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Extensions of locally bounded convolution operators in L^p-spaces

by

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PART I

1. Introduction. Given any measure space $(\Omega, \mathscr{A}, \omega)$ we denote by $M(\Omega) \equiv M(\Omega, \mathscr{A}, \omega)$ the class of all $\mathscr{A}$-measurable complex-valued functions on Ω, and by $L^p(\Omega) = L^p(\Omega, \mathscr{A}, \omega)$ the class of all functions f in $M(\Omega)$ such that $\|f\|_p = \left(\int_\Omega |f|^p d\omega\right)^{1/p}$ is finite. The numbers p and p' will be connected by $(1/p)+(1/p') = 1$.

In certain situations involving the study of operators in L^p-spaces it is convenient to show first that an operator is bounded on some subset of the L^p-space concerned; that is, given an operator T there is a subset S of $L^p(\Omega)$ and a constant k such that T maps S into $L^q(\Omega_1)$, and

$$\|T(f)\|_q \leqslant k\|f\|_p \quad (f \in S). \tag{1.1}$$

In cases where S is a dense subspace of $L^p(\Omega)$, $p \geqslant 1$, the extension of T to all of $L^p(\Omega)$ can be obtained by applying familiar arguments. There are, however, cases in which S is not even a subspace. For example, if $K_\alpha(f)$ represents the fractional integral defined on $(-\infty, \infty)$ by

$$K_\alpha(f)(x) = \int_{-\infty}^{\infty} |t-x|^{\alpha-1} f(t)\,dt, \tag{1.2}$$

then it is not difficult to show that, for all symmetrically non-increasing functions f in $L^p(-\infty, \infty)$, we have

$$\|K_\alpha(f)\|_q \leqslant k_{p,\alpha}\|f\|_p, \quad (1/q) = (1/p)-\alpha,\ 0 < \alpha < (1/p) < 1; \tag{1.3}$$

see for example [1], Theorem 383. The extension of (1.3) to all of $L^p(-\infty, \infty)$ can be justified by an inequality of Hardy and Littlewood also contained in [1]. It is our aim in this paper to prove theorems giving some conditions under which such extensions are possible. In fact, in the special cases considered below it will be sufficient to know that an inequality of the type given in (1.1) holds for a function satisfying certain conditions. In Part I we consider operators in L^p-spaces, and in Part II we prove estimates involving mixed norms.

We shall first prove a theorem which holds in a general measure space, and which contains a little more than the extension theorem we require. The main results will apply to operators which are representable as convolutions in the sense introduced by O'Neill in [5].

The bilinear operator σ defined on $M(\Omega)\times M(\Omega)$ will be called a *convolution* if the following conditions are satisfied:

(1.4) (i) if $f\in L^1(\Omega)$, $g\in L^1(\Omega)$, then $\sigma(f,g)$ exists [1] and

$$\|\sigma(f,g)\|_1\leqslant \|f\|_1\,\|g\|_1;$$

(ii) if $f\in L^p(\Omega)$, $g\in L^{p'}(\Omega)$, $1\leqslant p\leqslant\infty$, then $\sigma(f,g)$ exists, and

$$\|\sigma(f,g)\|_\infty\leqslant \|f\|_p\|g\|_{p'}.$$

Note. In O'Neill's definition, condition (ii) is replaced by two conditions involving the cases $p=1$ and $p=\infty$ only. However, from Corollary 1.8 of [5], it is seen that these conditions imply (ii).

Since $\|\sigma(f,g)\|_\infty\leqslant\|f\|_\infty\|g\|_1$, it follows by applying the Riesz convexity theorem (Theorem XII of [8]), that

(i′) for $g\in L^1(\Omega)$, $f\in L^p(\Omega)$, $p\geqslant 1$, we have $\|\sigma(f,g)\|_p\leqslant\|f\|_p\|g\|_1$.

By applying the convexity theorem once more to (ii) and (i′), we see that

(1.5) if $f\in L^p(\Omega)$, $g\in L^q(\Omega)$, $p\geqslant 1$, $q\geqslant 1$, $(1/r)=(1/p)+(1/q)-1$, $r\geqslant 1$, then

$$\|\sigma(f,g)\|_r\leqslant\|f\|_p\|g\|_q.$$

2. The general theorem. In this section we prove a general theorem in some fixed measure space $(\Omega,\mathscr{A},\omega)$. Let φ be a member of $M(\Omega)$. We define the operator T by

$$T(f)(x)=\sigma(\varphi,f)(x)\qquad (f\in M(\Omega),\ x\in\Omega). \tag{2.1}$$

We shall now define an operator closely related to T which maps a class of functions in $L^{p_0}(\Omega)$ into $L^{q_0}(\Omega)$ for some numbers p_0 and q_0.

Given a measurable function $\psi(x,t)$ on $\Omega\times\Omega$ we define the operator Ψ by

$$\Psi(f)(x)=\int_\Omega\psi(x,t)f(t)\,d\omega(t)\qquad(x\in\Omega,\ f\in M(\Omega)). \tag{2.2}$$

Further, let $\{u_a,\ a>0\}$ be a class of non-negative functions in $L^1(\Omega)\cap L^{p_0}(\Omega)$, $p_0\geqslant 1$ such that

$$\int_\Omega u_a\,d\omega=1,\qquad \|u_a\|_{p_0}=a^{\mu/p_0'}\|u_1\|_{p_0}, \tag{2.3}$$

where μ is some fixed real number.

[1] i.e. $\sigma(f,g)$ is a complex-valued function defined a.e. on Ω.

We impose the following conditions on φ, ψ and $\{u_a,\ a>0\}$:

(2.4) (i) There is a non-negative measurable function θ on Ω and a number $r_0\geqslant 1$ such that

$$\Big\{\int_\Omega|\psi(x,t)-\varphi(x)|^{r_0}\,d\omega(x)\Big\}^{1/r_0}\leqslant\theta(t),$$

where, for some fixed number $\alpha\geqslant 0$,

(ii) $\int_\Omega\theta(t)u_a(t)\,d\omega(t)\leqslant a^{-\mu\alpha}k_1<\infty$,

(iii) With $(1/q_0)=(1/p_0)-\beta$, $0\leqslant\beta\leqslant(1/p_0)$, we have

$$\|\Psi(u_a)\|_{q_0}\leqslant k_0\|u_a\|_{p_0}=k_0\|u_1\|_{p_0}a^{\mu/p_0'}\qquad(a>0),$$

where Ψ is defined as in (2.2), and k_0 is a finite constant.

The following are the main results of the paper.

(2.5) Theorem. *Let the functions φ, ψ and the class $\{u_a,\ a>0\}$ satisfy conditions* (2.3) *and* (2.4), *and let the operator T be defined as in* (2.1). *Further, let $(1/q)=(1/p)-\lambda$, where*

$$\lambda=\frac{\big(1+\beta-(1/p_0)\big)\alpha+\big(1-(1/p_0)\big)\big(1-(1/r_0)\big)}{1+\alpha-(1/p_0)},$$

and suppose that $1-\min\big((1/q_0),(1/r_0)\big)\leqslant(1/p)\leqslant 1$. Then there is a finite constant $k=k_p(p_0,r_0,\alpha,\beta)$ such that

$$\omega\big(\{x\in\Omega:|T(f)(x)|>s\}\big)\leqslant k^q s^{-q}\|f\|_p^q.$$

As a corollary of Theorem (2.5) we have

(2.6) Theorem. *Under the conditions of Theorem* (2.5) *we have, for* $1-\min\big((1/q_0),(1/r_0)\big)<(1/p)<1$, $(1/q)=(1/p)-\lambda$,

$$\|T(f)\|_q\leqslant k(p,\lambda)\|f\|_p\qquad(f\in L^p(\Omega)).$$

The proof of Theorem (2.5) depends on a lemma which involves the spliting of φ into two functions, one in $L^{q_0}(\Omega)$ and the other in $L^{r_0}(\Omega)$. The result is similar to that employed by Stein and Zygmund [6] in the treatment of closely related problems; see Lemma 1 of [6].

(2.7) Lemma. *Let the conditions of* (2.4) *be satisfied. Then, given any number $a>0$, there are functions φ_0 and φ_1 such that $\varphi=\varphi_0+\varphi_1$, and*

$$\|\varphi_0\|_{r_0}\leqslant k_1a^{-\mu\alpha},\qquad\|\varphi_1\|_{q_0}\leqslant ka^{\mu/p_0'}.$$

Proof. Given any number $a>0$, let

$$\varphi_0(x)=\varphi(x)-\int_\Omega\psi(x,t)u_a(t)\,d\omega(t)=\int_\Omega u_a(t)\{\varphi(x)-\psi(x,t)\}\,d\omega(t).$$

Then, by Minkowski's integral inequality and conditions (2.4) (i) and (ii), we have

$$\|\varphi_0\|_{r_0} \leqslant \int_{\Omega} u_a(t)\Big(\int_{\Omega} |\varphi(x)-\psi(x,t)|^{r_0}\,d\omega(x)\Big)^{1/r_0} d\omega(t)$$

$$\leqslant \int_{\Omega} u_a(t)\,\theta(t)\,d\omega(t) \leqslant k_1 a^{-\mu a}.$$

Now $\varphi_1(x) = \varphi(x)-\varphi_0(x) = \Psi(u_a)(x)$. Hence, by (2.4) (iii),

$$\|\varphi_1\|_{q_0} \leqslant k_2\|u_a\|_{p_0} = k_2\|u_1\|_{p_0} a^{\mu/p_0'}.$$

(2.8) **Proofs of Theorems (2.5) and (2.6).** First we prove Theorem (2.5). We shall write

$$(1/r_1) = (1/p)+(1/r_0)-1, \quad (1/q_1) = (1/p)+(1/q_0)-1, \quad b = a^{-\mu},$$

so that, by the conditions of the theorem, we have

$$0 \leqslant (1/r_1) \leqslant 1, \quad 0 \leqslant (1/q_1) \leqslant 1.$$

If $r_1 = \infty$, $q_1 = \infty$, we have

$$\|T(f)\|_\infty \leqslant \|\sigma(\varphi_0, f)\|_\infty + \|\sigma(\varphi_1, f)\|_\infty \leqslant (\|\varphi_0\|_{p'} + \|\varphi_1\|_{p'})\,\|f\|_p,$$

and this gives the required conclusion. Suppose that $r_1 < \infty$ and $q_1 < \infty$. Then for $s > 0$, we have

$$\{x \in \Omega: |T(f)(x)| > 2s\}$$
$$\subseteq \{x \in \Omega: |\sigma(\varphi_0, f)(x)| > s\} \cup \{x \in \Omega: |\sigma(\varphi_1, f)(x)| > s\},$$

so that

$$\omega\big(\{x \in \Omega: |T(f)(x)| > 2s\}\big)$$
$$\leqslant \omega\big(\{x \in \Omega: |\sigma(\varphi_0, f)(x)| > s\}\big) + \omega\big(\{x \in \Omega: |\sigma(\varphi_1, f)(x)| > s\}\big)$$
$$\leqslant s^{-r_1}\int_{\Omega} |\sigma(\varphi_0, f)|^{r_1} d\omega + s^{-q_1}\int_{\Omega} |\sigma(\varphi_1, f)|^{q_1} d\omega.$$

Hence by (1.5) and Lemma (2.7), we have

$$\omega\big(\{x \in \Omega: |T(f)(x)| > 2s\}\big) \leqslant s^{-r_1}\|\varphi_0\|_{r_0}^{r_1}\|f\|_p^{r_1} + s^{-q_1}\|\varphi_1\|_{q_0}^{q_1}\|f\|_p^{q_1}$$
$$\leqslant s^{-r_1}k_1^{r_1}b^{ar_1}\|f\|_p^{r_1} + s^{-q_1}k^{q_1}b^{-q_1/p_0'}\|f\|_p^{q_1}.$$

Now we choose b so that

$$s^{-r_1}k_1^{r_1}b^{ar_1}\|f\|_p^{r_1} = c^{-1}s^{-q_1}k^{q_1}b^{-q_1/p_0'}\|f\|_p^{q_1},$$

where c is some fixed positive real number. From this we see that

$$b = c^{-\mu}m^{\mu_1(q_1-r_1)}s^{-\mu_1(q_1-r_1)}\|f\|_p^{\mu_1(q_1-r_1)},$$

where $m^{(q_1-r_1)} = k^{q_1}k_1^{-r_1}$, and $(1/\mu_1) = ar_1 + (q_1/p_0')$.

Hence

$$\omega\big(\{x \in \Omega: |T(f)(x)| > 2s\}\big)$$
$$\leqslant c^{-a\mu_1 r_1}(1+c)\,k_1^{r_1}m^{a\mu_1 r_1(q_1-r_1)}s^{-(r_1+a\mu_1 r_1(q_1-r_1))}\|f\|_p^{(r_1+ar_1\mu_1(q_1-r_1))}.$$

The required conclusion in the case $r_1 < \infty$, $q_1 < \infty$ is easily verified by setting $q = r_1 + a\mu_1 r_1(q_1-r_1) = \mu_1 r_1 q_1\big(1+a-(1/p_0)\big)$.

If either $r_1 = \infty$ or $q_1 = \infty$, we proceed as above, using the fact that

$$\omega\big(\{x \in \Omega: |g(x)| > s\}\big) = 0 \quad \text{if } s \geqslant \|g\|_\infty.$$

Now we consider Theorem (2.6). Let $p^{(1)}, p^{(2)}$ be two numbers such that

$$1-\min\big((1/q_0), (1/r_0)\big) \leqslant 1/p^{(1)}, \quad 1/p^{(2)} \leqslant 1,$$

and let $1/q^{(i)} = 1/p^{(i)} - \lambda$ $(i = 1, 2)$. Further, let the number p satisfy $(1/p) = \delta/p^{(1)} + (1-\delta)/p^{(2)}$, $0 < \delta < 1$, and let $(1/q) = \delta/q^{(1)} + (1-\delta)/q^{(2)}$ so that $(1/q) = (1/p) - \lambda$. Since, by Theorem 2.5, we have

$$\omega\big(\{x \in \Omega: |T(f)(x)| > s\}\big)^{1/q^{(i)}} \leqslant k^{(i)}\|f\|_{p^{(i)}} \quad (i = 1, 2),$$

the conclusion of Theorem 2.6 is an immediate consequence of the Marcinkiewicz-Zygmund interpolation theorem; see [7], and also [8].

3. Convolution operators in E_n. We shall now apply the main results of section 2 to convolution operators defined on $L^p(E_n)$, where E_n, $n \geqslant 1$, represents the Euclidean space of dimension n, and ω represents Lebesgue measure in E_n. We shall write dt for $d\omega(t)$. For $x = (x_1, x_2, \ldots, x_n) \in E_n$, we shall write $|x|^2$ for $(x_1^2 + x_2^2 + \ldots + x_n^2)$.

On $M(E_n) \times M(E_n)$, σ will represent the usual convolution operator defined by

$$\sigma(f, g)(x) = (f * g)(x) = \int_{E_n} f(t)\,g(x-t)\,dt \quad (x \in E_n).$$

In this case the result (1.5) is the well-known inequality of Young (see [1], p. 201).

Let φ be in $M(E_n)$. Then we define

$$T(f)(x) = \int_{E_n} f(t)\,\varphi(x-t)\,dt. \tag{3.1}$$

Further, we set $\psi(x, t) = \varphi(x-t)$, so that

$$\Psi(f)(x) = \int_{E_n} f(t)\,\varphi(x-t)\,dt \equiv T(f)(x). \tag{3.2}$$

Then inequality of condition (2.4) (i) now becomes

$$\Big\{\int_{E_n} |\varphi(x-t) - \varphi(x)|^{r_0}\,dx\Big\}^{1/r_0} \leqslant \theta(t).$$

In constructing the class $\{u_a, a>0\}$ we fix a function $u \geqslant 0$ in $L^1(E_n) \cap L^{p_0}(E_n)$, $p_0 \geqslant 1$, such that $\|u\|_1 = 1$. Then we define

$$u_a(t) = a^{-n} u(t/a) \qquad (a>0,\, t \in E_n),$$

so that

$$\|u_a\|_1 = 1, \qquad \|u_a\|_{p_0} = a^{-n/p_0'} \|u\|_{p_0}.$$

Now we have

(3.3) Theorem. *Let φ be a measurable function on E_n, and let the operator T be defined by*

$$T(f)(x) = \int_{E_n} \varphi(x-t) f(t)\,dt \qquad (x \in E_n).$$

Suppose that there is a non-negative function u in $L^1(E_n) \cap L^{p_0}(E_n)$, $p_0 \geqslant 1$, with $\|u\|_1 = 1$ for which the following conditions are satisfied:

(i) $$\Big(\int_{E_n} |\varphi(x-t) - \varphi(x)|^{r_0}\,dx\Big)^{1/r_0} \leqslant \theta(t),$$

where $r_0 \geqslant 1$, and, for some fixed number $\alpha \geqslant 0$,

(ii) $$a^{-n} \int_{E_n} \theta(t) u(t/a)\,dt \leqslant a^{n\alpha} k_1 < \infty \qquad (a>0).$$

(iii) *With* $(1/q_0) = (1/p_0) - \beta$, $0 \leqslant \beta \leqslant (1/p_0) \leqslant 1$, *we have*

$$\|T(u_a)\|_{q_0} \leqslant k_2 \|u_a\|_{p_0}, \qquad u_a(t) = a^{-n} u(t/a), \qquad a>0.$$

Then there is a finite constant $k = k(p_0, r_0, \alpha, \beta)$ such that

(a) *if* $1 - \min\big((1/q_0), (1/r_0)\big) \leqslant (1/p) \leqslant 1$, $(1/q) = (1/p) - \lambda$,

$$\lambda = \frac{\big(1+\beta-(1/p_0)\big)\alpha + \big(1-(1/p_0)\big)\big(1-(1/r_0)\big)}{1+\alpha-(1/p_0)},$$

then

$$\omega\big(\{x \in E_n : |T(f)(x)| > s\}\big)^{1/q} \leqslant k s^{-1} \|f\|_p;$$

(b) *if* $1 - \min\big((1/q_0), (1/r_0)\big) < (1/p) < 1$, $(1/q) = (1/p) - \lambda$, *then*

$$\|T(f)\|_q \leqslant k \|f\|_p \qquad (f \in L^p(E_n)).$$

Proof. In view of the comments made earlier in this section it is easily seen that Theorem (3.3) is a consequence of Theorems (2.5) and (2.6).

(3.4) Remark. Conditions (i) and (iii) of Theorem (3.3) with $p_0 = 1$, $r_0 = q_0$ are rather similar to those given by Hörmander in [2]; see [2], p. 113-114.

(3.5) An example. As we mentioned in the introduction, the main conclusions of this paper can be applied in giving alternative proofs of the Hardy-Littlewood-Soboleff theorem for fractional integrals. We

consider first the 1-dimensional case where

$$\varphi(t) = |t|^{\alpha-1}, \qquad \theta(t) = \int_{-\infty}^{\infty} |\varphi(x-t) - \varphi(x)|\,dx = |t|^{\alpha} k_\alpha,$$

and k_α is finite if $0 < \alpha < 1$. Note that here $r_0 = 1$.

Now we set

$$u(t) = \begin{cases} e^{-t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leqslant 0. \end{cases}$$

Then

$$a^{-1} \int_{-\infty}^{\infty} \theta(t) u(t/a)\,dt = a^{-1} k_\alpha \int_0^{\infty} t^{\alpha} e^{-t/a}\,dt = m_\alpha a^{\alpha} \qquad (0<\alpha<1).$$

Also, with $u_a(t) = a^{-1} u(t/a)$, we have $K_\alpha(u_a)(x) = a^{\alpha} K_\alpha(u)(x/a)$. Hence to justify conditions (i)-(iii) of Theorem (3.3) we only need to prove the following

(3.5.1) Lemma. *If* $0 < \alpha < (1/p_0) < 1$, $(1/q_0) = (1/p_0) - \alpha$, *then*

$$\|K_\alpha(u)\|_{q_0} \leqslant k_{\alpha,p} \|u\|_{p_0}.$$

Proof. For completeness we give a short proof which can be applied to all symmetrically non-increasing functions. First we notice that

$$K_\alpha(u)(x) = \int_0^{\infty} |t-x|^{\alpha-1} u(t)\,dt,$$

so that, by a theorem for homogeneous kernels given in [4],

$$\Big(\int_{-\infty}^{0} |K_\alpha(u)(x)|^{q_0}\,dx\Big)^{1/q_0} = \Big(\int_0^{\infty} \Big|\int_0^{\infty} |t+x|^{\alpha-1} u(t)\,dt\Big|^{q_0} dx\Big)^{1/q_0}$$
$$\leqslant \Big\{\int_0^{\infty} t^{(\alpha-(1/p_0))/(1-\alpha)} (1+t)^{-1}\,dt\Big\}^{1-\alpha} \|u\|_{p_0}.$$

Now let x be a positive number, and fix a number m such that $0 < m < 1$. Then, on using the fact that u is non-increasing on $(0, \infty)$, we have

$$K_\alpha(u)(x) = \int_{x/m}^{\infty} (t-x)^{\alpha-1} u(t)\,dt + \int_{x}^{x/m} (t-x)^{\alpha-1} u(t)\,dt +$$
$$+ \int_{mx}^{x} (x-t)^{\alpha-1} u(t)\,dt + \int_0^{mx} (x-t)^{\alpha-1} u(t)\,dt$$
$$\leqslant (1-m)^{\alpha-1} \int_{x/m}^{\infty} t^{\alpha-1} u(t)\,dt + \alpha^{-1}\big((x/m)-x\big)^{\alpha} u(x) +$$
$$+ \alpha^{-1}(x-mx)^{\alpha} u(mx) + (x-mx)^{\alpha-1} \int_0^{mx} u(t)\,dt.$$

On using the fact that

$$u(x) \leqslant x^{-1}\int_0^x u(t)\,dt,$$

we see that

$$K_a(u)(x) \leqslant (1-m)^{a-1}\int_x^\infty t^{a-1}u(t)\,dt + k(m,a)x^{a-1}\int_0^x u(t)\,dt,$$

where $k(m,a) = a^{-1}(m^{-1}-1)+m^{-1}a^{-1}(1-m)^a+(1-m)^{a-1}$.

The fact that

$$\Big(\int_0^\infty |K_a(u)(x)|^{q_0}dx\Big)^{1/q_0} \leqslant k\|u\|_{p_0}$$

can now be verified by applying the result of [4] once more; see Theorem 7 of [4].

From the above lemma it follows by applying Theorem 3.3 with $r_0 = 1$, $\beta = a$, that

(3.5.2) THEOREM. (a) *If* $0 < a < (1/p) \leqslant 1$, $(1/q) = (1/p)-a$, *then*

$$\omega\big(\{x\in(-\infty,\infty): |K_a(f)(x)| > s\}\big)^{1/q} \leqslant ks^{-1}\|f\|_p.$$

(b) *If* $0 < a < (1/p) < 1$, $(1/q) = (1/p)-a$, *then*

$$\|K_a(f)\|_q \leqslant k\|f\|_p.$$

Observe that, in view of Lemma (3.5.1), we can make the number p_0 of Theorem (3.3) arbitrarily close to 1.

(3.5.3) Note. The case $p = 1$ of Theorem (3.5.2) (a) was proved by Zygmund in [7] by a different approach.

(3.5.4) The n-dimensional form of K_a given by

$$K_a'(f)(x) = \int_{E_n} |t-x|^{(a-1)n}f(t)\,dt \qquad (x\in E_n),$$

can be treated similarly. It is convenient in this case to choose

$$u(t) = \pi^{-n/2}e^{-|t|^2}.$$

Since $u(t) = \pi^{-\frac{1}{2}n}e^{-t_1^2}e^{-t_2^2}\ldots e^{-t_n^2}$, the n-dimensional analogue of Lemma (3.5.1) can be obtained by applying the 1-dimensional result n times.

PART II

ESTIMATES INVOLVING MIXED NORMS

In this part of the paper we consider boundedness results involving mixed norms and prove results similar to those given by Hörmander [1], Chapter III. In particular, we apply the main conclusions in giving an alternative treatment of the maximal functions of fractional order introduced in [3].

4. Notation. In this part of the paper we retain the notation of Part I, and introduce some additional definitions.

Let $(\Omega, \mathscr{A}, \omega)$ and $(\Lambda, \mathscr{B}, \gamma)$ be two measure spaces. As in Part I, we denote the class of $(\mathscr{A}\times\mathscr{B})$-measurable functions on $\Omega\times\Lambda$ by $M(\Omega\times\Lambda)$. Further, given any function U in $M(\Omega\times\Lambda)$, we define the norm $\|(U)_r\|_q$ by

$$\|(U)_r\|_q = \Big\{\int_\Omega\Big(\int_\Lambda |U(x,y)|^r d\gamma(y)\Big)^{q/r}d\omega(x)\Big\}^{1/q},$$

and denote by $L^{qr}(\Omega\times\Lambda) \equiv L^{qr}$ the class of functions U such that $\|(U)_r\|_q$ is finite. Also, given $U = U(x,y)$ we shall write U_y for $U(\cdot,y)$, and $(U)_r$ for $\{\int_\Lambda |U(x,y)|^r d\gamma(y)\}^{1/r}$.

Again σ will denote the convolution operator on $M(\Omega)\times M(\Omega)$ satisfying the conditions of (1.4). By (1.5), it follows that if $f\in L^p(\Omega)$, $g\in L^q(\Omega)$, $(1/r) = (1/p)+(1/q)-1$, $p\geqslant 1$, $q\geqslant 1$, $r\geqslant 1$, then $\sigma(f,g)$ exists, and

$$\|\sigma(f,g)\|_r \leqslant \|f\|_p\|g\|_q.$$

We shall further assume that if U is in $M(\Omega\times\Lambda)$, then for each fixed x in Ω, the function $\sigma(U_y,f)(x)$ is in $M(\Lambda)$, and for $r\geqslant 1$,

$$\Big\{\int_\Lambda |\sigma(U_y,f)|^r d\gamma(y)\Big\}^{1/r} \leqslant \big|\sigma\big((U)_r, |f|\big)\big|. \tag{4.1}$$

5. The general result invoving mixed norms. We shall now prove the mixed-norm analogue of Theorems (2.5) and (2.6). Given a function U in $M(\Omega\times\Lambda)$, we define the class of operators $\{T_y, y\in\Lambda\}$ by

$$T_y(f) = \sigma(U_y, f) \qquad (y\in\Lambda, f\in M(\Omega)). \tag{5.1}$$

Also, given the class of functions $\{\psi_y(x,t), y\in\Lambda\}$ we define the class of operators $\{\Psi_y, y\in\Lambda\}$ by

$$\Psi_y(f)(x) = \int_\Omega \psi_y(x,t)f(t)\,d\omega(t) \qquad (x\in\Omega). \tag{5.2}$$

Again we denote by $\{h_a, a>0\}$ a class of non-negative functions in $M(\Omega)$ such that, for some fixed number $p_0\geqslant 1$, we have

$$\|h_a\|_1 = 1, \qquad \|h_a\|_{p_0} = a^{\mu/p_0'}k^* \qquad (a>0), \tag{5.3}$$

where μ is some fixed real number, and k^* is a finite positive constant.

We impose the following conditions on the above functions and operators:

(5.4) (i) For some fixed numbers $r \geqslant 1$, $r_0 \geqslant 1$, we have

$$\Big\{\int_\Omega \Big(\int_\Lambda |\psi_y(x,t) - U(x,y)|^r d\gamma(y)\Big)^{r_0/r} d\omega(x)\Big\}^{1/r_0} \leqslant \theta(t),$$

where θ is a non-negative function in $M(\Omega)$ such that, for some fixed number $a \geqslant 0$, we have

(ii) $\int_\Omega h_a(t)\theta(t)\,d\omega(t) \leqslant k_1 a^{-\mu a}$ $(a > 0)$.

(iii) For $(1/q_0) = (1/p_0) - \beta$, $0 \leqslant \beta \leqslant (1/p_0)$, we have

$$\|(\Psi(h_a))_r\|_{q_0} \leqslant k_2\|h_a\|_{p_0} = ka^{\mu/p_0'} \qquad (a > 0),$$

where k is a finite constant.

Given the fixed number $r \geqslant 1$ of (5.4) (i) we now set

$$T(f)(x) = \Big\{\int_\Lambda |T_y(f)(x)|^r d\gamma(y)\Big\}^{1/r}. \tag{5.5}$$

Then we have the following results:

(5.6) THEOREM. *Let the operator T and the classes $\{\psi_y, y \in \Lambda\}$, $\{h_a, a > 0\}$ be defined as in (5.1)-(5.5), and suppose that the conditions of (5.4) are satisfied. If $1 - \min\big((1/q_0), (1/r_0)\big) \leqslant (1/p) \leqslant 1$, $(1/q) = (1/p) - \lambda$, where*

$$\lambda = \frac{\big(1+\beta-(1/p_0)\big)a + \big(1-(1/r_0)\big)\big(1-(1/p_0)\big)}{1+a-(1/p_0)},$$

then there is a finite constant $k = k(p_0, r_0, r, a, \beta)$ such that

$$\omega\big(\{x \in \Omega : |T(f)(x)| > s\}\big)^{1/q} \leqslant ks^{-1}\|f\|_p.$$

(5.7) THEOREM. *Suppose that the conditions of Theorem (5.6) are satisfied and that $1 - \min\big((1/q_0), (1/r_0)\big) < (1/p) < 1$, $(1/q) = (1/p) - \lambda$. Then there is a finite constant $k = k(p_0, r_0, r, a, \beta)$ such that*

$$\|T(f)\|_q \leqslant k\|f\|_p \qquad (f \in L^p(\Omega)).$$

We shall require the following analogue of Lemma (2.7):

(5.8) LEMMA. *Let U in $M(\Omega\times\Lambda)$ be such that the conditions of (5.4) are satisfied. Then, given any number $a > 0$, there are functions V and W such that $U = V+W$, and*

$$\|(V)_r\|_{r_0} \leqslant k_1 a^{-\mu a}, \qquad \|(W)_r\|_{q_0} \leqslant ka^{\mu/p_0'}.$$

Proof. Suppose that

$$V(x,y) = U(x,y) - \int_\Omega \psi_y(x,t)h_a(t)\,d\omega(t)$$
$$= \int_\Omega \{U(x,y) - \psi_y(x,t)\}h_a(t)\,d\omega(t).$$

Then by applying Minkowski's integral inequality, we have

$$\Big\{\int_\Lambda |V(x,y)|^r d\gamma(y)\Big\}^{1/r} \leqslant \int_\Omega h_a(t)\Big\{\int_\Lambda |U(x,y) - \psi_y(x,t)|^r d\gamma(y)\Big\}^{1/r} d\omega(t).$$

By applying Minkowski's inequality once more and using (5.4) (i) and (5.4) (ii), it follows that

$$\|(V)_r\|_{r_0} \leqslant \int_\Omega h_a(t)\theta(t)\,d\omega(t) \leqslant a^{-\mu a}k_1.$$

Now if we set $W(x,y) = U(x,y) - V(x,y) = \Psi_y(h_a)(x)$, we see by applying (5.4) (iii) that

$$\|(W)_r\|_{q_0} \leqslant k_2\|h_a\|_{p_0} = ka^{\mu/p_0'}.$$

(5.9) Proofs of Theorems 5.6 and 5.7. For each number $a > 0$, we set $U = V+W$, where V and W satisfy the conditions of Lemma (5.8). Then by applying (4.1), we have

$$T(f)(x) = \Big\{\int_\Lambda |\sigma(U_y, f)(x)|^r d\gamma(y)\Big\}^{1/r}$$
$$\leqslant \Big\{\int_\Lambda |\sigma(V_y, f)(x)|^r d\gamma(y)\Big\}^{1/r} + \Big\{\int_\Lambda |\sigma(W_y, f)(x)|^r d\gamma(y)\Big\}^{1/r}$$
$$\leqslant \big|\sigma\big((V)_r, |f|\big)(x)\big| + \big|\sigma\big((W)_r, |f|\big)(x)\big|.$$

To prove Theorem (5.6) we now proceed as in the proof of Theorem (2.5); see (2.8) with $(V)_r$ in place of φ_0 and $(W)_r$ in place of φ_1.

Theorem (5.7) follows from the Marcinkiewicz-Zygmund interpolation theorem by proceeding as in the proof of Theorem (2.6).

6. Convolution operators in Euclidean space. We now consider the mixed-norm analogue of the results involving convolution operators on $L^p(E_n)$ given in section 3. Using the notation of Part I we note that, in the Euclidean case, the class of operators $\{T_y, y \in \Lambda\}$ of (5.1) is given by

$$T_y(f)(x) = \int_{E_n} U(x-t, y)f(t)\,dt,$$

so that

$$T(f)(x) = \Big\{\int_\Lambda \Big|\int_{E_n} U(x-t, y)f(t)\,dt\Big|^r d\gamma(y)\Big\}^{1/r}. \tag{6.1}$$

As before, ω represents the translation invariant (Lebesgue) measure in E_n, and we write $d\omega(t) = dt$. Also, condition (4.1) is easily seen to be a direct consequence of Minkowski's integral inequality.

Now we set $\psi_y(x,t) = U(x-t, y)$, so that $\Psi_y(f) \equiv T_y(f)$, and $\|(\Psi(f))_r\|_{q_0} = \|T(f)\|_{q_0}$. Further, we define the class $\{h_a, a > 0\}$ by

$$h_a(t) = a^{-n}h(t/a) \quad (a > 0,\ t \in E_n), \qquad h \geqslant 0, \qquad \|h\|_1 = 1.$$

Then on making suitable substitution in Theorems (5.6) and (5.7) with $-\mu = n$, we have

(6.2) THEOREM. *Let U be in $M(E_n \times \Lambda)$ and, for some number $r \geq 1$, let the operator T be defined as in (6.1). Suppose that there is a non-negative function h in $L^1(E_n) \cap L^{p_0}(E_n)$, $p_0 \geq 1$, with $\|h\|_1 = 1$, and for which the following conditions are satisfied:*

(i) *For some fixed number $r_0 \geq 1$, we have*

$$\Big\{\int_{E_n}\Big(\int_{\Lambda} |U(x-t, y)-U(x, y)|^r d\gamma(y)\Big)^{r_0/r} dx\Big\}^{1/r_0} \leq \theta(t),$$

where θ is a non-negative function in $M(E_n)$ such that, for some fixed number $\alpha \geq 0$, we have

(ii) $a^{-n}\int_{E_n} h(t/a)\,\theta(t)\,dt \leq k_1 a^{n\alpha}$ $(a > 0)$.

(iii) *For* $(1/q_0) = (1/p_0)-\beta$, $0 \leq \beta \leq (1/p_0)$, *there is a finite constant k_2 such that*

$$\|T(h_a)\|_{q_0} \leq k_2\|h_a\|_{p_0} = ka^{-n/p_0'} \qquad (a > 0),$$

where $h_a(t) = a^{-n}h(t/a)$ $(t \in E_n)$.

Then there is a finite constant $k = k(p_0, r_0, r, \alpha, \beta)$ such that

(a) *if* $1-\min\big((1/q_0), (1/r_0)\big) \leq (1/p) \leq 1$, $(1/q) = (1/p)-\lambda$,

$$\lambda = \frac{\big(1+\beta-(1/p_0)\big)\alpha+\big(1-(1/p_0)\big)\big(1-(1/r_0)\big)}{1+\alpha-(1/p_0)},$$

then

$$\omega\big(\{x \in E_n : |T(f)(x)| > s\}\big)^{1/q} \leq s^{-1}k\|f\|_p;$$

(b) *if* $1-\min\big((1/q_0), (1/r_0)\big) < (1/p) < 1$, *then*

$$\|T(f)\|_q \leq k\|f\|_p \qquad (f \in L^p(E_n)).$$

7. Maximal functions of fractional order. In [3] we introduced the maximal function of fractional order defined for measurable functions on E_n by

$$(7.1)\qquad N_r(f)(x) = \Big\{\int_0^\infty \Big|y^{-n+((n-1)/r)}\int_{|t|\leq y} f(x-t)\,dt\Big|^r dy\Big\}^{1/r} \qquad (x \in E_n).$$

The well-known maximal function associated with a measurable function on E_n corresponds to $N_\infty(f)$. We proved, by assuming the known result involving $N_\infty(f)$, that if $r \geq p > 1$, $(1/q) = (1/p)-(1/r)$, then there is a constant k such that

$$\|N_r(f)\|_q \leq k\|f\|_p.$$

We shall now give an alternative proof of this in the case $n = 1$, $1 < p \leq r < \infty$, and prove a substitute result similar to that known to hold for fractional integrals in the case $p = 1$. Also, we give an indication of an alternative treatment of (7.1) in the case $n \geq 2$.

When $n = 1$, we have

$$(7.1.1)\qquad N_r(f)(x) = \Big\{\int_0^\infty \Big|\int_{-\infty}^\infty U(x-t, y)f(t)\,dt\Big|^r dy\Big\}^{1/r},$$

where

$$U(x, y) = \begin{cases} 1/y & \text{if} \quad 0 < |x| \leq y, \\ 0 & \text{if} \quad |x| > y. \end{cases}$$

Now we define the function $h(t)$ by

$$(7.1.2)\qquad h(t) = \begin{cases} e^{-t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leq 0, \end{cases}$$

and we set $h_a(t) = a^{-1}h(t/a)$, $a > 0$.

We shall now justify the conditions necessary for the application of Theorem (6.2). Firstly we notice that $|U(x-t, y)-U(x, y)| = 1/y$ if either $|x-t| < y < |x|$ or $|x| < y < |x-t|$, and is 0 otherwise, so that

$$\Big\{\int_{-\infty}^{\infty}\Big(\int_0^\infty |U(x-t, y)-U(x, y)|^r dy\Big)^{r_0/r} dx\Big\}^{1/r_0}$$
$$= (r-1)^{-1/r}\Big\{\int_{-\infty}^{\infty}\big||x-t|^{1-r}-|x|^{1-r}\big|^{r_0/r} dx\Big\}^{1/r_0}$$
$$= |t|^{(1/r)+(1/r_0)-1}k(r_0, r),$$

where $k(r_0, r)$ is finite if $r > 1$, $r_0 > 1$, $(1/r_0)+(1/r) > 1$. It is clear that if $r < \infty$, we can fix a number $r_0 > 1$ for which $k(r_0, r)$ is finite. Having fixed r_0, we set

$$\theta(t) = |t|^{(1/r)+(1/r_0)-1}k(r_0, r),$$

so that

$$\int_{-\infty}^{\infty} h_a(t)\,\theta(t)\,dt = k(r_0, r)\,a^{-1}\int_0^\infty e^{-t/a}t^{(1/r)+(1/r_0)-1}dt$$
$$= a^{(1/r)+(1/r_0)-1}k^*(r_0, r).$$

Hence the number α of Theorem (6.2), condition (ii), is given by $\alpha = (1/r)+(1/r_0)-1$.

Now it is easily seen by making suitable changes of variables that

$$N_r(h_a)(x) = a^{(1/r)}N_r(h)(x/a).$$

Hence for condition (iii) of Theorem (6.2) we now need the following

(7.2) LEMMA. *If* $(1/q_0) = (1/p_0)-(1/r)$, $r \geq p_0 > 1$, *then there is a finite constant k_2 such that*

$$\|N_r(h)\|_{q_0} \leq k_2\|h\|_{p_0}.$$

Proof. It is clearly sufficient to show that

$$\Big(\int_0^\infty |N_r(h)(x)|^{q_0}dx\Big)^{1/q_0} \leqslant k_2\|h\|_{p_0}.$$

A similar estimate involving $\big(\int_{-\infty}^0 |N_r(h)(x)|^{q_0}dx\big)^{1/q_0}$ can be obtained by making an obvious change of variables in the integral defining $N_r(h)$.

Let $x > 0$ be fixed. Then, by using the inequality $(a+b)^{1/r} \leqslant a^{1/r}+b^{1/r}$, $a > 0$, $b > 0$, $r \geqslant 1$, we have

$$N_r(h)(x) \leqslant \Big(\int_x^\infty \Big|y^{-1}\int_{x-y}^{x+y} h(t)\,dt\Big|^r dy\Big)^{1/r} + \Big(\int_0^x \Big|y^{-1}\int_{x-y}^{x+y} h(t)\,dt\Big|^r dy\Big)^{1/r}$$
$$= I_1+I_2.$$

Now since $h(t)$ is decreasing on $(0, \infty)$ we have, for $0 < y < x$,

$$y^{-1}\int_{x-y}^{x+y} h(t)\,dt = 2\int_0^1 h\big(x-xt+y(1-t)+t(x-y)\big)\,dt$$
$$\leqslant 2\int_0^1 h(x-xt)\,dt$$
$$= 2x^{-1}\int_0^x h(t)\,dt.$$

Hence $I_2 \leqslant 2x^{(1/r)-1}\int_0^x h(t)\,dt$, and the fact that $\|I_2\|_{q_0} \leqslant k\|h\|_{p_0}$ follows from the known inequality for homogeneous kernels ([4], Theorem 7).

Next we consider I_1. Since $h(t) = 0$ for $t \leqslant 0$, it follows by applying Minkowski's inequality that

$$I_1 = \Big(\int_x^\infty \Big|y^{-1}\int_0^{x+y} h(t)\,dt\Big|^r dy\Big)^{1/r}$$
$$\leqslant \Big(\int_x^\infty \Big|y^{-1}\int_x^{x+y} h(t)\,dt\Big|^r dy\Big)^{1/r} + \Big(\int_x^\infty \Big|y^{-1}\int_0^x h(t)\,dt\Big|^r dy\Big)^{1/r}$$
$$= \Big(\int_x^\infty \Big|y^{-1}\int_x^{x+y} h(t)\,dt\Big|^r dy\Big)^{1/r} + (r-1)^{-1/r}x^{(1/r)-1}\int_0^x h(t)\,dt$$
$$= I_{11}+I_{12}.$$

As in the case of I_2, we have $\|I_{12}\|_{q_0} \leqslant k\|h\|_{p_0}$. Hence to complete the proof of the lemma we consider I_{11}.

Let ϱ be any positive number such that $(\varrho-1)r+1 < 0$. Then since $t \leqslant x+y$ implies that $y^{-\varrho} \leqslant (t-x)^{-\varrho}$, we see that

$$\int_x^\infty \Big|y^{-1}\int_x^{x+y} h(t)\,dt\Big|^r dy \leqslant \Big(\int_x^\infty y^{(\varrho-1)r}dy\Big)\Big(\int_x^\infty (t-x)^{-\varrho}h(t)\,dt\Big)^r,$$

so that

$$I_{11} \leqslant \big((1-\varrho)r-1\big)^{-1/r}x^{\varrho-1+(1/r)}\int_x^\infty (t-x)^{-\varrho}h(t)\,dt.$$

By applying the inequality for homogeneous kernels ([4], Theorem 1), we see that

$$\Big(\int_0^\infty |I_{11}|^{q_0}dx\Big)^{1/q_0} \leqslant k_1^*\|h\|_{p_0},$$

where

$$k_1^* = \Big\{\int_1^\infty t^{((1/r)-(1/p_0))/(1-(1/r))}(t-1)^{-\varrho/(1-(1/r))}dt\Big\}^{1-(1/r)}.$$

Since the constant k_1^* is finite for any number ϱ satisfying $1-(1/r) > \varrho > 1-(1/p_0)$, this concludes the proof of the lemma in the case $p_0 < r$. When $r = p_0$, the inequality involving I_{11} is a direct consequence of the well-known Hardy's inequality; again, see [4], Theorem 7.

(7.3) Theorem. *Let $f \in L^p(-\infty, \infty)$, let $1 \leqslant p < \infty$, $(1/q) = (1/p)-(1/r)$, and let $N_r(f)$ be defined as in (7.1.1). Then there is a finite constant, $k = k(p, r)$ such that*

(a) *if $1 \leqslant p \leqslant r < \infty$ $(r > p$ if $p = 1)$, then*

$$\omega\big(\{x \in (-\infty, \infty): |N_r(f)(x)| > s\}\big)^{1/q} \leqslant s^{-1}k\|f\|_p;$$

(b) *if $1 < p \leqslant r < \infty$, then*

$$\|N_r(f)\|_q \leqslant k\|f\|_p.$$

Proof. From Lemma (7.2) and the earlier comments of this section, we fix a number p_0 satisfying $1 < p_0 \leqslant r < \infty$, and set $\alpha = (1/r)+(1/r_0)-1$, $\beta = (1/r)$ in Theorem (6.2), so that

$$\lambda = \frac{\big(1+(1/r)-(1/p_0)\big)\big((1/r)+(1/r_0)-1\big)+\big(1-(1/r_0)\big)\big(1-(1/p_0)\big)}{(1/r)+(1/r_0)-(1/p_0)} = (1/r).$$

The condition on the exponent p of Theorem (6.2) (a) becomes

$$1-\min\big((1/p_0)-(1/r),\,(1/r_0)\big) \leqslant (1/p) \leqslant 1.$$

Since $(1/r_0) > 1-(1/r)$ and since we can choose p_0 arbitrarily close to 1, we see that the conclusions of the theorem follow from (6.2). We note, however, that the case $r = p$ of (b) does not follow from (a), but can be justified independently by an application of the inequality for homogeneous kernels ([4], Theorem 1).

(7.4) *Note on the n-dimensional case*, $n \geqslant 2$. The n-dimensional form of $N_r(f)$ can be treated by applying an argument similar to that employed in the 1-dimensional case. Firstly, we note that, on putting y for y^n and dy for $ny^{n-1}dy$ in (7.1), we can express $N_r(f)$ alternatively as

$$N_r(f)(x) = n^{-1/r}\Big\{\int_0^\infty \Big|y^{-1}\int_{|t|^n \leqslant y} f(x-t)\,dt\Big|^r dy\Big\}^{1/r}.$$

Hence in this case the function $U(x, y)$ of Theorem (6.2) is given by

$$U(x,y) = \begin{cases} y^{-1} & \text{if } |x| \leqslant y^{1/n}, \\ 0 & \text{if } |x| > y^{1/n}. \end{cases}$$

To obtain the n-dimensional analogue of Lemma (7.2) we set $h(t) = \pi^{-\frac{1}{2}n} e^{-|t|^2}$.

Now if w_n represents a unit vector on $S_{n-1} = \{x \in E_n : |x| = 1\}$, then for $t \in E_n$ we can write $t = \eta w_n$, $\eta > 0$. Hence on expressing integrals over E_n in polar coordinates, we have

$$N_r(h)(x) = n^{-1/r}\Big\{\int_0^\infty \Big|y^{-1}\int_{S_{n-1}}\int_0^{y^{1/n}} \eta^{(n-1)} h(x-\eta w_n)\,dw_n\,d\eta\Big|^r dy\Big\}^{1/r}$$

$$= n^{-((1/r)+1)}\Big\{\int_0^\infty \Big|y^{-1}\int_{S_{n-1}}\int_0^{y} h(x-\eta^{1/n} w_n)\,dw_n\,d\eta\Big|^r dy\Big\}^{1/r},$$

and it follows by applying Minkowski's integral inequality that

$$N_r(h)(x) \leqslant n^{-(1+(1/r))}\int_{S_{n-1}}\Big\{\int_0^\infty \Big|y^{-1}\int_0^{y} h(x-\eta^{1/n} w_n)\,d\eta\Big|^r dy\Big\}^{1/r} dw_n.$$

The estimate involving

$$\Big\{\int_0^\infty \Big|y^{-1}\int_0^y e^{-(x_j-\eta^{1/n}\xi_j)^2}\,d\eta\Big|^r dy\Big\}^{1/r}, \quad j = 1, 2, \ldots, n,$$

can be obtained by proceeding as in the proof of Lemma (7.2). Hence the estimate involving $N_r(h)$ can be obtained by applying this 1-dimensional result n times, and then using Minkowski's integral inequality.

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge 1934.

[2] L. Hörmander, *Estimates for translation invariant operators in L^p-spaces*, Acta Math. 104 (1960), p. 93-140.

[3] G. O. Okikiolu, *On maximal functions of fractional order*, Studia Math. 30 (1968), p. 259-271.

[4] — *Bounded linear transformations in L^p-spaces*, J. London Math. Soc. 41 (1966), p. 407-414.

[5] R. O'Neill, *Convolution operators and $L(p,q)$-spaces*, Duke Math. J. 30 (1963), p. 129-142.

[6] E. M. Stein and A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and L^p-spaces*, Annals of Math. 85 (1967), p. 337-349.

[7] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operations*, J. Math. Pures Appl. 35 (1956), p. 223-248.

[8] — *Trigonometric series*, Vol. II. Cambridge 1959.

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