Also since f verifies condition (2) of the theorem,

$$|f(x_n)| \leqslant a\Phi(|x_n|) \leqslant a\Phi(|x|),$$

and since $f(x_n) \to f(x)$ a.e., it follows by Lebesgue theorem on dominated convergence that

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int f(x_n)\,d\mu = \int f(x)\,d\mu,$$

thus completing the representation of F. The uniqueness of f is verified as in theorem 2.

Conversely, if f is a real-valued continuous function on R satisfying conditions (1) and (2) of the theorem, then from Remark 2 it follows that the functional $F(x) = \int f(x)\,d\mu$ is well defined on L_Φ and is additive. Next we verify that F is continuous. Let x_n be a sequence in L_Φ converging to x. Thus by lemma 1 since f is continuous, $f(x_n) \to f(x)$ converges in measure on sets of finite measure and further the inequality $\int f(x_n \chi_E)\,d\mu \leqslant a \int \Phi(|x_n \chi_E|)\,d\mu$ implies that $\{f(x_n)\}_{n\geqslant 1}$ are of uniformly absolutely continuous L_1-norms. Hence $\int f(x_n)\,d\mu \to \int f(x)\,d\mu$. Thus $F(x_n) \to F(x)$.

In conclusion it might be mentioned that the problem of representing additive functionals on Orlicz spaces L_Φ, when the space is not of absolutely continuous norm, is not considered here and it is conjectured that non-trivial continuous additive functionals do not exist in such spaces.

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Banach spaces of functions satisfying a modulus of continuity condition*

by

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1. Introduction and terminology. A function $\beta: [0, \infty) \to [0, \infty]$ will be called a *modulus of continuity* if it is monotone increasing, continuous at zero, and zero at zero only. Note that it need not be subadditive. For pseudometric spaces (X, d) and (Y, e), a function $f: (X, d) \to (Y, e)$ will be said *to satisfy a modulus of continuity condition* β (locally) if there is some positive real M (and some positive real ε) such that $e(f(x), f(y)) \leqslant Md(x, y)$ (whenever $d(x, y) < \varepsilon$) for all x and y in X. Obviously, such a function is uniformly continuous.

Let F denote the real or complex numbers with the usual metric. For a pseudometric space (X, d), let $\mathrm{Lip}(X, \beta\circ d)$ be the set of bounded F-valued functions on X which satisfy a modulus of continuity condition β locally. When $\beta(t) = t$, we will denote the set by $\mathrm{Lip}(X, d)$. If only one metric is being considered on X, we will denote $\mathrm{Lip}(X, \beta\circ d)$ by $\mathrm{Lip}(X, \beta)$. It is known that if β is subadditive (so that $\beta\circ d$ is a pseudometric) and the functions satisfy the modulus of continuity condition β globally, then $\mathrm{Lip}(X, \beta\circ d)$ is a Banach space with a natural norm [4].

Let (X, d), (X, d') and (Y, e) be pseudometric spaces. If there exist $M, \varepsilon > 0$ such that $d(x, y) \leqslant Md'(x, y)$ whenever $d'(x, y) < \varepsilon$, we indicate it by writing $d \ll d'$. Then to say that $f: (X, d) \to (Y, e)$ satisfies a local Lipschitz condition can be denoted $e\circ f_2 \ll d$, where $f_2(x, y) = (f(x), f(y))$. If $d \ll d'$ and $d' \ll d$, we say that d and d' are *strongly equivalent* (in contrast to topologically or uniformly equivalent) and denote it by $d \approx d'$.

We attempt to describe how the various spaces $\mathrm{Lip}(X, \beta\circ d)$ are related, if one considers different pseudometrics on X or different moduli of continuity. In the first section, we give a natural norm for $\mathrm{Lip}(X, \beta\circ d)$, under which it is a Banach space. Then we show that $\mathrm{Lip}(X, d)$ is con-

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tinuously imbedded in $\mathrm{Lip}(X, d')$ iff $d' \preccurlyeq d$. In the second section, we show that if (X, d) is compact and $\lim_{t\to 0} t/\beta(t) = 0$, then the unit ball of $\mathrm{Lip}(X, d)$ is compact in $\mathrm{Lip}(X, \beta)$. Finally, we investigate $\mathrm{Lip}(X, \beta)$ for compact spaces which are "uniformly locally starlike" and show that

$$\bigcap \{\mathrm{Lip}(X, \beta) \mid \lim_{t\to 0} t/\beta(t) = 0\} = \mathrm{Lip}(X, d).$$

2. The Banach space $\mathrm{Lip}(X, \beta\circ d)$. As usual, we denote $\sup\{|f(x)| \mid x \in X\}$ by $\|f\|_\infty$. For $f \in \mathrm{Lip}(X, \beta\circ d)$, set

$$\|f\|_\beta = \sup\left\{\frac{|f(x)-f(y)|}{\beta\circ d(x, y)} \mid d(x, y) > 0\right\}.$$

2.1. PROPOSITION. *If $f \in \mathrm{Lip}(X, \beta\circ d)$, then $\|f\|_\beta < \infty$.*

Proof. Let $\varepsilon, M > 0$ be such $|f(x)-f(y)| \leqslant M\beta\circ d(x, y)$ whenever $d(x, y) < \varepsilon$. Then

$$\sup\left\{\frac{|f(x)-f(y)|}{\beta\circ d(x, y)} \mid d(x, y) > 0\right\}$$

$$= \sup\left\{\frac{|f(x)-f(y)|}{\beta\circ d(x, y)} \mid d(x, y) \geqslant \varepsilon\right\} \vee \sup\left\{\frac{f(x)-f(y)}{\beta\circ d(x, y)} \mid 0 < d(x, y) < \varepsilon\right\}$$

$$\leqslant \frac{2\|f\|_\infty}{\beta(\varepsilon)} \vee M < \infty.$$

The standard arguments show that $\|\cdot\|_\beta$ is a pseudonorm for $\mathrm{Lip}(X, \beta)$. Setting $\|f\| = \|f\|_\infty \vee \|f\|_\beta$, we obtain a norm for $\mathrm{Lip}(X, \beta)$. It is well-known that $\mathrm{Lip}(X, \beta)$ is complete when β is subadditive. The proofs remain valid when β is not subadditive, so we omit them.

We shall be interested in a particular subspace of $\mathrm{Lip}(X, \beta)$, namely $\mathrm{lip}(X, \beta) = \{f \in \mathrm{Lip}(X, \beta) \mid |f(x)-f(y)| \text{ is } o\big(\beta(d(x, y))\big) \text{ as } d(x, y) \to 0\}$. It is readily seen that $\mathrm{lip}(X, \beta)$ is a closed subspace of $\mathrm{Lip}(X, \beta)$.

2.2. LEMMA. *Let (X, d) be a pseudometric space and set $d'(x, y) = d(x, y) \wedge 1$. Then $d' \approx d$ and*

$$d'(x, y) \leqslant \sup\{|f(x)-f(y)| \mid \|f\|_d \vee \|f\|_\infty \leqslant 1\}.$$

Proof. Define $f_z(x) = d(x, z) \wedge 1$. Since $|f_z(x)-f_z(y)| \leqslant d(x, y)$, we have $\|f_z\|_d \vee \|f_z\|_\infty \leqslant 1$. Now $|f_y(x)-f_y(y)| = |d(x, y) \wedge 1 - d(y, y) \wedge 1| = d(x, y) \wedge 1 = d'(x, y)$. Thus $\sup\{|f(x)-f(y)| \mid \|f\|_d\, \|f\|_\infty \leqslant 1\} \geqslant d'(x, y)$. That $d' \approx d$ is obvious.

2.3. THEOREM. *Let X be a space with pseudometrics d and e. Then $d \preccurlyeq e$ iff $\mathrm{Id}\colon \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, e)$ is a continuous imbedding.*

Proof. Let $d \preccurlyeq e$. Then there exist $K, \delta > 0$ such that $d(x, y) \leqslant Ke(x, y)$ whenever $e(x, y) < \delta$. For $f \in \mathrm{Lip}(X, d)$,

$$\|f\|_e = \sup\left\{\frac{|f(x)-f(y)|}{e(x, y)} \mid e(x, y) > 0\right\}$$

$$= \sup\left\{\frac{|f(x)-f(y)|}{d(x, y)}\,\frac{d(x, y)}{e(x, y)} \mid 0 < e(x, y) < \delta,\ d(x, y) > 0\right\}$$

$$\vee \sup\left\{\frac{|f(x)-f(y)|}{e(x, y)} \mid e(x, y) \geqslant \delta\right\}$$

$$\leqslant \|f\|_d K \vee \frac{2\|f\|_\infty}{\delta} \leqslant (K + 2/\delta)(\|f\|_d \vee \|f\|_\infty).$$

Hence $f \in \mathrm{Lip}(X, e)$ and the identity is continuous.

Conversely, suppose $\mathrm{Id}\colon \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, e)$ is continuous. Let U denote the unit ball of $\mathrm{Lip}(X, d)$. By continuity, there is an $M > 0$ such that $\|f\|_\infty \vee \|f\|_e \leqslant M$ for every $f \in U$. In particular, $f \in U$ implies that $|f(x)-f(y)| \leqslant Me(x, y)$. Setting $d'(x, y) = d(x, y) \wedge 1$, we apply (2.2) to see that $d'(x, y) \leqslant \sup\{|f(x)-f(y)| \mid f \in U\} \leqslant Me(x, y)$. Since $d \approx d'$ and $d' \preccurlyeq e$, we have $d \preccurlyeq e$.

2.4. COROLLARY. *Let X be a space with pseudometrics d and e. Then $d \approx e$ iff $\mathrm{Lip}(X, d) = \mathrm{Lip}(X, e)$.*

Proof. Assume that $\mathrm{Lip}(X, d) = \mathrm{Lip}(X, e)$. Then we have $\mathrm{Lip}(X, d+e) = \mathrm{Lip}(X, d) = \mathrm{Lip}(X, e)$. Since the norms of $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(X, e)$ are comparable with the norm of $\mathrm{Lip}(X, d+e)$, we can apply the closed graph theorem to obtain that all three spaces are norm equivalent. Then apply (2.3).

The above result sharpens a result of Sherbert [6], p. 1392, by dropping the restriction that the metrics be bounded.

2.5. COROLLARY. *Let (X, d) be a metric space and β, γ two moduli of continuity (with β subadditive). Then $\mathrm{Id}\colon \mathrm{Lip}(X, \beta) \to \mathrm{Lip}(X, \gamma)$ is a continuous imbedding if (and only if) $\limsup_{t\to 0} \beta(t)/\gamma(t) < \infty$.*

Proof. The proof of (2.5) is obtained by substituting $\beta\circ d$ for d and $\gamma\circ d$ for e in (2.3) and in (2.2).

Note that the assertion of (2.5) is considerably weaker than that of (2.3). Unless subadditivity of β is assumed (or at least subadditivity in a neighborhood of zero), we are unable to prove the converse of (2.5). The inequality $|\beta\circ d(x, y) - \beta\circ d(x, z)| \leqslant \beta\circ d(y, z)$ seems to be necessary to construct non-constant functions which satisfy the modulus of continuity condition β. We are unable to remove this restriction.

3. Lip(X, β) **when** $\lim_{t\to 0}\beta(t)/t = 0$. Glaeser [1], p. 91-97, has investigated Lip(X, β) when $\lim_{t\to 0}\beta(t)/t = 0$ for the special case of X being a regular compact subset of E^n and β being subadditive. His proofs rely on sophisticated results from the theory of distributions and the fact that $\beta\circ d$ is a metric if β is subadditive.

3.1. THEOREM. *Let (X, d) be a compact metric space and $\lim_{t\to 0}\beta(t)/t = 0$. Then the unit ball of* Lip(X, β) *is compact in* lip(X, d).

Proof. Let $f\in \mathrm{Lip}(X, \beta)$ and $\varepsilon > 0$ be given. Then

$$\frac{|f(x)-f(y)|}{d(x, y)} = \frac{|f(x)-f(y)|}{\beta\circ d(x, y)}\cdot\frac{\beta\circ d(x, y)}{d(x, y)}$$
$$\leqslant \|f\|_\beta \beta\circ d(x, y)/d(x, y) \to 0$$

as $d(x, y)\to 0$. Hence $f\in\mathrm{lip}(X, d)$.

Let $W = X\cup(X\times X-\Delta)$, endowed with the disjoint union topology. For any $f\in\mathrm{Lip}(X,d)$, define $f^*\colon W\to F$ by $f^*(x) = f(x)$ for $x\in X$ and $f^*(x, y) = \big(f(x)-f(y)\big)/d(x, y)$ for $(x, y)\in X\times X-\Delta$. Each $f\in\mathrm{lip}(X, d)$ can be extended continuously to $W' = X\cup X\times X$ by defining $f^*(x, x) = 0$ for $x\in X$.

Since $\mathrm{Lip}(X,\beta)\approx\mathrm{Lip}(X,\beta\wedge 1)$, we assume without loss of generality that β is bounded by one. Thus $\beta(t)/t$ is a bounded function, so that

$$\frac{|f(x)-f(y)|}{d(x, y)} \leqslant \|f\|_\beta\frac{\beta\circ d(x, y)}{d(x, y)} \leqslant K\|f\|_\beta \quad \text{for some } K > 0.$$

We see then that $U = \{f^* \mid \|f\|_\beta\vee\|f\|_\infty\leqslant 1\}$ is a set of uniformly bounded continuous functions on W'. Restricted to X, the f^* in U clearly are an equicontinuous family. We will show that they are an equicontinuous family when restricted to $X\times X$.

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that $d(x, y) < 3\delta$ implies $\beta\circ d(x, y)/d(x, y) < \varepsilon/2$. Choose $0 < \delta' < \delta$ so that $\delta' < \delta^2/4(1+\delta\varepsilon)$. We will show that if $d(x, u) < \delta'$ and $d(y, v) < \delta'$, then $|f^*(x, y)-f^*(u, v)| < \varepsilon$ for any $f^*\in U$.

Case (i). $d(x, y) < \delta$ (or $d(u, v) < \delta$). Then $d(u, v)\leqslant d(u, x)+d(x, y)+d(y, v)\leqslant 3\delta$. For $f^*\in U$,

$$|f^*(x, y)-f^*(u, v)| \leqslant |f^*(x, y)|+|f^*(u, v)|$$
$$\leqslant \frac{|f(x)-f(y)|}{\beta\circ d(x, y)}\,\frac{\beta\circ d(x, y)}{d(x, y)} + \frac{|f(u)-f(v)|}{\beta\circ d(u, v)}\,\frac{\beta\circ d(u, v)}{d(u, v)}$$
$$\leqslant \|f\|_\beta\varepsilon/2+\|f\|_\beta\varepsilon/2\leqslant\varepsilon.$$

Case (ii). $d(x, y)\geqslant\delta$ and $d(u, v)\geqslant\delta$. Then

$$|f^*(x, y)-f^*(u, v)| \leqslant \left|\frac{f(x)}{d(x, y)}-\frac{f(u)}{d(u, v)}\right| + \left|\frac{f(y)}{d(x, y)}-\frac{f(u)}{d(u, v)}\right|.$$

Now

$$\left|\frac{f(x)}{d(x, y)}-\frac{f(u)}{d(u, v)}\right| \leqslant \left|\frac{f(x)}{d(x, y)}-\frac{f(x)}{d(u, v)}\right| + \left|\frac{f(x)}{d(u, v)}-\frac{f(u)}{d(u, v)}\right|$$
$$\leqslant |f(x)|\frac{|d(u, v)-d(x, y)|}{d(x, y)\cdot d(u, v)} + \frac{d(u, x)}{d(u, v)}\,\frac{|f(x)-f(u)|}{d(u, x)}$$
$$\leqslant\frac{d(x, u)+d(y, v)}{\delta^2}+\frac{d(u, x)}{\delta}\cdot\frac{\varepsilon}{2}\leqslant\left(\frac{1}{\delta^2}+\frac{\varepsilon}{\delta}\right)\big(d(x, u)+d(y, v)\big) < \varepsilon/2.$$

Similarly,

$$\left|\frac{f(y)}{d(x, y)}-\frac{f(u)}{d(u, v)}\right| < \varepsilon/2.$$

Since δ' does not depend on the particular $f^*\in U$, U is an equicontinuous family. Hence U is totally bounded in $C(W')$, the space of bounded continuous functions on W'. But de Leeuw [3], p. 57, showed that lip(X, d) is isometrically imbedded in $C(W')$ under the mapping $f\to f^*$. Hence the unit ball of Lip(X, β) is precompact in lip(X, d).

Let $\{f_n\}$ be a sequence of functions from the unit ball of Lip(X, β) such that $f_n\to f$ in lip(X, d). Then

$$\frac{|f(x)-f(y)|}{\beta\circ d(x, y)} = \lim_{n\to\infty}\frac{|f_n(x)-f_n(y)|}{\beta\circ d(x, y)}\leqslant 1$$

for $(x, y)\in X\times X-\Delta$. Hence f is in the unit ball of Lip(X, β), so the unit ball of Lip(X, β) is compact in lip(X, d).

3.2. COROLLARY. *Let X, d and β be as in* (3.1). *Then* Id: Lip$(X, \beta)\to$ lip(X, d) *is a compact operator.*

If X is a space with pseudometrics d and d', we will say that d' is $o(d)$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d'(x, y)\leqslant\varepsilon d(x, y)$. (Note the analogy to functions in lip(X, d).)

3.3. COROLLARY. *Let (X, d) be a compact metric space and d' a metric on X which is $o(d)$. Then the unit ball of* Lip(X, d') *is compact in* lip(X, d).

Proof. The proof is the same as the proof of (3.1), except that $d'(x, y)$ is substituted for $\beta\circ d(x, y)$ wherever the latter appears.

3.4. COROLLARY. *Let (X, d) be a compact metric space and β, γ two moduli of continuity. If γ is subadditive in a neighborhood of zero and $\lim_{t\to 0}\beta(t)/\gamma(t) = 0$, then* Id: Lip$(X, \beta)\to$ lip(X, γ) *is a compact operator.*

Proof. Suppose $\gamma(t)$ is subadditive for $t \leqslant \delta$. Letting $\alpha(t) = \gamma(t) \wedge \delta$, $\alpha(t)$ is subadditive for all t. By (2.5), $\mathrm{Lip}(X, \alpha)$ is isomorphic under the identity to $\mathrm{Lip}(X, \gamma)$. Apply the proof of (3.1) again, substituting $\alpha \circ d$ for d.

The proof of (3.1) relies heavily on Ascoli's theorem. Thus our results are only valid if X is compact. It would be interesting to know if that restriction can be dropped.

4. $\bigcap\{\mathrm{Lip}(X, \beta) | \lim_{t\to 0} \beta(t)/t = \infty\}$. A metric space (X, d) is a *starlike from a point p (in X)* if given $x \in X$ and $a \in (0, 1)$, there exists $y \in X$ with $d(p, y) + d(y, x) = d(p, x)$ and $d(p, y) = ad(p, x)$. A space (X, d) is *uniformly locally starlike* if there exists $\delta > 0$ such that the δ-neighborhood of each point $p \in X$ is starlike from p. Trivially, any convex metric space is uniformly locally starlike.

4.1. LEMMA. *Let (X, d) be a uniformly locally starlike metric space. Let f be an F-valued uniformly continuous function and β its modulus of continuity. Then β is subadditive in a neighborhood of zero, and if $\beta(t)/t$ is not bounded in that neighborhood, then $\lim_{t\to 0} \beta(t)/t = \infty$.*

Proof. Let δ be such that the δ-neighborhood of each point p is starlike from p. If $\delta_1 + \delta_2 < \delta$, then $\beta(\delta_1 + \delta_2) = \sup\{|f(x) - f(y)| \,|d(x, y) \leqslant \delta_1 + \delta_2\}$. When $d(x, y) \leqslant \delta_1 + \delta_2$, there exists a point z such that $d(x, z) \leqslant \delta_1$ and $d(z, y) \leqslant \delta_2$. Thus $|f(x) - f(z)| \leqslant \beta(\delta_1)$, $|f(z) - f(y)| \leqslant \beta(\delta_2)$, and $|f(x) - f(y)| \leqslant \beta(\delta_1) + \beta(\delta_2)$. Taking the sup over the left-hand side, we have $\beta(\delta_1 + \delta_2) \leqslant \beta(\delta_1) + \beta(\delta_2)$, so β is subbadditive.

Consider the sequence $\{2^n\beta(1/2^n)\}_{n=0}^{\infty}$. Since $\beta(2t) \leqslant 2\beta(t)$ for $2t < \delta$, we have $2^n\beta(1/2^n) \leqslant 2^{n+1}\beta(1/2^{n+1})$ ultimately. Also for $1/2^{n+1} < t < 1/2^n < \delta$, we have

$$2^n\beta(1/2^{n+1}) \leqslant \beta(t)/t \leqslant 2^{n+1}\beta(1/2^n).$$

Hence, if $\beta(t)/t$ is not bounded, then $\lim_{t\to 0} \beta(t)/t = \infty$.

4.2. THEOREM. *Let (X, d) be a uniformly locally starlike metric space. Let f be an F-valued function on X. If for any modulus of continuity β such that $\lim_{t\to 0} \beta(t)/t = \infty$, there exists $K > 0$ such that*

$$|f(x) - f(y)| \leqslant K\beta \circ d(x, y) \quad \textit{for all } (x, y) \in X \times X,$$

then $f \in \mathrm{Lip}(X, d)$.

Proof. Set $\alpha(t) = \sup\{|f(x) - f(y)| \,|d(x, y) \leqslant t\}$. Suppose f does not satisfy a Lipschitz condition. Since f must be bounded, we may assume that f does not satisfy a local Lipschitz condition. Thus $\alpha(t) \leqslant Kt$ for $t < \delta$ is untrue for any K and any $\delta > 0$. That is, $\alpha(t)/t$ is not bounded in any interval $(0, \delta)$. By (4.1), $\lim_{t\to 0} \alpha(t)/t = \infty$.

Let $\gamma(t) = [t\alpha(t)]^{1/2}$. Then $\gamma(t)$ is a modulus of continuity, and $\lim_{t\to 0} \gamma(t)/t = \infty$. Thus there exists K such that $|f(x) - f(y)| \leqslant K\gamma \circ d(x, y)$. Since $\alpha(t)/t \to \infty$ as $t \to 0$, there exists $\varepsilon > 0$ such that $[\alpha(t)/t]^{1/2} > 2K$ for $t < \varepsilon$. Then

$$|f(x) - f(y)| \leqslant K\gamma \circ d(x, y) = K[d(x, y)\alpha \circ d(x, y)]^{1/2}$$

$$\leqslant \tfrac{1}{2}\left[\frac{\alpha \circ d(x, y)}{d(x, y)}\right]^{1/2} [d(x, y)\alpha \circ d(x, y)]^{1/2} = \tfrac{1}{2}\alpha \circ d(x, y).$$

But $\alpha(t) = \sup\{|f(x) - f(y)| \,|d(x, y) \leqslant t\}$. The contradiction arose from the assumption that f did not satisfy a local Lipschitz condition.

4.3. COROLLARY. *$\bigcap\{\mathrm{Lip}(X, \beta) | \lim_{t\to 0} \beta(t)/t = \infty\}$ is $\mathrm{Lip}(X, d)$ whenever (X, d) is a uniformly locally starlike metric space.*

It should be noted that the theorem is false if one considers only a countable number of moduli of continuity β_n. By a result in [2], p. 12, we can construct a function β which has an infinite derivative at zero, but satisfies $\lim_{t\to 0} \beta_n(t)/\beta(t) = \infty$ for each β_n. The constructed function can even be chosen subadditive and piecewise linear.

The proofs given above depend strongly on the fact that (X, d) is uniformly locally starlike. This seems to be an unnatural restriction, but we are unable to remove it.

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