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Universal bases

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1. Introduction. A sequence (e_n) is called a *basis* in a Banach space X if $e_n \in X$ $(n = 1, 2, \ldots)$ and each x in X can be represented in a unique way in the form $x = \sum_n c_n e_n$. A basis is called *unconditional* if this series converges unconditionally, i.e. if for every sequence $(g(n))$ with $g(n) = \pm 1$ the series $\sum_n g(n) c_n e_n$ converges. A basis is called *normalized* if $\|e_n\| = 1$ for $n = 1, 2, \ldots$ and *seminormalized* if

$$0 < \inf_n \|e_n\| \leqslant \sup_n \|e_n\| < +\infty.$$

Two bases (x_n) and (y_n) in Banach spaces X and Y, respectively, are said to be *equivalent* if a series $\sum_n c_n x_n$ converges if and only if the series $\sum_n c_n y_n$ converges. If (n_k) is an increasing sequence of indices, then the sequence (e_{n_k}) is called a *subbasis* of a basis (e_n). Clearly a subbasis (e_{n_k}) is a basis in the closed linear subspace which it spans. A sequence (e_{n_k}) is said to be a *complemented subbasis* of a basis (e_n) if for every real sequence (c_n) the convergence of the series $\sum_{n=1}^{\infty} c_n e_n$ implies the convergence of the series $\sum_{k=1}^{\infty} c_{n_k} e_{n_k}$.

In the present paper we study the following concept:

Definition. Let $\mathfrak{B}$ be a family of bases. A basis (e_n) is said to be [*complementably*] *universal* for $\mathfrak{B}$ if every basis in $\mathfrak{B}$ is equivalent to a [complemented] subbasis of (e_n).

It is natural to ask whether, for a given family $\mathfrak{B}$ of bases, there exists a universal basis which belongs to $\mathfrak{B}$. The following theorem, which is the main result of the present paper, gives a positive answer to this question in the case of two important families of bases:

THEOREM 1. *The following families of bases contain complementably universal elements:*

the family of all seminormalized bases,

the family of all seminormalized unconditional bases.

It is easy to show that every seminormalized basis is equivalent to a normalized one. This follows also from the proof of Theorem 1 because the universal elements which we actually construct are normalized bases. Thus Theorem 1 may be stated equivalently in a "modest" form with the word "seminormalized" replaced by "normalized". On the other hand, if (x_n) is a basis in a Banach space, then $(\|x_n\|^{-1}x_n)$ is a normalized basis in the same space. Hence the assumption of Theorem 1 that bases are seminormalized is not too restrictive.

The next corollary is a simple consequence of Theorem 1.

COROLLARY 1. *There exists a separable Banach space B [resp. U] with a basis [an unconditional basis] such that every separable Banach space with a basis [an unconditional basis] is isomorphic to a complemented subspace of B [resp. of U].*

We recall that a Banach space X is said to be *isomorphic* to a Banach space X_1 if there exists an *isomorphism* (= a linear homeomorphism) from X onto X_1. A closed linear subspace F of a Banach space E is said to be *complemented* in E if there exists a bounded linear projection from E onto F.

Theorem 1 and Corollary 1 will be proved in Section 2. Section 3 is devoted to a study of Banach spaces which have universal bases. We shall show that in the space $C(0;1)$ of continuous real-valued (or complex-valued) functions on the closed interval $[0;1]$ there exists a basis which is universal for the family of all seminormalized bases. The same property characterizes any Banach space with a basis which contains a closed linear subspace isomorphic to $C(0;1)$. This enables us to show that there exists a family of power continuum consisting of seminormalized bases which are universal for the family of all seminormalized bases and are such that every two of them span non-isomorphic Banach spaces. On the other hand, we shall show that there exists a Banach space unique up to an isomorphism, which has a normalized [unconditional] basis complementably universal for the family of all normalized [unconditional] bases. Section 4 contains some open problems and some remarks. We shall show there that the family of all normalized shrinking bases (see Section 4 for the definition) does not have any universal element in the family of all shrinking bases.

2. Spaces of norms and the proof of Theorem 1. Let R^∞ denote the space of all real sequences $t = (t(1), t(2), \ldots)$. Let $\pi_n\colon R^\infty \to R^\infty$ be defined by $(\pi_n t)(i) = t(i)$ for $i \leqslant n$ and $(\pi_n t)(i) = 0$ for $n < i$. Put $R^n = \pi_n(R^\infty)$. Clearly $R^1 = R$ may be regarded as the real line and R^∞ as the product of $\aleph_0$ copies of R's. Furthermore, let $I^\infty = \{t \in R^\infty\colon |t(i)| \leqslant 1 \text{ for } i = 1, 2, \ldots\}$ and $I^n = I^\infty \cap R^n$. Define $e_n \in R^\infty$ by $e_n(i) = 0$ for $i \neq n$ and $e_n(n) = 1$ $(n = 1, 2, \ldots)$.

Let $\mathscr{B}_n$ (resp. $\mathscr{B}$) denote the set of all non-negative functions $p(\cdot)$ on R^n (resp. on $\bigcup_{n=1}^{\infty} R^n$) satisfying the following conditions:

(i) $p(t) = 0$ iff $t = 0$; $p(ct) = |c|p(t)$; $p(t+s) \leqslant p(t)+p(s)$ for $t, s \in R^n$ (resp. $t, s \in \bigcup_{n=1}^{\infty} R^n$) and for $c \in R$,

(ii) $p(e_i) = 1$ for $i \leqslant n$ (resp. for $i = 1, 2, \ldots$),

(iii) if $|p(s)| < 1$, then $s \in I^\infty$,

(iv) $p(\pi_m s) \leqslant p(s)$ for $s \in R^n$ and for $m \leqslant n$ (resp. $s \in \bigcup_{n=1}^{\infty} R^n$ and $m = 1, 2, \ldots$).

The elements of $\mathscr{B}_n$ and $\mathscr{B}$ will be called *norms*.

A *sign-automorphism* is any map $g\colon R^\infty \to R^\infty$ defined by $(gt)(i) = g(i)t(i)$ for $t \in R^\infty$, where $g(i) = \pm 1$ $(i = 1, 2, \ldots)$. By G we shall denote the set of all sign-automorphisms.

Let us set

$$\mathscr{U}_n = \{p \in \mathscr{B}_n\colon p(gs) = p(s) \text{ for } s \in R^n;\ g \in G\} \qquad (n = 1, 2, \ldots),$$

$$\mathscr{U} = \{p \in \mathscr{B}\colon p(gs) = p(s) \text{ for } s \in \bigcup_{n=1}^{\infty} R^n;\ g \in G\}.$$

Finally for $n = 1, 2, \ldots$ and for $p, q \in \mathscr{B}_n$ let

$$d_n(p, q) = \log\Big(\sup_{t \in \partial I^n} p(t)\cdot\big(q(t)\big)^{-1}\cdot \sup_{t \in \partial I^n} q(t)\cdot\big(p(t)\big)^{-1}\Big),$$

where $\partial I^n = \{t \in I^n\colon \max_{i \leqslant n}|t(i)| = 1\}$.

LEMMA 1. *d_n is a metric on $\mathscr{B}_n$. The metric space $(\mathscr{B}_n, d_n)$ is compact.*

Proof. The first assertion is trivial. To prove the second denote by $\hat{p}$ the restriction of a $p \in \mathscr{B}_n$ to ∂I^n. Then the family $\hat{\mathscr{B}}_n = \{\hat{p}\colon p \in \mathscr{B}_n\}$ consists of functions uniformly bounded (by $2n$) and equicontinuous. (This follows immediately from the inequality $|p(s)-p(t)| \leqslant p(s-t) \leqslant \sum_{i=1}^{n}|s(i)-t(i)| \leqslant 2n$ for $s, t \in \partial I^n$ and for $p \in \mathscr{B}_n$, which is a simple consequence of (i) and (ii).) Furthermore, if a sequence $(\hat{p}_m)$ of elements of $\hat{\mathscr{B}}_n$ converges uniformly to a function f, then $f = \hat{p}$ where $p(0) = 0$ and $p(s) = \max_{i \leqslant n}|s(i)|\cdot f(s\cdot[\max_{i \leqslant n}|s(i)|]^{-1})$ for $s \neq 0$. Thus, by the Ascoli theorem $\hat{\mathscr{B}}_n$ is a compact set in the space of all continuous real-valued functions on ∂I^n. Hence the map $p \to \hat{p}$ from $\mathscr{B}_n$ onto $\hat{\mathscr{B}}_n$, being obviously one-to-one and continuous, is a homeomorphism. This completes the proof.

For a fixed index n and for p belonging either to $\mathscr{B}_m$ with $m \geqslant n$ or to $\mathscr{B}$ we denote by $J_n(p)$ the restriction of p to R^n.

LEMMA 2. *The restriction operator J_n has the following properties:*

(1) *if $m \geqslant n$, then $d_n(J_n(p), J_n(q)) \leqslant d_m(p, q)$ for $p, q \in \mathscr{B}_m$;*

(2) $J_n(\mathscr{B}_{n+1}) = \mathscr{B}_n$; $J_n(\mathscr{U}_{n+1}) = \mathscr{U}_n$ $(n = 1, 2, \ldots)$;

(3) *if $p \in \mathscr{B}_{n+1}$ and $\tilde{q} \in \mathscr{B}_n$, then there exists $q \in \mathscr{B}_{n+1}$ such that $\tilde{q} = J_n(q)$ and $d_n(J_n(p), J_n(q)) = d_{n+1}(p, q)$. Moreover, if $p \in \mathscr{U}_{n+1}$ and $\tilde{q} \in \mathscr{U}_n$, then q may be taken from $\mathscr{U}_{n+1}$.*

Proof. (1) is an obvious consequence of the definition of d_n and d_{n+1}. Clearly (2) follows from (3).

To prove (3), let

$$a = \inf_{t \in \partial I^n} p(t) \cdot (\tilde{q}(t))^{-1}, \qquad b = \sup_{t \in \partial I^n} p(t) \cdot (\tilde{q}(t))^{-1}.$$

Then $b \geqslant 1 \geqslant a > 0$ and $d_n(J_n(p), q) = \log b - \log a$. Clearly, we have

(4) $$b^{-1} p(t) \leqslant \tilde{q}(t) \leqslant a^{-1} p(t) \quad \text{for } t \in R^n.$$

Let us set

$$\tilde{Q} = \{t \in R^n : |\tilde{q}(t)| \leqslant 1\},$$
$$P = \{s \in R^{n+1} : |p(s)| \leqslant 1\}.$$

Let Q be the smallest convex set in R^{n+1} such that

$$Q \supset \tilde{Q}, \quad Q \supset aP, \quad \pm e_{n+1} \in Q.$$

Define q as the Minkowski functional of Q, i.e.

$$q(s) = \inf\{c > 0 \mid c^{-1} s \in Q\} \quad \text{for } s \in R^{n+1}.$$

Observe that (4) is equivalent to the following inclusion:

(5) $$bP \cap R^n \supset \tilde{Q} \supset aP \cap R^n.$$

Thus

(6) $$Q \cap R^n = \tilde{Q} \quad \text{and} \quad bP \supset Q \supset aP.$$

Hence q satisfies condition (i) of the norm because Q is a symmetric convex neighbourhood of zero in R^{n+1}. By (6), $J_n(q) = \tilde{q}$ and

(7) $$b^{-1} p(s) \leqslant q(s) \leqslant a^{-1} p(s) \quad \text{for } s \in R^{n+1}.$$

Combining (7) with (1) we get $d_{n+1}(p, q) = d_n(J_n(p), J_n(q)) = \log b - \log a$. Condition (iii) restated in terms of convex bodies means simply that $Q \subset I^{n+1}$. Since p and $\tilde{q}$ satisfy (iii), we get $aP \subset P \subset I^{n+1}$; $\pm e_{n+1} \in I^{n+1}$; $\tilde{Q} \subset I^n \subset I^{n+1}$. Thus $Q \subset I^{n+1}$; equivalently q satisfies (iii). Since $e_{n+1} \in Q$

and $c \cdot e_{n+1} \notin I^{n+1} \supset Q$ for $|c| > 1$, we infer that $q(e_{n+1}) = 1$. Furthermore, $q(e_i) = \tilde{q}(e_i) = 1$ for $i \leqslant n$ because $\tilde{q}$ satisfies (ii). Hence q satisfies (ii). Next we shall check that q satisfies (iv). To this end we first show that

(8) $$q(s) \geqslant q(\pi_n s) = \tilde{q}(\pi_n s) \quad \text{for } s \in R^{n+1}.$$

Let $L_{n+1} = \{r \in R^{n+1} : \pi_n r = 0\}$. Then (8) is equivalent to the inclusion

(9) $$\tilde{Q} \oplus L_{n+1} \supset Q$$

(if $T_i \subset R$ $(i = 1, 2)$, then $T_1 \oplus T_2 = \{t \in R : t = t_1 + t_2 : t_1 \in T_1, t_2 \in T_2\}$).

Since p satisfies (iv), we have $(P \cap R^n) \oplus L_{n+1} \supset P$. Thus, by (5),

(10) $$\tilde{Q} \oplus L_{n+1} \supset (aP \cap R^n) \oplus L_{n+1} \supset aP.$$

Obviously

(11) $$\tilde{Q} \subset \tilde{Q} \oplus L_{n+1} \quad \text{and} \quad \pm e_{n+1} \in \tilde{Q} \oplus L_{n+1}.$$

Clearly (10) and (11) imply (9) and therefore (8). Combining (8) with the assumption that q satisfies (iv) for $m = 1, 2, \ldots, n$ we get

$$q(s) \geqslant \tilde{q}(\pi_n s) \geqslant \tilde{q}(\pi_m s) = q(\pi_m s) \quad \text{for } s \in R^{n+1}$$

(because $\pi_m \pi_n = \pi_m$ for $m \leqslant n$). Thus q satisfies (iv).

Finally, we will show that if $p \in \mathscr{U}_{n+1}$ and $\tilde{q} \in \mathscr{U}_n$, then $q \in \mathscr{U}_{n+1}$. To this end observe that our assumption implies that $gP = P$ and $g\tilde{Q} = \tilde{Q}$ for each g in G. Since $g\pi_n = \pi_n g$, it follows from the definition of Q that $gQ = Q$. Equivalently, $q(gs) = q(s)$ for $s \in R^{n+1}$ and for $g \in G$. This completes the proof.

PROPOSITION 1. *Let $\varepsilon > 0$. There exists a sequence $(\mathscr{A}_n)$ such that*

(12) *$\mathscr{A}_n$ is a finite subset of $\mathscr{B}_n$ [resp. of $\mathscr{U}_n$],*

(13) *$\mathscr{A}_{n+1}$ is an $\varepsilon \cdot (1 - 2^{-n-1})$-net for $\mathscr{B}_{n+1}$ [resp. $\mathscr{U}_{n+1}$],*

(14) $$J_n(\mathscr{A}_{n+1}) = \mathscr{A}_n.$$

Proof. Put $\mathscr{B}_1 = \mathscr{U}_1 = \mathscr{A}_1$ (observe that $\mathscr{B}_1 = \mathscr{U}_1$ is a one-point set). Suppose that for some $m \geqslant 1$ the sets $\mathscr{A}_i$ $(i \leqslant m)$ have already been defined to satisfy conditions (12)-(14). Pick in $\mathscr{B}_{m+1}$ [resp. in $\mathscr{U}_{m+1}$] a finite set, say $\mathscr{F}$, which is an $\varepsilon \cdot 2^{-m-1}$-net for $\mathscr{B}_{m+1}$ [resp. for $\mathscr{U}_{m+1}$] (this is possible in view of Lemma 1). In view of (3) of Lemma 2 for each pair $(p, \tilde{q})$ such that $p \in \mathscr{F}$ and $\tilde{q} \in \mathscr{A}_m$ we construct a norm $q = q(p, \tilde{q})$ in $\mathscr{B}_{m+1}$ [resp. in $\mathscr{U}_{m+1}$] so that $J_m(q) = \tilde{q}$ and $d_{m+1}(p, q) = d_m(J_m(p), \tilde{q})$. Since (by the inductive hypothesis) $\mathscr{A}_m$ is an $\varepsilon(1 - 2^{-m})$-net for $\mathscr{B}_m$ [resp. for $\mathscr{U}_m$] and since $\mathscr{F}$ is a $2^{-m-1}\varepsilon$-net for $\mathscr{B}_{m+1}$ [resp. for $\mathscr{U}_{m+1}$], the set

$$\mathscr{A}_{m+1} = \{q = q(p, \tilde{q}) : \tilde{q} \in A_m, p \in \mathscr{F}\}$$

is an $\varepsilon(1-2^{-m-1})$-net for $\mathscr{B}_{m+1}$ [resp. for $\mathscr{U}_{m+1}$]. Since $J_m(q(p,\tilde{q}))=\tilde{q}$ for each $q(p,\tilde{q})$ in $\mathscr{A}_{m+1}$, we infer that $J_m(\mathscr{A}_{m+1})=\mathscr{A}_m$. This completes the induction and the proof of the Proposition.

Our next lemma is a modification of a result of Gurariĭ and Kadec ([10], Theorem 1).

LEMMA 3. *Let $1\leqslant k\leqslant n$ and let $\tilde{p}\in\mathscr{B}_n$ and $q\in\mathscr{B}_{k+1}$. Suppose that there exists a sequence $1\leqslant i_1<i_2<\ldots<i_k<i_{k+1}=n+1$ such that*

$$q(t)=\tilde{p}\Big(\sum_{j=1}^{k}t(j)e_{i_j}\Big)\quad \textit{for } t\in R^k,$$

$$\tilde{p}(s)\geqslant\tilde{p}\Big(\sum_{j=1}^{k}s(i_j)e_{i_j}\Big)\quad \textit{for } s\in R^n.$$

Then there exists a $p\in\mathscr{B}_{n+1}$ such that $J_n(p)=\tilde{p}$ and

$$q(t)=p\Big(\sum_{j=1}^{k+1}t(j)e_{i_j}\Big)\quad \textit{for } t\in R^{k+1},$$

$$p(s)\geqslant p\Big(\sum_{j=1}^{k+1}s(i_j)e_{i_j}\Big)\quad \textit{for } s\in R^{n+1}.$$

Moreover, if $\tilde{p}\in\mathscr{U}_n$ and $q\in\mathscr{U}_{k+1}$, then p may be taken from $\mathscr{U}_{n+1}$.

Proof. Let us set for $r\in R^k$, $s\in R^n$ and $c\in R$

$$F(r,s,c)=\tilde{p}\Big(s-\sum_{j=1}^{k}r(j)e_{i_j}\Big)+q(r+ce_{k+1}),$$

$$p(s+ce_{n+1})=\inf_{r\in R^k}F(r,s,c).$$

To show that p satisfies (i) observe that p may be regarded as the quotient norm of the quotient space X/Y, where X is the space of all pairs (s,t) $(s\in R^n, t\in R^{k+1})$ with the norm $\|(s,t)\|=\tilde{p}(s)+q(t)$ and

$$Y=\Big\{(s,t)\in X\colon s+\sum_{j=1}^{k}t(j)e_{i_j}=0;\ t(k+1)=0\Big\}.$$

Next observe that

$$(15)\qquad p(s)=\tilde{p}(s)\ \text{for}\ s\in R^n\quad\text{and}\quad p\Big(\sum_{j=1}^{k+1}t(j)e_{i_j}\Big)=q(t)\ \text{for}\ t\in R^{k+1}.$$

Indeed, we have $p(s)\leqslant F(0,s,0)=\tilde{p}(s)$. While for arbitrary $r\in R^k$ we have

$$\tilde{p}\Big(s-\sum_{j=1}^{k}r(j)e_{i_j}\Big)+q(r)\geqslant\tilde{p}(s)-\tilde{p}\Big(\sum_{j=1}^{k}r(j)e_{i_j}\Big)+q(r)=\tilde{p}(s).$$

Hence $p(s)\geqslant\tilde{p}(s)$ and we get $p(s)=\tilde{p}(s)$. Similarly, if $s=\sum_{j=1}^{k}t(j)e_{i_j}$ for some $t\in R^k$, then for arbitrary $c\in R$ we have

$$p\Big(\sum_{j=1}^{k}t(j)e_{i_j}+ce_{i_{k+1}}\Big)\leqslant F(t,s,c)=q(t+ce_{k+1}),$$

while for arbitrary $r\in R^k$ we have

$$F\Big(r,\sum_{j=1}^{k}t(j)e_{i_j},c\Big)=\tilde{p}\Big(\sum_{j=1}^{k}(t-r)(j)e_{i_j}\Big)+q(r+ce_{k+1})$$
$$=q(t-r)+q(r+ce_{k+1})\geqslant q(t+ce_{k+1}).$$

Thus $p\big(\sum_{j=1}^{k}t(j)e_{i_j}+ce_{i_{k+1}}\big)\geqslant q(t+ce_{k+1})$, and this completes the proof of (15). Clearly (15) implies that $p(e_{n+1})=q(e_{k+1})=1$ and $p(e_m)=\tilde{p}(e_m)=1$ for $m\leqslant n$, because q and $\tilde{p}$ satisfy (ii). Thus p satisfies (ii). From the definition of p and the fact that q satisfies (iv) we infer that

$$p(s+ce_{n+1})\geqslant p(s)=\tilde{p}(s)\qquad\text{for}\ s\in R^n\ \text{and}\ c\in R.$$

Combining this inequality with (15) we conclude as in the proof of Lemma 2 that p satisfies (iv). The last inequality also implies that if $p(s+ce_{n+1})\leqslant 1$ for some $s\in R^n$ and for $c\in R$, then $\tilde{p}(s)\leqslant 1$. Thus $s\in I^n$ because $\tilde{p}$ satisfies (iii). On the other hand, the definition of p implies that if $p(s+ce_{n+1})\leqslant 1$, then there exists an $r\in R^k$ such that

$$q\Big(\sum_{j=1}^{k}r(j)e_j+ce_{k+1}\Big)\leqslant 1.$$

Thus $|c|\leqslant 1$ (because q satisfies (iii)). Hence p satisfies (iii) and this proves that $p\in\mathscr{B}_{n+1}$. Next if $s\in R^{n+1}$, then (15) implies that for every $r\in R^k$ we have

$$\tilde{p}\Big(\sum_{i=1}^{n}s(i)e_i-\sum_{j=1}^{k}r(j)e_{i_j}\Big)+q\big(r+s(n+1)e_{k+1}\big)$$
$$\geqslant\tilde{p}\Big(\sum_{j=1}^{k}\big(s(i_j)-r(j)\big)e_{i_j}\Big)+q\big(r+s(n+1)e_{k+1}\big)$$
$$=q\Big(\sum_{j=1}^{k}s(i_j)e_j-r\Big)+q\big(r+s(n+1)e_{k+1}\big)$$
$$\geqslant q\Big(\sum_{j=1}^{k+1}s(i_j)e_j\Big)=p\Big(\sum_{j=1}^{k+1}s(i_j)e_{i_j}\Big).$$

Thus $p(s) \geqslant p\left(\sum_{j=1}^{k+1} s(i_j)e_{i_j}\right)$ for $s \in R^{n+1}$.

Finally, assume that $\tilde{p} \in \mathscr{U}_n$ and $q \in \mathscr{U}_{k+1}$. For each $g \in G$ pick g^* in G so that $g^*(j) = g(i_j)$ for $j = 1, 2, \ldots, k+1$. Then, since $\tilde{p} \in \mathscr{U}_n$ and $q \in \mathscr{U}_{k+1}$, we get

$$\tilde{p}\left(gs - \sum_{j=1}^{k}(g^*r)(j)e_{i_j}\right) = \tilde{p}\left(g\left[s - \sum_{j=1}^{k} r(j)e_{i_j}\right]\right) = \tilde{p}\left(s - \sum_{j=1}^{k} r(j)e_{i_j}\right),$$

$$q\left(g^*r + g(n+1)ce_{k+1}\right) = q\left(g^*[r + ce_{k+1}]\right) = q(r + ce_{k+1}).$$

Hence $F\left(g^*r, gs, g(n+1)c\right) = F(r, s, c)$. Therefore

$$\begin{aligned} p(g[s + ce_{n+1}]) &= \inf_{r \in R^k} F\left(r, gs, g(n+1)c\right) \\ &= \inf_{g^*r \in R^k} F\left(g^*r, gs, g(n+1)c\right) \\ &= \inf_{g^*r \in R^k} F(r, s, c) = p(s + ce_{n+1}). \end{aligned}$$

Hence $p \in \mathscr{U}_{n+1}$. This completes the proof.

The next proposition gives in fact the construction of the universal basis.

PROPOSITION 2. *For every $\varepsilon > 0$ there exists a $p \in \mathscr{B}$ [resp. $p \in \mathscr{U}$] such that for every $\tilde{q} \in \mathscr{B}$ [resp. $\tilde{q} \in \mathscr{U}$] there exists an increasing sequence of indices $i_1 < i_2 < \ldots$ such that*

$$(16) \qquad (1+\varepsilon)^{-1}\tilde{q}(t) \leqslant p\left(\sum_{j=1}^{k} t(j)e_{i_j}\right) \leqslant (1+\varepsilon)\tilde{q}(t) \qquad (t \in R^k;\ k = 1, 2, \ldots),$$

and

$$(16a) \qquad p(s) \geqslant p\left(\sum_{j=1}^{\infty} s(i_j)e_{i_j}\right) \quad \textit{for } s \in \bigcup_{k=1}^{\infty} R^k.$$

Proof. Observe first that Proposition 1 implies that there exists an increasing sequence of indices $1 = N(1) < N(2) < \ldots$ and a sequence (q_n) such that

(17) if $N(k) \leqslant n < N(k+1)$, then $q_n \in \mathscr{B}_k$ [resp. $q_n \in \mathscr{U}_k$],

(18) the set $\mathscr{A}_k = \bigcup_{n=N(k)}^{N(k+1)-1} \{q_n\}$ forms a $\log(1+\varepsilon)$-net for $\mathscr{B}_k$ [resp. $\mathscr{U}_k$],

$$(19) \qquad J_k(\mathscr{A}_{k+1}) = \mathscr{A}_k.$$

Next we shall define inductively a sequence of norms (p_n) and a sequence of finite increasing sequences of indices $(\alpha(n))$ such that

$$(20) \qquad p_n \in \mathscr{B}_n \text{ [resp. } p_n \in \mathscr{U}_n], \qquad J_n(p_{n+1}) = p_n;$$

(21) if $N(k) \leqslant n < N(k+1)$ and $\alpha(n) = \{i_1(n), i_2(n), \ldots, i_{k(n)}(n)\}$, then $k(n) = k$ and $i_{k(n)}(n) = n$;

(22) if $1 \leqslant m \leqslant n$, $N(l) \leqslant m \leqslant N(l+1)$ and $J_l(q_n) = q_m$, then $i_j(m) = i_j(n)$ for $j = 1, 2, \ldots, l(m)$;

(23) if $1 \leqslant m \leqslant n$, then $p_n\left(\sum_{j=1}^{k(m)} t(j)e_{i_j(m)}\right) = q_{k(m)}(t)$ for $t \in R^{k(m)}$ and

$$p_n(s) \geqslant p_n\left(\sum_{j=1}^{k(m)} s\left(i_j(m)\right)e_{i_j(m)}\right) \quad \text{for } s \in R^n.$$

Let us set $p_1 = q_1$ and $\alpha(1) = \{1\}$. Suppose that for some $n \geqslant 1$ the norms p_m and the finite increasing sequences of indices $\alpha(m)$ have been defined to satisfy conditions (20)-(23) for each $m \leqslant n$. Let us choose k so that $N(k+1) \leqslant n+1 < N(k+2)$. Then it follows from (19) that there exists an m such that $q_m \in \mathscr{A}_k$ and $J_k(q_{n+1}) = q_m$. By the inductive hypothesis (conditions (21) and (23)), we have $k = k(m) \leqslant m \leqslant n$ and for every $t \in R^k$

$$p_n\left(\sum_{j=1}^{k(m)} t(j)e_{i_j(m)}\right) = q_m(t) = q_{n+1}(t)$$

and

$$p_n(s) \geqslant p_n\left(\sum_{j=1}^{k(m)} s\left(i_j(m)\right)e_{i_j(m)}\right) \quad \text{for } s \in R^n.$$

Thus we are in the position of Lemma 3. By this Lemma, there exists a norm p' in $\mathscr{B}_{n+1}$ [resp. in $\mathscr{U}_{n+1}$] such that

$$p'\left(\sum_{j=1}^{k} t(j)e_{i_j(m)} + t(k+1)e_{n+1}\right) = q_{n+1}(t), \qquad t \in R^{k+1},$$

and

$$p'(s) \geqslant p'\left(\sum_{j=1}^{k} s\left(i_j(m)\right)e_{i_j(m)} + s(n+1)e_{n+1}\right) \quad \text{for } s \in R^{n+1}.$$

Let us put $p_{n+1} = p'$ and $\alpha(n+1) = \{i_1(m), i_2(m), \ldots, i_k(m), n+1\}$. Clearly, so defined p_{n+1} and $\alpha(n+1)$ satisfy conditions (20)-(23). This completes the induction.

Let us define the norm p by

$$p(t) = p_n(t) \quad \text{for } t \in R^n.$$

It follows from (20) that p is well defined and belongs to $\mathscr{B}$ [resp. to $\mathscr{U}$]. Now pick $\tilde{q} \in \mathscr{B}$ [resp. $\tilde{q} \in \mathscr{U}$]. It follows from (18) that there exists a sequence (q'_k) such that

(24) $$q_k' = q_{n_k} \in \mathscr{A}_k \quad (k = 1, 2, \ldots),$$

(25) $$d_k\big(q_k', J_k(\tilde{q})\big) < \log(1+\varepsilon).$$

Observe that without loss of generality we may also assume that

(26) $$J_k(q_n') = q_k' \quad \text{for } k \leqslant n.$$

(Indeed, let (q_n') be an arbitrary sequence of norms such that $q_n' \in \mathscr{A}_n$ for $n = 1, 2, \ldots$ By (19), $J_k(q_n') \in \mathscr{A}_k$ for $n \geqslant k$. Hence the set $\mathscr{F}_k = \bigcup_{n \geqslant k} \{J_k(q_n')\}$ is finite and therefore compact in the discrete topology. Thus the Cartesian product $F = \mathsf{P}\mathscr{F}_k$ is compact. Let

$$Z_n = \{(q_k'') \in F \colon J_k(q_n'') = q_k'' \text{ for } k \leqslant n\}.$$

Then (Z_n) is a decreasing sequence of non-empty closed subsets of F. Thus, by compactness of F, there exists a sequence (q_k'') in F which belongs to all Z_n; equivalently the sequence (q_k'') satisfies (26). By definition of F, for every index k there exists an index $n = n(k)$ such that $J_k(q_n') = q_k''$ and $q_n' \in \mathscr{A}_n$. Hence, if the sequence (q_n') satisfies (25), then (by (1))

$$d_k\big(q_k'', J_k(\tilde{q})\big) = d_k\Big(J_k(q_m'), J_k\big(J_m(\tilde{q})\big)\Big) \leqslant d_m\big(q_m', J_m(\tilde{q})\big) < \log(1+\varepsilon).$$

Hence the sequence (q_k'') satisfies conditions (24)-(26)).

Finally, we define the increasing sequence of indices $i_1 < i_2 < \ldots$ by $i_k = n_k$ $(k = 1, 2, \ldots)$, where n_k is defined by (24).

It follows from (26), (21), (22) and (24) that $a(n_k) = \{i_1, i_2, \ldots, i_k\}$. Thus, by (23), we get (16a) and

(27) $$p\Big(\sum_{j=1}^{k} t(j) e_{i_j}\Big) = p_{n_k}\Big(\sum_{j=1}^{k} t(j) e_{i_j}\Big) = q_{n_k}(t) = q_k'(t) \quad \text{for } t \in R^k.$$

Combining (25) with (27) we obtain (16). This completes the proof.

Proof of Theorem 1. Fix $\varepsilon > 0$ and denote by $\tilde{E} = \tilde{E}_\varepsilon$ the normed linear space $\big(\bigcup_{n=1}^{\infty} R^n, p\big)$, where $p \in \mathscr{B}$ [resp. $p \in \mathscr{U}$] is that of Proposition 2. Let $E = E_\varepsilon$ denote the completion of $\tilde{E}$ (We identify $\tilde{E}$ with its canonical image in E). The unit vectors (e_i) form a basis [resp. an unconditional basis] in E, because the linear combinations of (e_i) are dense in E and, by (iv),

$$p\Big(\sum_{i=1}^{n} c_i e_i\Big) \leqslant p\Big(\sum_{i=1}^{n+m} c_i e_i\Big) \quad \text{for real } c_1, c_2, \ldots, c_{n+m} \ (n, m = 1, 2, \ldots).$$

[Moreover, if $p \in \mathscr{U}$, then $p\big(\sum_{i=1}^{n+m} c_i e_i\big) \geqslant p\Big(g\big(\sum_{i=1}^{n} c_i e_i\big)\Big)$ for $g \in G$] (cf. e.g. [2], p. 127, and [11]). We show that this basis is complementably universal for the family of all seminormalized bases [resp. for the family of all seminormalized unconditional bases].

Let X be a Banach space with a seminormalized basis [resp. a seminormalized unconditional basis], say (x_j). Let

$$a = \inf_j \|x_j\| \quad \text{and} \quad b = \sup_j \|x_j\|.$$

Then $0 < a \leqslant b < +\infty$, because the basis is seminormalized. Let us set

$$\tilde{q}(t) = \sup_n \Big(\max\Big(\Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \cdot b^{-1};\ |t(n)|\Big)\Big) \quad \text{for } t \in \bigcup_{n=1}^{\infty} R^n$$

$$\text{[resp. } \tilde{q}(t) = \sup_{g \in G} \sup_n \Big(\max\Big(\Big\|\sum_{j=1}^{n} g t(j) x_j\Big\| \cdot b^{-1};\ |t(n)|\Big)\Big) \text{ for } t \in \bigcup_{n=1}^{\infty} R^n\text{]}.$$

Since (x_j) is a basis [resp. an unconditional basis], there is a $K \geqslant 1$ such that

$$\sup_n \Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \leqslant K \Big\|\sum_{k=1}^{\infty} t(j) x_j\Big\| \quad \text{for } t \in \bigcup_{k=1}^{\infty} R^k$$

$$\text{[resp. } \sup_n \sup_{g \in G} \Big\|\sum_{j=1}^{n} g t(j) x_j\Big\| \leqslant K \Big\|\sum_{j=1}^{\infty} t(j) x_j\Big\| \quad \text{for } t \in \bigcup_{k=1}^{\infty} R^k\text{]}$$

(cf. [5], p. 69, and [11]). Hence

$$|t(n)| \leqslant 2Ka^{-1} \Big\|\sum_{j=1}^{\infty} t(j) x_j\Big\|.$$

Thus we get

(28) $$b^{-1} \Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \leqslant \tilde{q}(t) \leqslant 2Ka^{-1} \Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \quad \text{for } t \in R^n \ (n = 1, 2, \ldots).$$

We omit the standard verification that $\tilde{q} \in \mathscr{B}$ [resp. $\tilde{q} \in \mathscr{U}$]. Next we apply Proposition 2 to define the increasing sequence of indices $i_1 < i_2 < \ldots$ for which inequalities (16) and (16a) are satisfied. Combining (16) with (28) we get

$$b^{-1}(1+\varepsilon)^{-1} \Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \leqslant p\Big(\sum_{j=1}^{n} t(j) e_{i_j}\Big)$$

$$\leqslant 2Ka^{-1}(1+\varepsilon) \Big\|\sum_{j=1}^{n} t(j) x_j\Big\| \quad \text{for } t \in R^n \ (n = 1, 2, \ldots).$$

Clearly, the last inequality and (16a) imply that the basis (x_j) is equivalent to the complemented subbasis $(e_{i_\cdot})$. This completes the proof.

Proof of Corollary 1. This Corollary is an obvious consequence of Theorem 1 and the following well-known facts:

LEMMA 4. *If a basis (x_j) in a Banach space X is equivalent to a subbasis (e_{i_j}) of a basis (e_i) in a Banach space E, then the operator $u: X \to E$ defined by*

$$ux = \sum_{j=1}^{\infty} c_j e_{i_j} \quad \text{for} \quad x = \sum_{j=1}^{\infty} c_j x_j \in X$$

is an isomorphism from X onto a closed linear subspace of E which is spanned by the subbasis (e_{i_j}).

LEMMA 5. *If (e_{i_j}) is a complemented subbasis of a basis (e_i) in a Banach space E, then the operator $P: E \to E$ defined by*

$$Px = \sum_{j=1}^{\infty} c_{i_j} e_{i_j} \quad \text{for} \quad x = \sum_{i=1}^{\infty} c_i e_i$$

is a bounded linear projection from E onto the closed linear subspace which is spanned by the subbasis (e_{i_j}).

Remark. Observe that a basis (e_i) is unconditional if and only if each of its subbases is complemented (cf. [5], pp. 58-59 and 73). Hence a universal unconditional basis is automatically complementably universal.

3. Banach spaces which have universal bases. We recall that a Banach space E is said to be *isomorphically universal for all separable Banach spaces* if each separable Banach space is isomorphic to a closed linear subspace of E.

THEOREM 2. *Let E be a Banach space with a basis. Then the following conditions are equivalent:*

(o) *there exists a semi normalized basis in E which is universal for the family of all seminormalized bases;*

(oo) *E is isomorphically universal for all separable Banach spaces.*

Proof. The implication (o) $\Rightarrow$ (oo) is easy to prove. Since the space $C(0;1)$ has a basis ([5], p. 69) and is isomorphically universal for all separable Banach spaces ([5], p. 93), this implication is an obvious consequence of Lemma 4.

Proof of the implication (oo) $\Rightarrow$ (o) is much more sophisticated. The crucial point of the proof is the following fact:

PROPOSITION 3. *The space $C(0;1)$ has a normalized basis, say (z_j), which is universal for the family of all seminormalized bases.*

The derivation of the implication (oo) $\Rightarrow$ (o) *from Proposition* 3. Let E be a Banach space which is isomorphically universal for all separable Banach spaces. Then E contains a subspace isomorphic to $C(0;1)$.

Hence, by [15], there is another subspace of E, say E_1, which is isomorphic to $C(0;1)$ and is complemented in E. Thus E is isomorphic to the Cartesian product $E_0 \oplus C(0;1)$, where E_0 is the kernel of a projection from E onto E_1. Since $C(0;1)$ is isomorphic to its Cartesian square ([1], p. 184), E is isomorphic to the space $E_0 \oplus C(0;1) \oplus C(0;1)$ and therefore E is isomorphic to the space $E \oplus C(0;1)$. Now let $(\tilde{e}_j)$ be an arbitrary basis in E. We define a basis (x_i) in the space $E \oplus C(0;1)$ by

$$x_{2j-1} = (\tilde{e}_j, 0), \quad x_{2j} = (0, z_j) \quad (j = 1, 2, \ldots),$$

where (z_j) is the basis in $C(0;1)$ of Proposition 3. Finally, if u is an isomorphism from $E \oplus C(0;1)$ onto E, then the basis (e_i) defined by $e_i = ux_i$ $(i = 1, 2, \ldots)$ has the desired properties.

Before proving Proposition 3 we recall that the *Schauder basis* $(\varphi_n)_{n=0}^{\infty}$ in $C(0;1)$ is defined by (cf. [4] and [7])

$$\varphi_0(t) \equiv 1, \quad \varphi_1(t) = t,$$

$$\varphi_{2^k+r}(t) = \begin{cases} 0 & \text{for } t \notin \left(2^{-k-1}(2r-2); 2^{-k-1}(2r)\right), \\ 1 & \text{for } t = 2^{-k-1}(2r-1), \\ \text{linear for the other } t \end{cases}$$

$$(r = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots).$$

For every x in $C(0;1)$ we have $x = \sum_{n=0}^{\infty} f_n(x) \cdot \varphi_n$, where the coefficient functionals $f_n(\cdot)$ are defined by (cf. [4], [7])

$$f_0(x) = x(0), \quad f_1(x) = x(1) - x(0),$$

$$f_{2^k+r}(x) = x\left(2^{-k-1}(2r-1)\right) - 2^{-1}x\left(2^{-k-1}(2r-2)\right) - 2^{-1}x(2r2^{-k-1})$$

$$(r = 1, 2, \ldots, 2^k;\ k = 0, 1, \ldots).$$

Let us set

$$D_0 = \{0\}, \quad D_1 = \{0\} \cup \{1\},$$

$$D_{2^k+r} = \bigcup_{\nu=0}^{2^k} \{\nu 2^{-k}\} \cup \bigcup_{\nu=1}^{2r-1} \{\nu 2^{-k-1}\} \quad (r = 1, 2, \ldots, 2^k;\ k = 0, 1, \ldots).$$

Clearly, we have

LEMMA 6. *Let $x \in C(0;1)$. If for a certain index n we have $x(t) = 0$ for all $t \in D_n$, then $f_k(x) = 0$ for $k \leqslant n$, i.e. $x = \sum_{k=n+1}^{\infty} f_k(x)\varphi_k$.*

Now we are ready for the proof of the main lemma. We recall that a basis (x_n) in a Banach space X is said to be *monotone* if

$$\left\|\sum_{i=1}^{n} c_i x_i\right\| \leqslant \left\|\sum_{i=1}^{n+1} c_i x_i\right\| \quad (c_1, c_2, \ldots, c_n \text{ real};\ n = 1, 2, \ldots).$$

LEMMA 7. *Let $(x_k)_{k=1}^{\infty}$ be a normalized monotone basis in a Banach space X, then for every $\varepsilon > 0$ there exist an isometric isomorphism $u\colon X \xrightarrow[\text{into}]{} C(0;1)$ and an increasing sequence of indices $1 = n_0 < n_1 < \ldots$ such that*

$$ux_k = \sum_{i=n_{k-1}+1}^{\infty} f_i(ux_k)\varphi_i, \qquad \Big\|\sum_{i=n_k+1}^{\infty} f_i(ux_k)\varphi_i\Big\| < \varepsilon 2^{-k}.$$

Proof. Let $\Delta \subset [0;1]$ be an arbitrary set homeomorphic to the Cantor discontinuum and such that $\Delta \cap D_n = \emptyset$ for $n = 0, 1, \ldots$ By [5], p. 93, there exists an isometric isomorphism $w\colon X \xrightarrow[\text{into}]{} C(\Delta)$. Let us put for convenience: $x_0 = 0 \in X$, $y_0 = 0 \in C(0;1)$ and $n_0 = 1$. Next we define inductively a sequence of functions (y_k) in $C(0;1)$ and an increasing sequence of indices (n_k) such that

$$y_k(t) = \begin{cases} (wx_k)(t) & \text{for } t \in \Delta, \\ 0 & \text{for } t \in D_{n_{k-1}}, \\ \text{linear for the other } t, & \end{cases} \qquad (k = 1, 2, \ldots) \tag{29}$$

$$\Big\|\sum_{i=n_k+1}^{\infty} f_i(y_k)\varphi_i\Big\| < 2^{-k}\varepsilon \qquad (k = 1, 2, \ldots). \tag{30}$$

We set

$$ux = \sum_{k=1}^{\infty} c_k y_k \quad \text{for } x = \sum_{k=1}^{\infty} c_k x_k \in X.$$

Clearly, in order to prove that u is an isometric isomorphism from X into $C(0;1)$ it is enough to check that

$$\Big\|\sum_{i=0}^{k} c_i y_i\Big\| = \Big\|\sum_{i=0}^{k} c_i x_i\Big\| \qquad (c_0, c_1, \ldots, c_k \text{ real; } k = 0, 1, \ldots). \tag{31}$$

Formula (31) is obvious for $k = 0$ because $y_0 = 0$ and $x_0 = 0$. Suppose that (31) holds for some $m \geqslant 0$. Choose arbitrary real numbers $c_0, c_1, c_2, \ldots, c_{m+1}$ and put $x = \sum_{i=0}^{m} c_i x_i$; $c_{m+1} = c$. Since each y_i is an extension of wx_i and since w is an isometric embedding, we have

$$\|ux + cy_{m+1}\| \geqslant \sup_{t\in\Delta} |(ux + cy_{m+1})(t)| = \|wx + cwx_{m+1}\| = \|x + cx_{m+1}\|. \tag{32}$$

On the other hand, formula (29) implies that the function $ux + cy_{m+1}$ is linear on each interval of the set $[0;1] \setminus (\Delta \cup D_{n_m})$. Therefore

$$\|ux + cy_{m+1}\| \leqslant \sup_{t\in\Delta\cup D_{n_m}} |(ux + cy_{m+1})(t)| = \max\Big(\sup_{t\in\Delta} |(ux + cy_{m+1})(t)|, \sup_{t\in D_{n_m}} |(ux + cy_{m+1})(t)|\Big). \tag{33}$$

Since $ux + cy_{m+1}$ is an extension of the function $w(x + cx_{m+1})$, we get (taking into account that w is an isometry)

$$\sup_{t\in\Delta} |(ux + cy_{m+1})(t)| = \|w(x + cx_{m+1})\| = \|x + cx_{m+1}\|. \tag{34}$$

Since $y_{m+1}(t) = 0$ for $t \in D_{n_m}$, we get

$$\sup_{t\in D_{n_m}} |(ux + cy_{m+1})(t)| = \sup_{t\in D_{n_m}} |(ux)(t)| \leqslant \|ux\|. \tag{35}$$

It follows from the inductive hypothesis and the monotonicity os the basis (x_k) that

$$\|ux\| = \|x\| \leqslant \|x + cx_{m+1}\|. \tag{36}$$

Comparing (33) with (34), (35) and (36), we get $\|ux + cy_{m+1}\| \leqslant \|x + cx_{m+1}\|$. Hence, by (32), $\|ux + cy_{m+1}\| = \|x + cx_{m+1}\|$. This completef the induction.

Finally, observe that $y_k = ux_k$ $(k = 1, 2, \ldots)$. By (29), $(ux_k)(t) = 0$ for $t \in D_{n_{k-1}}$. Thus Lemma 6 implies that

$$ux_k = \sum_{i=n_{k-1}+1}^{\infty} f_i(ux_k)\varphi_i.$$

By (30), we get

$$\Big\|\sum_{i=n_k+1} f_i(ux_k)\varphi_i\Big\| < 2^{-k}\varepsilon.$$

This completes the proof.

The next lemma is an obvious consequence of Lemma 7 and of [3], Theorem 1.

LEMMA 8. *Each monotone basis (x_k) in a Banach space X is equivalent to a sequence of disjoint blocks with respect to the Schauder basis in $C(0;1)$,*

$$\tilde{y}_k = \Big\|\sum_{i=n_{k-1}+1}^{n_k} f_i(ux_k)\varphi_i\Big\|^{-1} \sum_{i=n_{k-1}+1}^{n_k} f_i(ux_k)\varphi_i,$$

where the increasing sequence (n_k) of indices and the isometrically isomorphic embedding $u\colon X \to C(0;1)$ are those of Lemma 7 (with $0 < \varepsilon < 1$).

Finally, we need the following recent result due to Zippin ([17], Lemma 3):

LEMMA 9. *Let $(e_n)_{n=1}^{\infty}$ be a basis in a Banach space E. Let $(n_k)_{k=0}^{\infty}$, with $n_0 = 0$, be an increasing sequence of indices. Let*

$$\tilde{y}_k = \sum_{i=n_{k-1}+1}^{n_k} c_i e_i \neq 0 \qquad (k = 1, 2, \ldots).$$

Then there is a basis in E, say $(\tilde{e}_n)_{n=1}^{\infty}$, such that $\tilde{e}_{n_k} = \tilde{y}_k$ $(k = 1, 2, \ldots)$. Moreover, if $\|\tilde{y}_k\| = 1$ $(k = 1, 2, \ldots)$, then $\|\tilde{e}_n\| = 1$ for $n = 1, 2, \ldots$

Proof of Proposition 3. Since every basis is equivalent to a monotone basis ([5], p. 67), it follows from Lemmas 8 and 9 that every normalized basis is equivalent to a subbasis of a normalized basis in $C(0;1)$. In particular, the universal basis for the family of all seminormalized bases (constructed in the proof of Theorem 1) is equivalent to a subbasis of a normalized basis in $C(0;1)$, say (z_j). Clearly (z_j) is the desired basis in $C(0;1)$. This completes the proof.

COROLLARY 2. *There exist $2^{\aleph_0}$ normalized bases which are universal for the family of all seminormalized bases and which span mutually non-isomorphic Banach spaces. Hence these bases are mutually non-equivalent.*

Proof. According to Theorem 2 and Lemma 4 it is enough to show that there exist $2^{\aleph_0}$ mutually non-isomorphic Banach spaces which are isomorphically universal for all separable Banach spaces and which have bases. For $1 < a < +\infty$ let $X_a = C \oplus l_a$, where $C = C(0;1)$. Clearly, each X_a is isomorphically universal for all separable Banach spaces because it contains a closed linear subspace isometrically isomorphic to C. Since both spaces C and l_a have bases, X_a also has a basis (cf. the derivation of the implication (oo) $\Rightarrow$ (o) from Proposition 3). Finally we shall show that if $a \neq \beta$, then X_a is not isomorphic to X_β. To this end we shall show that if $a \neq \beta$, then X_a does not have complemented subspaces isomorphic to l_β. (Clearly the subspace $0 \oplus l_\beta$ of X_β is complemented in X_β and is isomorphic to l_β.) Let $u\colon l_\beta \to X_a$ be an isomorphic embedding and let P_1 and P_2 denote the natural projections from X_a onto its subspaces $C \oplus 0$ and $0 \oplus l_a$ respectively. Let $l_{\beta,n}$ denote the subspace of l_β consisting of those sequences whose first n coordinates are zeros. By a result of Banach ([1], p. 205), the restriction of $P_2 u$ to $l_{\beta,n}$ does not have any bounded inverse $(n = 1, 2, \ldots)$. Hence there is an $x_n \in l_{\beta,n}$ such that $\|P_2 u x_n\| < n^{-1}$ and $\|x_n\| = 1$ $(n = 1, 2, \ldots)$. Observe that, by the well-known characterization of the weak convergence in l_β, the sequence (x_n) weakly converges to zero. Now suppose that there exists a bounded linear projection, say P, from X_a onto ul_β. Then

$$ux_n = Pux_n = PP_1 ux_n + PP_2 ux_n \quad (n = 1, 2, \ldots). \tag{37}$$

Let v denote the restriction of PP_1 to $C \oplus 0$. Clearly v may be regarded as a bounded linear operator from C into the space ul_β which is isomorphic to l_β. Since l_β is reflexive, v is weakly compact (cf. [6], p. 482, for the definition). Thus, by a result of Grothendieck [9] ([6], p. 494), v takes weakly convergent sequences into convergent sequences. Since the sequence (x_n) weakly converges to zero, the sequence $(P_1 ux_n)$ has the same property. Since $vP_1 ux_n = PP_1 P_1 ux_n = PP_1 ux_n$ for $n = 1, 2, \ldots$, we infer that $\lim_n PP_1 ux_n = 0$. Since $\|P_2 ux_n\| < n^{-1}$ for $n = 1, 2, \ldots$, we get $\lim_n PP_2 ux_n = 0$. Thus, by (37), $\lim_n ux_n = 0$. Hence $\lim_n x_n = 0$ because u is an isomorphic embedding. But this contradicts the condition $\|x_n\| = 1$ for $n = 1, 2, \ldots$ That completes the proof.

Observe that complementably universal bases for the family of all normalized bases exist neither in $C(0;1)$ nor in the spaces X_a of Corollary 2. Indeed, it follows from the proof of Corollary 2 that the space X_a and therefore $C(0;1)$ (being a complemented subspace of X_a) do not have complemented subspaces isomorphic to l_{2a}. Hence the desired conclusion follows from Lemmas 4 and 5.

Next we shall show that there exists a Banach space unique up to an isomorphism, which has a complementably universal basis for the family of all normalized bases. This will follow from Theorem 1 and the fact that all normalized bases which are complementably universal for the family of all normalized bases are in a certain sense equivalent. The same result holds for unconditional bases.

To formulate the next theorem we shall need the following concept:

Definition 2. Bases (x_n) and (y_n) in Banach spaces X and Y respectively are said to be *permutatively equivalent* if there exists a permutation $\sigma(\cdot)$ of the set of indices onto itself such that the sequence $(x_{\sigma(n)})$ is a basis in X and the bases $(x_{\sigma(n)})$ and (y_n) are equivalent.

Observe that a basis (x_n) is unconditional if and only if for every permutation $\sigma(\cdot)$ of the indices the sequence $(x_{\sigma(n)})$ is a basis (cf. [5], p. 73).

One can easily see that the relation of permutative equvalence of bases is reflexive, symmetric and transitive. In the sequel we use German letters $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ for denoting the equivalence classes with respect to this relation.

Now we are ready to state the next result:

THEOREM 3. *Every two seminormalized [unconditional] bases which are complementably universal for the family of all seminormalized [unconditional] bases are permutatively equivalent. Hence they span isomorphic Banach spaces.*

The proof of the theorem requires some lemmas and notation. We recall first the following well-known result (cf. [2], p. 127, and [11]):

LEMMA 10. *Let (x_n) be a sequence of non-zero elements of a Banach space X. Suppose that linear combinations of x_n are dense in X. Then*

(a) *(x_n) is a basis in X if and only if there exists a $K \geqslant 1$ such that*

$$\Big\|\sum_{i=1}^{n} c_i x_i\Big\| \leqslant K \Big\|\sum_{i=1}^{n+m} c_i x_i\Big\| \quad (c_i \text{ real}; i = 1, 2, \ldots, n+m; n, m = 1, 2, \ldots) \tag{38a}$$

(b) (x_n) *is an unconditional basis in* X *if and only if there exists a constant* $K^* \geqslant 1$ *such that*

$$\left\|\sum_{i=1}^{n} g(i)c_i x_i\right\| \leqslant K^* \left\|\sum_{i=1}^{n} c_i x_i\right\| \tag{38b}$$

$$(c_i \textit{ real};\ g(i) = \pm 1;\ i = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

If X and Y are Banach spaces and $1 \leqslant a < +\infty$, then $(X \oplus Y)_a$ denotes the Cartesian product of X and Y with the norm $\|(x, y)\| = (\|x\|^a + \|y\|^a)^{1/a}$. If $(X_a)_{a \in A}$ is a (countable) family of Banach spaces, then by $(\underset{a \in A}{\mathsf{P}} X_a)_2$ we denote the Banach space of functions $\mathbf{x} = (x(a))_{a \in A}$ such that $x(a) \in X_a$ for $a \in A$ and $\|\mathbf{x}\| = \left(\sum_{a \in A} \|x(a)\|^2\right)^{1/2} < +\infty$.

The next two lemmas enable us to define operations of "addition" and of "infinite power" on the equivalence classes of permutatively equivalent bases. For this purpose we need only a special case of Lemma 12 (with $X_i = X$ and $x_j^{(i)} = x_j$ for $i, j = 1, 2, \ldots$). However, in the present formulation Lemma 12 will be applied in the proof of Theorem 4.

LEMMA 11. *If* (x_i) *and* (y_i) *are normalized [unconditional] bases in Banach spaces* X *and* Y *respectively, then the sequence* (e_k) *defined by*

$$e_{2i-1} = (x_i, 0), \qquad e_{2i} = (0, y_i) \qquad (i = 1, 2, \ldots) \tag{39}$$

is a normalized [unconditional] basis in the space $(X \oplus Y)_a$ $(1 \leqslant a < +\infty)$. *Moreover, if* (x_i) *and* (y_i) *satisfy* (38a) *[resp.* (38b)*] with constants* K_1 *and* K_2 *respectively [with* K_1^* *and* K_2^* *respectively], then* (e_k) *satisfies* (38a) *[resp.* (38b)*] with the constant* $\max(K_1, K_2)$ *[resp.* $\max(K_1^*, K_2^*)$*].*

LEMMA 12. *If* $(x_j^{(i)})$ *is a normalized [unconditional] basis in a Banach space* X_i *satisfying* (38a) *[resp.* (38b)*] with a constant* K_i *[resp.* K_i^**]* $(i = 1, 2, \ldots)$ *and if* $\sup_i K_i < +\infty$ *[resp.* $\sup_i K_i^* < +\infty$*], then the sequence* $(\mathbf{e}_k)$ *defined by*

$$\tag{40} \begin{aligned} e_{N^2+j}(i) &= \begin{cases} 0 & \text{for } i \neq N+1, \\ x_j^{(N+1)} & \text{for } i = N+1; \end{cases} \quad (1 \leqslant j \leqslant N+1;\ N = 0, 1, \ldots), \\ e_{N^2+j}(i) &= \begin{cases} 0 & \text{for } i \neq j-N-1, \\ x_{N+1}^{(j-N-1)} & \text{for } i = j-N-1 \end{cases} \quad (N+1 < j \leqslant 2N+1;\ N = 0, 1, \ldots) \end{aligned}$$

is a normalized [unconditional] basis in the space $(\underset{1 \leqslant i < +\infty}{\mathsf{P}} X_i)_2$ *satisfying* (38a) *[resp.* (38b)*] with the constant* $\sup_i K_i$ *[resp.* $\sup_i K_i^*$*].*

The proof of Lemmas 11 and 12 consists in verifying the assumptions of Lemma 10. We omit the details.

Let (x_j) and (y_j) be normalized bases in Banach spaces X and Y respectively. Let $\mathfrak{x}$ and $\mathfrak{y}$ denote the equivalence classes of all bases which are permutatively equivalent to the bases (x_j) and (y_j) respectively. Then by $\mathfrak{x} + \mathfrak{y}$ we denote the equivalence class of the basis (e_k) defined by (39). By $\mathfrak{x}^\infty$ we denote the equivalence class of the basis $(\mathbf{e}_k)$ defined by (40) in the Banach space $(\underset{1 \leqslant i < +\infty}{\mathsf{P}} X_i)_2$ where $X_i = X$ and $x_j^{(i)} = x_j$ for $i, j = 1, 2, \ldots$ One can easily see that the definition of the classes $\mathfrak{x} + \mathfrak{y}$ and $\mathfrak{x}^\infty$ does not depend on the particular choice of representatives from the classes $\mathfrak{x}$ and $\mathfrak{y}$.

LEMMA 13. *The operations "+" and "∞" have the following properties*:

$$\tag{41} \begin{aligned} \mathfrak{x} + \mathfrak{y} &= \mathfrak{y} + \mathfrak{x}, \qquad \mathfrak{x} + (\mathfrak{y} + \mathfrak{z}) = (\mathfrak{x} + \mathfrak{y}) + \mathfrak{z}, \\ \mathfrak{x}^\infty + \mathfrak{x} &= \mathfrak{x}^\infty, \qquad (\mathfrak{x} + \mathfrak{y})^\infty = \mathfrak{x}^\infty + \mathfrak{y}^\infty. \end{aligned}$$

If an element of a class $\mathfrak{y}$ *is a complemented subbasis of a basis belonging to a class* $\mathfrak{x}$, *then there exists a class* $\mathfrak{z}$ *such that* $\mathfrak{x} = \mathfrak{y} + \mathfrak{z}$.

We omit the direct verification of this lemma. We only mention that to prove the second part of the Lemma we use Lemma 5.

Proof of Theorem 3. Since every seminormalized basis is equivalent to a normalized one, in the sequel we restrict our attention to the case of normalized bases. Let $\mathfrak{x}$ be the equivalence class of a normalized [unconditional] basis which is complementably universal for the family of all normalized [unconditional] bases. Then, by the second part of Lemma 13, for the class $\mathfrak{x}^\infty$ and for a class $\mathfrak{y}$ of an arbitrary normalized [unconditional] basis there are classes $\mathfrak{z}$ and $\mathfrak{z}_1$ such that

$$\mathfrak{x} = \mathfrak{x}^\infty + \mathfrak{z} = \mathfrak{y} + \mathfrak{z}_1.$$

Thus using (41) we get

$$\mathfrak{x} = \mathfrak{x}^\infty + \mathfrak{z} = (\mathfrak{y} + \mathfrak{z}_1)^\infty + \mathfrak{z} = \mathfrak{y}^\infty + \mathfrak{z}_1^\infty + \mathfrak{z} = \mathfrak{y} + (\mathfrak{y}^\infty + \mathfrak{z}_1^\infty + \mathfrak{z}) = \mathfrak{y} + \mathfrak{x}.$$

Now assuming that $\mathfrak{y}$ is also a class of a normalized [unconditional] basis which is complementably universal for the family of all normalized [unconditional] bases we find, by symmetry, that $\mathfrak{y} = \mathfrak{x} + \mathfrak{y}$. Hence $\mathfrak{x} = \mathfrak{y}$. This proves the first part of Theorem 3. The second part of this Theorem is an obvious consequence of Lemma 4.

COROLLARY 3. *Every two seminormalized unconditional bases which are universal for the family of all seminormalized unconditional bases are permutatively equivalent. Hence they span isomorphic Banach spaces.*

Proof. This is an immediate consequence of Theorem 3 and the Remark to Lemma 5.

We shall denote by B the Banach space unique up to an isomorphism, which has a seminormalized basis complementably universal for the family of all seminormalized bases. By U we shall denote the Banach space unique up to an isomorphism, which has a seminormalized unconditional basis universal for the family of all seminormalized unconditional bases. Our next corollary shows that the property described in Corollary 1 also characterizes B and U uniquely up to an isomorphism.

COROLLARY 4. *Let X be a Banach space with a basis [resp. an unconditional basis]. If every separable Banach space with a basis [with an unconditional basis] is isomorphic to a complemented subspace of X, then X is isomorphic to B [resp. X is isomorphic to U].*

Proof. Let $\mathfrak{y}$ denote the equivalence class of a normalized basis in B which is complementably universal for the family of all seminormalized bases and let $\mathfrak{x}$ be a class of a normalized basis in X. Then, by Theorem 3, $\mathfrak{y} = \mathfrak{y}+\mathfrak{y} = \mathfrak{y}+\mathfrak{x}$. Thus, by Lemma 4, the spaces $B\oplus B$ and $B\oplus X$ are isomorphic to B. On the other hand, by the assumption on X, there exists a Banach space Z such that X is isomorphic to the space $B\oplus Z$. Since B is isomorphic to $B\oplus B$, we infer that X is isomorphic to $B\oplus(B\oplus Z)$ and therefore to $B\oplus Z$. Hence the spaces B and X are isomorphic. The proof in the case of U is analogous.

4. Concluding remarks.

1° Theorem 1 provokes the following:

PROBLEM 1. Does there exist a universal basis for the family of all bases?

2° We recall that a basis (x_i) in a Banach space X is called *shrinking* ([5], p. 69) if

$$\lim_n \left(\sup\left\{|x^*(z)|\ \|z\| = 1;\ z = \sum_{k=n}^{\infty} c_k z_k\right\}\right)$$

for every bounded linear functional x^* on X.

In contrast with Theorem 1 we have

THEOREM 4. *There is no shrinking basis universal for the family of all normalized shrinking bases.*

Proof. Let ω_1 denote the first uncountable ordinal number. We define inductively a family (X_a) of Banach spaces (indexed by countable ordinal numbers $0 \leqslant a < \omega_1$) as follows:

$$X_0 = l_2; \quad \text{if } a = b+1, \text{ then } X_a = (X_b\oplus l_2)_1;$$

$$\text{if } a \text{ is a limit ordinal number, then } X_a = (\mathsf{P}_{0\leqslant b<a} X_b)_2.$$

Since the unit vector basis in l_2 is a normalized unconditional basis satisfying (38b) with $K^* = 1$, it follows from Lemmas 11 and 12 (by easy transfinite induction) that each X_a has a normalized unconditional basis satisfying (38b) with $K^* = 1$. Since all X_a are reflexive, the bases are shrinking ([5], p. 71). Now, by a recent result due to Szlenk ([16], p. 121-122), if a Banach space E contains for every $a < \omega_1$ a subspace isomorphic to X_a, then the space E^*, dual to E, is non-separable. Thus the desired conclusion follows from Lemma 4 and the fact that if a Banach space E has a shrinking basis, then E^* has a basis, and consequently E^* is separable ([5], p. 70).

We conjecture that a similar result to Theorem 4 holds for boundedly complete bases (cf. [5], p. 69, for the definition)

3° Let us observe that the Schauder basis $(\varphi_n)_{n=0}^{\infty}$ in $C(0;1)$ (defined before Lemma 6) is not universal for the family of all seminormalized bases. In fact, subbases of the Schauder basis represent only a very narrow class of bases. It is an obvious consequence of the following result:

PROPOSITION 4. *Every subbasis of the Schauder basis in $C(0;1)$ contains a subbasis equivalent either to the unit vector basis in the space c_0 or to the basis $(1,1,1,\ldots),(0,1,1,1,\ldots),(0,0,1,1,1,\ldots),\ldots$ in the space c.*

The proof of this Proposition is similar to the proof of Proposition 8 and Lemma 4 of [14].

PROBLEM 2. Does there exist in $C(0;1)$ an orthogonal system which is a universal basis for the family of all seminormalized bases?

4° The next problem is closely related to Corollary 1.

PROBLEM 3. Does there exist a separable Banach space E such that every separable Banach space is isomorphic to a complemented subspace of E?

One can easily show (by an argument similar to that of Corollary 4) that if there exists such an E, then it will be unique up to an isomorphism. Moreover, if E has a basis, then it will be isomorphic to B. Clearly, the negative answer to Problem 3 implies (by Corollary 1) the existence of a separable Banach space which does not have any basis.

5° The last two problems concern the space U (of Corollary 4).

PROBLEM 4. Are all normalized bases in the space U permutatively equivalent?

PROBLEM 5. Let a Banach space X with an unconditional basis contain a closed linear subspace isomorphic to U. Is X isomorphic to U?

In connection with Problem 5 observe that there is no "unconditional analogue" of (Zippin's) Lemma 9, as is shown in the following example:

Example. Let $1 < a < 2$ and let (x_n) be an unconditional basis in the space $L_a = L_a(0;1)$ (e.g. the Haar basis [13], [8]). By a result of Kadec [12] for a fixed β with $a < \beta < 2$ there exists a sequence (f_n) in L_a which is equivalent to the unit vector basis in l_β. Clearly the sequence (f_n) weakly converges to zero. Hence, by [3], p. 156, there is a sequence of blocks (z_k) where

$$z_k = \sum_{i=m(k-1)+1}^{m(k)} c_i^{(k)} x_i, \quad 0 = m(0) < m(1) < m(2) < \ldots,$$

which is equivalent to a subsequence (f_{n_k}). Thus, by the well-known property of the unit vector basis in l_β, the sequence (z_k) is equivalent to the unit vector basis in l_β. Since for $1 < a < \beta < 2$ the space L_β does not have complemented subspaces isomorphic to l_β (cf. [12]), Lemmas 4 and 5 imply that there is no unconditional basis in L_a having a subbasis equivalent to (z_k).

This example answers in the negative a question of Ivan Singer.

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Additive functionals on Orlicz spaces

by

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This paper is concerned with obtaining integral representations of a class of non-linear functionals on Orlicz spaces. These functionals are known as additive functionals and their representation has been studied in Martin and Mizel [6], Mizel and Sundaresan [7]. For the importance of this class of functionals in generalized random processes we refer to Gel'fand and Vilenkin [2]. Further the representation theorems obtained here are of intrinsic interest and provide generalizations of results established in Halmos [3], Bartle and Joichi [1] and Krasnosel'skii [4].

We start with few definitions, remarks and establish a theorem useful in subsequent discussion.

Throughout this paper (T, Σ, μ) is a complete non-atomic totally σ-finite positive measure space. Φ (with or without a suffix) denotes a continuous non-zero Young function. L_Φ denotes the Banach space of real-valued measurable functions f on T such that for a positive number K (depending on f) $M(kf) = \int_T \Phi(k|f|)\,d\mu < \infty$ equipped with the norm

$$\|f\| = \inf\left\{\frac{1}{\xi} \mid \xi > 0,\ M(\xi f) \leqslant 1\right\}.$$

For a detailed discussion of this class of Banach spaces and for the undefined terms in this paper we refer to Luxemburg [5].

Next we proceed to define additive functionals. Throughout the rest of the paper $\int f d\mu$ denotes the definite integral $\int_T f d\mu$.

Definition. Let $\mathscr{F}$ be a linear space of measurable functions on a measure space (T, Σ, μ). A real-valued function F on $\mathscr{F}$ is said to be *additive* if (1) $F(x+y) = F(x)+F(y)$ for $x, y \in \mathscr{F}$ such that $\mu\{t \mid x(t)y(t) \neq 0\} = 0$ and (2) $F(x) = F(y)$ if x, y are equimeasurable functions in $\mathscr{F}$, i.e. $\mu(x^{-1}(B)) = \mu(y^{-1}(B))$ for all Borel sets B in R, the real line.

Remark 1. If x, y are integrable equimeasurable functions, it is verified that $\int x d\mu = \int y d\mu$ and further if f is a Borel measurable function