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A modern version of the E. Noether's theorems in the calculus of variations, II

by

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INTRODUCTION

In this part (for the part I, see [2]) we investigate the consequences of the invariance of the integral functional $\mathscr{J}$ with respect to variations given by vector fields belonging to an infinite-dimensional subspace of T_M. At first we formulate the Second Theorem of Noether for a simple case and then its general variant.

PRELIMINARIES

Let M be a vector bundle with a base E (an orientable, n-dimensional differentiable manifold of class C^∞) and with a standard fibre $F = \boldsymbol{R}^m$. By J we denote the jet-bundle of order j generated by cross-sections of the bundle M. For the sake of simplicity we have limited our considerations in part I to the case of $m = j = 1$.

Therefore, in the preliminaries we have an opportunity both to recall the results of the previous part and to give their brief formulation in the general case.

Let $\Gamma(M)$ be the set of cross-sections of the bundle M with the relatively compact domains. By D_u (resp. R_u) we denote the domain (resp. graph) of a cross-section $u \in \Gamma(M)$.

An *integral functional* is a function on $\Gamma(M)$ defined as

$$\Gamma(M) \ni u \to \mathscr{J}(u) := \int_{D_u} \mathscr{L}_u \in \boldsymbol{R}^1, \tag{1}$$

where $\mathscr{L}_u$ is given by a differentiable map

$$J \ni [u]_p \to L([u]_p) \in \bigwedge^n T_p^*(E) \subset \bigwedge^n T^*(E)$$

as

$$E \ni p \to \mathscr{L}_u(p) := L([u]_p) \in \bigwedge^n T^*(E).$$

The real algebra spanned by the integral functionals is denoted by $\mathscr{F}$. We define a vector subspace T_M as

$$T_M := \Big\{X \in \Gamma\big(M, T(M)\big) : \bigwedge_{m, m' \in M} \big(\pi(m) = \pi(m')\big) \Rightarrow (\pi_* X_m = \pi_* X_{m'})\Big\}.$$

Let $u \in \Gamma(M)$ and $X \in T_M$; then in a neighborhood of $R_u \subset M$ there exist integral curves Ψ of the vector field, where Ψ is a differentiable map

$$]-\varepsilon, \varepsilon[\times R_u \ni (t, m) \to \Psi_t(m) \in M.$$

Thus we have obtained a functional, $\mathfrak{X}_u$, on $\mathscr{F}$ defined as

$$\langle \mathscr{J}, \mathfrak{X}_u \rangle := \frac{d}{dt} \mathscr{J}(u_t)|_{t=0},$$

where $u_t \in \Gamma(M)$ is given by

$$R_{u_t} := \Psi_t(R_u).$$

We have also a vector field $\boldsymbol{X} \in \Gamma\big(\tilde{u}, T(J)\big)$, where $\tilde{u} := \{[u]_p \in J : p \in D_u\}$ and the vector $\boldsymbol{X}_{[u]_p} \in T_{[u]_p}(J)$ is represented by a curve

$$]-\varepsilon, \varepsilon[\ni t \to [u_t]_{\pi\Psi_t(u(p))} \in J.$$

It has been shown that both $\mathfrak{X}_u$ and $\boldsymbol{X}$ depend on $X|_{R_u}$ only. Hence we have three maps:

$$T_M \ni X \to h_u(X) := X|_{R_u},$$

$$T_M \ni X \to I \circ h_u(X) := \boldsymbol{X},$$

$$T_M \ni X \to H \circ I \circ h_u(X) := \mathfrak{X}_u.$$

Let us define vector spaces $W_M(u) := h_u(T_M)$,

$$W_J(u) := I \circ h_u(T_M), \qquad \mathfrak{W}_u := H \circ I \circ h_u(T_M).$$

It can be seen that for every $u \in \Gamma(M)$ the maps h_u, H are homomorphisms and the map I is an isomorphism. We define maps $\pi_{\tilde{u}}$, $i_{\tilde{u}}$ as

$$E \ni p \to \pi_{\tilde{u}}(p) := [u]_p \in J,$$

$$\tilde{u} \ni [u]_p \to i_{\tilde{u}}([u]_p) := [u]_p \in J.$$

If $\boldsymbol{X} \in H^{-1}(\mathfrak{X}_u)$, then

$$\langle \mathscr{J}, \mathfrak{X}_u \rangle = \int_{D_u} \pi_{\tilde{u}}^* \circ i_{\tilde{u}}^* \underset{\hat{\boldsymbol{X}}}{\pounds} \mathfrak{L}, \tag{2}$$

where $\mathfrak{L} \in \Gamma\big(J, \overset{n}{\bigwedge} T^*(J)\big)$ is the lift of L (see (1) in the first part), and $\hat{\boldsymbol{X}} \in \Gamma\big(J, T(J)\big)$ is such that

$$\bigwedge_{j \in J} \big(\pi(j) = p \in D_u\big) \Rightarrow (\pi_* \hat{\boldsymbol{X}}_j = \pi_* \boldsymbol{X}_{[u]_p}).$$

The vector spaces T_M and $\Gamma\big(E, \overset{l}{\bigwedge} T^*(E)\big)$ have the structure of a module over the ring $C^\infty(E)$.

The integrand in (2) defines a map from T_M to $\Gamma\big(E, \overset{n}{\bigwedge} T^*(E)\big)$, where $\boldsymbol{X} = I \circ h_u(X)$. This map can be uniquely expressed as the sum of two maps $[\mathscr{L}_u]$ and $d\mathscr{N}(\mathscr{L}_u)$, where

$$T_M \ni X \to [\mathscr{L}_u]_X \in \Gamma\big(E, \overset{n}{\bigwedge} T^*(E)\big),$$

$$T_M \ni X \to \mathscr{N}_X(\mathscr{L}_u) \in \Gamma\big(E, \overset{n-1}{\bigwedge} T^*(E)\big)$$

and the first one is linear in the sense of the module structure, i.e.

$$[\mathscr{L}_u]_{X+Y} = [\mathscr{L}_u]_X + [\mathscr{L}_u]_Y, \tag{3}$$

$$[\mathscr{L}_u]_{f \circ \pi X} = f[\mathscr{L}_u]_X \tag{4}$$

for $X, Y \in T_M, f \in C^\infty(E)$.

It can be shown that $[\mathscr{L}_u]_X$ depends only on the vertical component $X_\perp$ of $h_u(X)$; by horizontal vectors we mean vectors tangent to R_u.

Thus (2) can be written as

$$\langle \mathscr{J}, H \circ I \circ h_u(X) \rangle = \int_{D_u} [\mathscr{L}_u]_X + d\mathscr{N}_X(\mathscr{L}_u).$$

THE SECOND NOETHER'S THEOREM (SIMPLE CASE)

Let

$$S := \{X_f \in T_M : X_f = \pi^* f X + \pi^*(\underset{Y}{\pounds} f) X_1, f \in C^\infty(E)\}$$

where $X, X_1 \in T_M$, $Y \in \Gamma\big(E, T(E)\big)$, are fixed vector fields. Let us notice that for $X \in T_M$, $Y \in \Gamma\big(E, T(E)\big)$, $f \in C^\infty(E)$

$$(\underset{Y}{\pounds} f)[\mathscr{L}_u]_X = \underset{Y}{\pounds} f[\mathscr{L}_u]_X - f \underset{Y}{\pounds}[\mathscr{L}_u]_X$$

$$= Y \lrcorner df[\mathscr{L}_u]_X + d(Y \lrcorner f[\mathscr{L}_u]_X) - f \underset{Y}{\pounds}[\mathscr{L}_u]_X = d(Y \lrcorner f[\mathscr{L}_u]_X) - f \underset{Y}{\pounds}[\mathscr{L}_u]_X.$$

Hence, if $X_f \in S$, then

$$\begin{aligned}\langle \mathscr{J}, H\circ I\circ h_u(X_f)\rangle &= \int_{D_u} [\mathscr{L}_n]_{X_f} + d\mathscr{N}_{X_f}(\mathscr{L}_u)\\ &= \int_{D_u} [\mathscr{L}_u]_{\pi^* f X} + [\mathscr{L}_u]_{\pi^*(\underset{Y}{\pounds} f) X_1} + d\mathscr{N}_{X_f}(\mathscr{L}_u)\\ &= \int_{D_u} f[\mathscr{L}_u]_X + (\underset{Y}{\pounds} f)[\mathscr{L}_u]_{X_1} + d\mathscr{N}_{X_f}(\mathscr{L}_u)\\ &= \int_{D_u} f([\mathscr{L}_u]_X - \underset{Y}{\pounds}[\mathscr{L}_u]_{X_1}) + \int_{\partial D_u} \mathscr{N}_{X_f}(\mathscr{L}_u) + Y \lrcorner f[\mathscr{L}_u]_{X_1}.\end{aligned}$$

Taking into account that f is arbitrary and can vanish, together with its derivatives, on ∂D_u, we get

THE SECOND NOETHER'S THEOREM

$$\left(\begin{array}{l}\mathscr{J}\in\mathscr{F} \textit{ is invariant at}\\ \textit{a point } u\in\Gamma(M) \textit{ with}\\ \textit{respect to every } X_f\in S.\end{array}\right) \Rightarrow \Big([\mathscr{L}_u]_X - \underset{Y}{\pounds}[\mathscr{L}_u]_{X_1} \underset{D_u}{\equiv} 0\Big).$$

THE SECOND NOETHER'S THEOREM (GENERAL CASE)

Let $\Upsilon = (Y_1,\dots,Y_k)\in\prod^k \Gamma(E,T(E))$, $\omega\in\Gamma(E,\bigwedge^l T^*(E))$; then we define

$$\underset{\Upsilon}{\pounds}\omega := \underset{Y_1}{\pounds}\dots\underset{Y_k}{\pounds}\omega,\qquad \Upsilon^{-1} := (-Y_k,\dots,-Y_1).$$

It is easily seen that if $\Upsilon\in\prod^k\Gamma(E,T(E))$, $f\in C^\infty(E)\cong\Gamma(E,\bigwedge^0 T^*(E))$, $\omega\in\Gamma(E,\bigwedge^n T^*(E))$, $\dim E=n$, then there exists an $\omega'\in\Gamma(E,\bigwedge^{n-1}T^*(E))$ such that

(5) $$(\underset{\Upsilon}{\pounds} f)\omega = f\underset{\Upsilon^{-1}}{\pounds}\omega + d\omega'.$$

Let r be a positive integer; then, with the notation

$$\mathscr{Y} := (\Upsilon_1,\dots,\Upsilon_r)\in\prod_{k=1}^r\Big(\prod^k\Gamma(E,T(E))\Big),$$

where $\Upsilon_k\in\prod^k\Gamma(E,T(E))$, $k=1,\dots,r$, and $\mathscr{X} := (X,X_1,\dots,X_r)\in\prod^{r+1}T_M$, we define

$$\mathscr{Y}^{-1} := (\Upsilon_1^{-1},\dots,\Upsilon_r^{-1}),$$

$$[\mathscr{L}_u]_{\mathscr{X}} := ([\mathscr{L}_u]_X, [\mathscr{L}_u]_{X_1},\dots,[\mathscr{L}_u]_{X_r})\in\prod^{r+1}\Gamma(E,\bigwedge^n T^*(E)).$$

If $(\omega,\omega_1,\dots,\omega_r)\in\prod^{r+1}\Gamma(E,\bigwedge^l T^*(E))$, $l=0,1,\dots,n$, then

(6) $$\underset{\mathscr{Y}}{\pounds}(\omega,\omega_1,\dots,\omega_r) := (\omega, \underset{\Upsilon_1}{\pounds}\omega_1,\dots,\underset{\Upsilon_r}{\pounds}\omega_r).$$

An element $(\omega,\omega,\dots,\omega)\in\prod^{r+1}\Gamma(E,\bigwedge^l T^*(E))$ will be denoted, for simplicity, by ω. Because of $C^\infty(E)\cong\Gamma(E,\bigwedge^0 T^*(E))$, it follows from (6) that

$$\underset{\mathscr{Y}}{\pounds} f = (f,\underset{\Upsilon_1}{\pounds}f,\dots,\underset{\Upsilon_r}{\pounds}f)\in\prod^{r+1}C^\infty(E).$$

Let A and B be vector spaces with an external operation $A\times B\ni(a,b)\to ab\in B$; then we define

$$(\alpha|\beta) := \sum_{i=1}^k a_ib_i\in B \quad\text{and}\quad \theta\alpha := \sum_{i=1}^k a_i\in A,$$

where $\alpha=(a_1,\dots,a_k)\in\prod^k A$, $\beta=(b_1,\dots,b_k)\in\prod^k B$, $k=1,2,\dots$ Thus for $f\in C^\infty(E)$ we have

$$(\pi^*\underset{\mathscr{Y}}{\pounds}f|\mathscr{X}) = \pi^* fX + \sum_{k=1}^r \pi^*(\underset{\Upsilon_k}{\pounds}f)X_k\in T_M.$$

Let s be a positive integer and $\mathscr{Y}_i\in\prod_{k=1}^r\big(\prod^k\Gamma(E,T(E))\big)$, $\mathscr{X}_i\in\prod^{r+1}T_M$, $i=1,\dots,s$, are given; then we define

$$S := \Big\{X_f = \sum_i(\pi^*\underset{\mathscr{Y}_i}{\pounds}f|\mathscr{X}_i)\in T_M : f\in C^\infty(E)\Big\}.$$

It is easily seen that S is a vector space.

Now we can formulate

THE SECOND NOETHER'S THEOREM.

$$\left(\begin{array}{l}\mathscr{J}\in\mathscr{F} \textit{ is invariant at}\\ \textit{a point } u\in\Gamma(M) \textit{ with}\\ \textit{respect to every } X\in S.\end{array}\right) \Rightarrow \Big(\sum_i\theta\underset{\mathscr{Y}_i^{-1}}{\pounds}[\mathscr{L}_u]_{\mathscr{X}_i}\underset{D_u}{\equiv}0\Big).$$

Proof. Making use of (3), (4) and (5) we have

$$[\mathscr{L}_u]_X = \sum_i(\underset{\mathscr{Y}_i}{\pounds}f|[\mathscr{L}_u]_{\mathscr{X}_i}) = \sum_i(f|\underset{\mathscr{Y}_i^{-1}}{\pounds}[\mathscr{L}_u]_{\mathscr{X}_i}) + d\omega,$$

where $\omega \in \Gamma(E, \overset{n-1}{\bigwedge} T^*(E))$ depends on f and $\operatorname{supp}\omega \subset \operatorname{supp} f$. Obviously $\operatorname{supp}\mathscr{N}_X(\mathscr{L}_u) \subset \operatorname{supp} f$. Thus

$$\begin{aligned}\langle \mathscr{J}, H\circ I\circ h_u(X)\rangle &= \int_{D_u} [\mathscr{L}_u]_X + d\mathscr{N}_X(\mathscr{L}_u)\\ &= \int_{D_u} f \sum_i \theta \underset{\mathscr{Y}_i^{-1}}{\pounds} [\mathscr{L}_u]_{x_i} + \int_{\partial D_u} \mathscr{N}_X(\mathscr{L}_u) + \omega.\end{aligned}$$

Since f is arbitrary, the proof is completed.

The identity obtained is called the *generalized Bianchi identity*.

Remark. It is easy to notice that the set C of "parameter" functions must satisfy only the following conditions: 1° $C \subset C_0^\infty(E)$; 2° the set $\{\operatorname{supp} f : f \in C\}$ form a basis of neighbourhoods in E.

EXAMPLES

We shall apply both theorems of Noether to electrodynamics.

1. Let E be the Minkowski space (a Riemannian manifold) with a Riemannian metric $g \in \Gamma(E, \overset{2}{\otimes} T(E))$. There is a canonical field Δ of 4-forms on E which can be defined with a coordinate chart as

$$\Delta = |\det(\langle g, dx^i \otimes dx^j\rangle)|^{1/2} dx^1 \wedge \ldots \wedge dx^4.$$

We define the automorphism (forming a dual) of the algebra of tensor fields on E.

At first let us denote by g^* such an element of $\Gamma(E, \overset{2}{\otimes} T^*(E))$ that $c_1^1(g\otimes g^*)$ (cf. [1]) is the unity of the algebra $\Gamma(E, T(E)\otimes T^*(E))$ with a product given by $ab = c_1^2(a\otimes b)$, where $a, b \in \Gamma(E, T(E)\otimes T^*(E))$.

Let $v \in \Gamma(E, T(E))$ and $v^* \in \Gamma(E, T^*(E))$; then the dual elements $\tilde{v}$ and $\widetilde{v^*}$ are defined as

$$\tilde{v} := c_1^1(g^*\otimes v), \qquad \widetilde{v^*} := c_1^1(g\otimes v^*).$$

Since $\sim$ has to be an automorphism of the algebra of tensor fields, our definition is complete. It is easily seen that $\tilde{g} = g^*$.

2. In the theory called electrodynamics by states we mean sections F of the bundle $\overset{2}{\bigwedge} T^*(\mathcal{O})$ which satisfy the condition $dF = 0$ (the first pair of the Maxwell equations), where $\mathcal{O}$ is an open domain in E. For simplicity we assume that $\mathcal{O}$ is contractible. Thus for every electromagnetic field F there exists a global potential $u \in \Gamma(\mathcal{O}, T^*(\mathcal{O}))$ such that $F = du$.

In the following we shall deal with potentials $u \in \Gamma(\mathcal{O}, T^*(\mathcal{O}))$ as fundamental elements of electrodynamics. However, the results which have a physical interpretation will be formulated by using $F \in \Gamma(\mathcal{O}, \overset{2}{\bigwedge} T^*(\mathcal{O}))$. So we state $M = T^*(\mathcal{O})$.

In the fundamental structure of electrodynamics we have the variational principle given by the Lagrangian

$$J^1(M) \ni [u]_p \to L([u]_p) := \langle \widetilde{du}, du\rangle_p \Delta_p \in \overset{4}{\bigwedge} T_p^*(\mathcal{O}).$$

If we consider electromagnetic fields in the presence of currents we add to the above Lagrangian the map

$$J^1(M) \ni [u]_p \to L_1([u]_p) := \langle j, u\rangle_p \Delta_p \in \overset{4}{\bigwedge} T^*(\mathcal{O}),$$

where a current $j \in \Gamma(\mathcal{O}, T(\mathcal{O}))$ is given.

We state $K = L + L_1$ for electrodynamics with currents.

Let $X \in W_M(u)$; then we have a decomposition of X into the field tangent to R_u and the vertical field $X_\perp$. Since $X_\perp(u_p)$ is a vector tangent to the vector space M_p, it can be represented by $\omega_p \in M_p$. Let $\omega \in \Gamma(\mathcal{O}, M)$ be the field representing the vector field $X_\perp$; then $[\omega] \in \Gamma(\mathcal{O}, J^1(M))$ is the field representing the vector field $\boldsymbol{X}_\perp := I(X_\perp) \in W_J(u)$ in a similar way.

Since

$$\mathscr{L}_u(p) = L([u]_p) = \langle \widetilde{du}, du\rangle_p \Delta_p,$$

we have

$$\mathfrak{L}([u]_p) = \langle \widetilde{du}, du\rangle_p \pi^* \Delta_p,$$

where $\pi : J^1(M) \to E$.

We notice that $\underset{\boldsymbol{X}_\perp}{\pounds} \pi^*\Delta = \pi^* \underset{\pi_*\boldsymbol{X}_\perp}{\pounds} \Delta = 0$.

Thus

$$\begin{aligned}(\underset{\boldsymbol{X}_\perp}{\pounds} \mathfrak{L})([u]_p) &= \frac{d}{dt}\langle \widetilde{du} + t\widetilde{d\omega}, du + t\,d\omega\rangle|_{t=0}\Delta\\ &= 2\langle \widetilde{du}, d\omega\rangle \Delta.\end{aligned}$$

It is easy to verify that the map

$$\Gamma(\mathcal{O}, M) \ni \omega \to 2\langle \widetilde{du}, d\omega\rangle \Delta + 4d\langle \widetilde{du}\otimes\omega\otimes\Delta\rangle$$

is linear in the sense of the module structure $(\Gamma(\mathcal{O}, M), C^\infty(\mathcal{O}))$. $\langle\ \rangle$ denotes complete contraction, i.e. $\langle \widetilde{du}\otimes\omega\otimes\Delta\rangle = c_1^1\circ c_1^1(\widetilde{du}\otimes\omega\otimes\Delta)$. Thus

$$[\mathscr{L}_u]_X = [\mathscr{L}_u]_{X_\perp} = 2\langle \widetilde{du}, d\omega\rangle \Delta + 4d\langle \widetilde{du}\otimes\omega\otimes\Delta\rangle$$

and the Noether expression has the form

$$\mathscr{N}_X(\mathscr{L}_u) = 4\langle \omega\otimes\widetilde{du}\otimes\Delta\rangle + \langle \widetilde{du}, du\rangle\, Y \lrcorner \Delta,$$

where $Y = \pi_* X$.

In the presence of currents we have

$$[\mathscr{K}_u]_X = [\mathscr{L}_u]_X + \langle j, \omega\rangle \Delta .$$

3. Making use of the First Theorem of Noether we get the energy-momentum conservation law in electrodynamics as a consequence of the invariance of the action functional with respect to translations in E.

Let $a \in \Gamma(E, T(E))$ and $\varphi_t : E \to E$ be the diffeomorphisms generated by a.

We assign to $a \in \Gamma(E, T(E))$ a field $A \in T_M$ as follows: the vector $A_{u_p} \in T_{u_p}(M)$ is given by the curve

$$t \to \varphi^*_{-t}(u_p) = (\varphi^*_{-t}u)_{\varphi_t(p)} \in M_{\varphi_t(p)}.$$

It can be seen that A is projectible and $\pi_* A = a$.

Let $u \in \Gamma$; then the field $h_u(A) \in W_M(u)$ has the vertical component $A_\perp$ represented by $-\underset{a}{\pounds}\, u \in \Gamma(\mathcal{O}, M)$ (cf. 2). This can be shown as follows: let $f \in C^\infty(M)$; then

$$\begin{aligned}\langle h_u(A), df\rangle_{u_p} &= \frac{d}{dt} f\big((\varphi^*_{-t}u)_{\varphi_t(p)}\big)|_{t=0}\\ &= \frac{d}{dt} f\big((\varphi^*_{-t}u)_p\big)|_{t=0} + \frac{d}{dt} f(u_{\varphi_t(p)})|_{t=0}\\ &= \langle A_\perp, df\rangle_{u_p} + \langle Z, df\rangle_{u_p},\end{aligned}$$

where the vector field Z is tangent to R_u and the field $A_\perp$ is given by the curves $t \to (\varphi^*_{-t}u)_p$ which determines, in the vector spaces M_p, $p \in \mathcal{O}$, the vectors $(-\underset{a}{\pounds}\, u)_p \in M_p$. Thus from part I we have

$$\begin{aligned}\mathscr{N}_A(\mathscr{L}_u) &= N_{A_\perp}(\mathscr{L}_u) + a \,\lrcorner\, \pi_u^* \circ i_u^* \Omega\\ &= 4\langle \widetilde{du} \otimes \underset{a}{\pounds}\, u \otimes \Delta\rangle + \langle \widetilde{du}, du\rangle\, a \,\lrcorner\, \Delta .\end{aligned}$$

If a_i, $i = 1, \ldots, 4$, are independent translations in E, then from the First Theorem of Noether we get the energy-momentum conservation law

$$d\mathscr{N}_{A_i}(\mathscr{L}_u) = 0 \text{ on } \mathcal{O}, \quad i = 1, \ldots, 4.$$

4. Now we are going to find the generalized Bianchi identity in electrodynamics, connected with the gauge invariance.

The gauge transformations determine vertical fields $X_f \in T_M$, $f \in C^\infty(E)$ represented by $df \in \Gamma(\mathcal{O}, M)$ (cf. 2).

Let $Y_i \in \Gamma(\mathcal{O}, T(E))$, $i = 1, \ldots, 4$, form at every point $p \in \mathcal{O}$ a base in $T_p(E)$; let $\omega^i \in \Gamma(\mathcal{O}, T^*(E))$ be such that $\langle Y_i, \omega^k\rangle = \delta_i^k$, and let $X_i \in T_M$

be the field represented by ω^i, $i = 1, \ldots, 4$. Since $df = (\underset{Y_i}{\pounds} f)\,\omega^i$, we have

$$X_f = \sum_i (\pi^* \underset{Y_i}{\pounds} f) X_i .$$

In accordance with the previous notation

$$\mathscr{Y}_i = \Upsilon_i = Y_i, \quad \mathscr{X}_i = (0, X_i),$$

$$X_f = \sum_i (\pi^* \underset{\mathscr{Y}_i}{\pounds} f | \mathscr{X}_i),$$

$$\mathscr{Y}_i^{-1} = \Upsilon_i^{-1} = -Y_i, \quad [\mathscr{L}_u]_{\mathscr{X}_i} = (0, [\mathscr{L}_u]_{\mathscr{X}_i}),$$

$$S = \Big\{X_f = \sum_i (\pi^* \underset{\mathscr{Y}_i}{\pounds} f | \mathscr{X}_i) : f \in C^\infty(E)\Big\}.$$

It follows directly from the definitions that

$$\begin{aligned}\sum_i \theta \underset{\mathscr{Y}_i^{-1}}{\pounds} [\mathscr{L}_u]_{\mathscr{X}_i} &= \sum_i \theta \underset{\mathscr{Y}_i^{-1}}{\pounds} (0, [\mathscr{L}_u]_{X_i})\\ &= \sum_i \theta (0, -\underset{Y_i}{\pounds} [\mathscr{L}_u]_{X_i}) = -\sum_i \underset{Y_i}{\pounds} [\mathscr{L}_u]_{X_i}\\ &= -\sum_i \underset{Y_i}{\pounds} (2\langle \widetilde{du}, d\omega^i\rangle \Delta + 4d\langle du \otimes \omega^i \otimes \Delta\rangle).\end{aligned}$$

Thus as a result of the Second Theorem of Noether we have the identity

$$\sum_i \underset{Y_i}{\pounds} [\mathscr{L}_u]_{X_i} = 0. \tag{7}$$

We recall that this relation holds for every potential $u \in \Gamma$ whether it describes a real electromagnetic field (i.e. a field satisfying the Euler-Lagrange condition) or not.

Using identity (7) we can get the continuity equation for a current. For this purpose we return to electrodynamics with currents. Let a potential $u \in \Gamma$ describe a real electromagnetic field; then for every $X \in T_M$ we have

$$[\mathscr{K}_u]_X = [\mathscr{L}_u]_X + \langle j, \omega\rangle \Delta \underset{\mathcal{O}}{\equiv} 0,$$

where $\omega \in \Gamma(\mathcal{O}, M)$ represents $X_\perp$. Obviously

$$0 \underset{\mathcal{O}}{\equiv} \sum_i \underset{Y_i}{\pounds} [\mathscr{K}_u]_{X_i} = \sum_i \underset{Y_i}{\pounds} [\mathscr{L}_u]_{X_i} + \sum_i \underset{Y_i}{\pounds} \langle j, \omega^i\rangle \Delta .$$

Hence taking into account (7) we get the continuity equation

$$\sum_i \underset{Y_i}{£} \langle j, \omega^i \rangle \underset{\omega}{\Delta} = 0.$$

References

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A differentiable structure in the set of all bundle sections over compact subsets

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In several branches of mathematics (e.g. in the calculus of variations, mathematical physics etc.) we have to deal with sets of maps, e.g. with families of parametrized curves or, more generally, with k-cubes in a finite-dimensional differentiable manifold.

The case of the set of C^k-maps from a compact Banach C^∞-manifold into a separable Banach C^∞-manifold has been investigated by Eells [1, 2]. He has shown that this set has the structure of a C^∞-manifold modelled on a separable Banach space.

A particular case was worked out by Palais [5]. The construction of the Hilbert manifold of parametrized curves was one of the main items of his general Morse theory.

The above-mentioned results are inadequate for many important problems. For example, in the modern formulation of the classical field theory the states are described by sections of the respective bundles; besides, the compact sections play a fundamental role.

In the present note we prove that the set of compact sections of finite-dimensional differentiable bundle can be naturally equipped with the structure of a differentiable manifold modelled on a Fréchet space. Some other problems of this kind are solved, e.g. a differentiable structure in a set of non-parametrized curves or, more generally, in a set of compact submanifolds; the results will be published in this journal.

We want to emphasize that in the construction of a differentiable structure in such sets there are difficulties which do not occur in "parametrized" cases. The set of homotopic C^k-submanifolds which are boundaries of relatively compact domains in a given finite-dimensional C^∞-manifold has a canonical structure of a topological manifold modelled on a Banach space $C^k(\Omega)$, where Ω is one of those C^k-submanifolds, but the coordinate maps are not differentiable (the formally calculated derivative of a coordinate map contains differential operators; cf. Remark on p. 200). To overcome this difficulty, in the present paper we consider C^∞-submanifolds and we take as a model space the space $C^\infty(\Omega)$ in