by Minkowski's integral inequality

$$\|(Z\circ Q^{-1})_p^{\delta r}(f)\|_p \leqslant \text{const}\int_\delta^r\int_{H_B}\|By\|\,dn(y)\,dt .$$

Since $E(\|By\|)$ is dominated by the Hilbert-Schmidt norm of B (see [3]), $(Z\circ Q^{-1})_p^{\delta r}(f)$ tends to zero in p-norm as δ and r tend to zero. $\|(Z\circ Q^{-1})_p^{\varrho R}(f)\|_p \leqslant \int_{H_B}\|\int_R^\varrho T_{tQBy}f\,dt/t\|_p\,|A(y)|\,dn(y)$ by Fubini's theorem and Minkowski's integral inequality. Let V denote the finite-dimensional orthogonal projection onto the base space of f. Suppose that $Q > V$. Then $T_{tQBy}f = (T_{tVBy}f)D_p\big(\cdot, t(Q-V)By\big)$, where $D_p(x,y) = \exp[(x,y)/p - \|y\|^2/2p]$. For the remainder of the proof we may assume without loss of generality that $f\geqslant 0$. For each y in H_B, the functions on the right of this last equation are independent positive tame functions. The first is based on VH and the second is based in $(Q-V)H$. The product is based in QH; we write the normal distribution on QH as a product of the normal distributions on VH and $(Q-V)H$ and apply Minkowski's integral inequality to the integral over $(Q-V)H$ to conclude that

$$\Big\|\int_\varrho^R T_{tQBy}f\,dt/t\Big\|_p \leqslant \Big\|\int_\varrho^R T_{tVBy}f\,dt/t\Big\|_p .$$

Set $\omega = VBy\|VBy\|^{-1}$, use the fact that f is tame and based in VH and the fact that the normal distribution is rotationally invariant to

write

$$\Big\|\int_\sigma^R T_{tVBy}f\,dt/t\Big\|_p^p$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big|\int_{\varrho\|VBy\|}^{R\|VBy\|} f(x_1-t, x_2.., x_n)\exp\Big(\frac{x_1t}{p}-\frac{t^2}{2p}\Big)dt/t\Big|^p dn(x_1)\dots dn(x_n).$$

Let $1/p = 1/\alpha+1/\beta-1$ $(\alpha, \beta > 1)$ and apply Young's inequality to get

$$\|(Z\circ Q^{-1})_p^{\varrho R}(f)\|_p \leqslant \delta(\varrho, R)\int_{H_B}\|VBy\|^{(1-\alpha)/\alpha}|A(y)\,dn|(y)$$
$$\leqslant \delta(\varrho, R)\int_{H_B}\|VBy\|^{(1-\alpha)/\alpha}dn(y),$$

where $\delta(\varrho, R)$ tends to zero as $\varrho, R\to\infty$ and where $\delta(\varrho, R)$ is independent of Q. Let K_1 denote the kernel of VB on H. On $K_1^\perp$, VB is a one-one finite-dimensional operator mapping into H. So there is a constant C such that $\|y\|\leqslant C\|VBy\|$ for y in $K_1^\perp$. Write the normal distribution on $K_1^\perp$ in polar coordinates; if V is sufficiently large, the last integral is finite. It is easy to see from the definition of V that we may always choose V to be sufficiently large that this last integral is finite. Thus as ϱ and R tend to infinity, $\|(Z\circ Q^{-1})_p^{\varrho R}(f)\|_p$ converges to zero uniformly in Q for sufficiently large Q when f is a bounded tame function.

Thus if f is a bounded boundedly differentiable tame function and if $A(y)$ is bounded, then

$$\|Z_p(f)-(Z\circ Q^{-1})(f)\|_p$$
$$\leqslant \|Z_p(f)-Z_p^{\delta\varrho}(f)\|_p+\|Z_p^{\delta\varrho}(f)-(Z\circ Q^{-1})_p^{\delta\varrho}(f)\|_p+\|(Z\circ Q^{-1})_p^{\delta\varrho}(f)-(Z\circ Q^{-1}(f)\|_p .$$

For $\varepsilon > 0$ there is a δ_0 and a ϱ_0 such that first and third terms on the right side of this last inequality are each $< \varepsilon/3$ when $\varrho\geqslant\varrho_0$ and $\delta\leqslant\delta_0$. Fix $\varrho\geqslant\varrho_0$ and $\delta\leqslant\delta_0$. By the strong continuity of the regular representation of H acting on $L^p(H)$ and by the bounded convergence theorem, the second term on the right converges to zero as Q tends strongly to the identity through the directed set of finite-dimensional projections on H. Hence $\lim_{Q\to 1}\{(Z\circ Q^{-1})_p(f)\,|\,Q\in F\} = Z_p(f)$. Since the bounded boundedly differentiable tame functions are dense in $L^p(H)$, Z_p is the strong limit of the net $\{(Z\circ Q^{-1})_p\,|\,Q\in\mathscr{F}\}$.

Let $A(y)$ be an absolutely integrable odd function or an r-power integrable even tame function $(r > 1)$ satisfying $E(A) = 0$. For definiteness, let $A(y)$ be odd. Let $A_n(y)$ be a sequence of bounded Borel measurable odd functions on H_B which converge in $L^1(H)$ to $A(y)$. Let Z_p^n and $(Z\circ Q^{-1})_p^n$ denote the singular integral and tame singular integral operators determined by A_n and let Z_p and $(Z\circ Q^{-1})_p$ denote the singular integral and tame singular integrals determined by A. For f in $L^p(H)$,

$$\|Z_p(f)-(Z\circ Q^{-1})_p(f)\|_p$$
$$\leqslant \|Z_p(f)-Z_p^n(f)\|_p+\|Z_p^n(f)-(Z\circ Q^{-1})_p^n(f)\|_p+\|(Z\circ Q^{-1})_p^n(f)-(Z\circ Q^{-1})_p(f)\|_p.$$

As has been shown in [1], the first and third terms on the right are each dominated by a constant multiple of $\|A-A_n\|_1$. So for $\varepsilon>0$ there is an integer N such that for $n\geqslant N$, the first and third terms on the right of this inequality are each $<\varepsilon/3$. Fix $n\geqslant N$. By the above argument we know that the second term on the right converges to zero as Q tends strongly to the identity through $\mathscr{F}$. Thus Z_p is the strong limit of the net $\{(Z\circ Q^{-1})_p\,|\,Q\in\mathscr{F}\}$ when $A(y)$ is an absolutely integrable odd function. A similar argument completes the proof for even r-power integrable $(r\geqslant 1)$ tame functions $A(y)$ with $E(A)=0$.

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Some remarks on the multiple Weierstrass transform and Abel summability of multiple Fourier-Hermite series

by

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INTRODUCTION

The purpose of this paper is to extend to the m-dimensional case some theorems given in [2], [3] and [4] concerning the inversion formula of the Weierstrass Transform and the Abel summability of Fourier-Hermite series. The theorems of the present paper are referred to the measure

$$e^{-|x|^2}dx = e^{-\sum_1^m x_j^2}dx_1\ldots dx_m,$$

case which is not included in [2], [3], [4] and [6]; on the other hand, we also give maximal theorems with respect to Abel Summability of multiple Fourier-Hermite series and to the inversion formula for the multiple Weierstrass Transform.

The first part of the paper is devoted to the study of theorems of general character concerning differentiation of multiple integrals which have to be used in the second part, the specific problem.

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NOTATION

1. By x we denote a point $(x_1,\ldots,x_m)$ of the Euclidean m-dimensional space:

$$|x| = \Big(\sum_{j=1}^m x_j^2\Big)^{1/2}.$$

2. If μ is an elementary measure defined on $\boldsymbol{R}^m$, it is an additive function of the subsets of $\boldsymbol{R}^m$ which are finite union of m-dimensional intervals. The variation W of μ on a cube $Q\subset\boldsymbol{R}^m$ is defined in the following way:

$$W(Q) = \sup_{S\subset Q}\sum_{j=1}^l|\mu(S_j)|,\qquad S=\bigcup_{j=1}^l S_j,\qquad S_i\cap S_j=\emptyset\quad\text{if } i\neq j,$$