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On functions and distributions with a vanishing derivative

by

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1. The main purpose of this note is to give some existence and unicity theorems for the equation $f^{(m)} = 0$, where f is a distribution or function of q real variables, and $f^{(m)}$ denotes the mixed derivative of order $m = (\mu_1, \ldots, \mu_q)$. The results presented here are closely related to papers [3] and [4].

We shall first fix the notation. If $x = (\xi_1, \ldots, \xi_q)$ and $s = (\sigma_1, \ldots, \sigma_q)$, where ξ_j are real numbers and σ_j are non-negative integers, then we use the notation $x^s = \xi_1^{\sigma_1} \ldots \xi_q^{\sigma_q}$ (if $\xi_j = 0$ and $\sigma_j = 0$, then we read $\xi_j^{\sigma_j} = 1$); thus the "power" of the vector x to the vector exponent s is a real number. By a *polynomial* of x of degree m we understand $\sum_{0 \leqslant s \leqslant m} a_s x^s$, where the coefficients a_s are real numbers.

Let $I = (A, B)$; in other terms, we assume that $A = (A_1, \ldots, A_q)$ and $B = (B_1, \ldots, B_q)$ are given points of the q-dimensional Euclidean space $\boldsymbol{R}^q$, such that $A_j < B_j$, and I is the set of points x satisfying $A < x < B$, i.e., $A_j < \xi_j < B_j$ ($j = 1, \ldots, q$). Given the order $m = (\mu_1, \ldots, \mu_q)$, we assume that, for every $j = 1, \ldots, q$, the interval I is cut by μ_j different hyperplanes $\xi_j = \xi_{j1}, \ldots, \xi_j = \xi_{j\mu_j}$; the intersection of the hyperplane $\xi_j = \xi_{jk}$ with I will be denoted by H_{jk}. Throughout this section, we assume that the interval I, the order $m = (\mu_1, \ldots, \mu_q)$ and the numbers ξ_{jk} ($j = 1, \ldots, q$; $k = 1, \ldots, \mu_j$) are fixed. If $\mu_j = 0$ for some index j, then we understand that no number ξ_{jk} with that index j is given. The union of all H_{jk} will be denoted by U. Thus we may say that U is the intersection of I with the union of all hyperplanes $\xi = \xi_{jk}$.

By x_s $(0 \leqslant s \leqslant m)$ we shall understand $x_s = (\xi_{1\sigma_1}, \ldots, \xi_{q\sigma_q})$, where ξ_{j0} denotes ξ_j. We see that the set of points $x = x_s$ is a hyperplane whose number of dimensions is $q - \operatorname{sgn}\sigma_1 - \ldots - \operatorname{sgn}\sigma_q$, where $\operatorname{sgn}\sigma_j = 0$, if $\sigma_j = 0$, and $\operatorname{sgn}\sigma_j = 1$, if $\sigma_j \geqslant 1$. Thus, in particular, x_0 denotes the variable x. The intersection of the hyperplane $x = x_s$ with I will be denoted by K_s. In particular, $K_0 = I$. Evidently, if $s \neq 0$, then K_s is included in some H_{jk}. This implies that the union of all K_s is U.

THEOREM 1. *There are polynomials* $w_s(x)$, $0 \leqslant s \leqslant m$, *of degree* $\leqslant (\tau_1, \dots, \tau_q)$ *with* $\tau_j = (\mu_j - 1)\operatorname{sgn}\sigma_j$, $w_0(x) = 1$, *such that, for every function* f, *continuous in* I, *the equality*

(1) $$f^{(m)}(x) = 0 \quad \textit{in } I$$

(*where the derivation is understood in the distributional sense*) *implies*

(2) $$\sum_{0 \leqslant s \leqslant m} w_s(x) f(x_s) = 0 \quad \textit{in } I,$$

and conversely.

Proof. By a direct verification, it is easily seen that (2) implies (1). Thus we have to show that (1) implies (2). We shall first prove this fact for functions f which have their derivatives continuous up to the order m.

Let $m_k = (\mu_{k1}, \dots, \mu_{kq})$, where $\mu_{kj} = \mu_j$ for $j \leqslant k$ and $\mu_{kj} = 0$ for $j > k$. In particular, we have $m_0 = 0$ and $m_q = m$. Evidently, assertion (2) reduces, for $m = m_0$, to $f(x) = 0$, as well as (1). Thus the theorem is trivially true for $m = m_0$. Assume that (2) holds for $m = m_{p-1}$ with some $1 \leqslant p \leqslant q$. We shall show that (2) then holds also for $m = m_p$. We may assume that $\mu_p \geqslant 1$, because in the case $\mu_p = 0$ there would be nothing to prove.

Let $g = f^{(\mu_p e_p)}$, where e_p denotes the vector whose p-th coordinate is 1, and all the remaining ones are 0. Then $f^{(m_p)} = 0$ implies $g^{(m_{p-1})} = 0$, thus

$$\sum_{0 \leqslant s \leqslant m_{p-1}} w_s(x) g(x_s) = 0$$

for some polynomials w_s which depend only on the numbers ξ_{jk} with $j \leqslant p-1$. We may also write

(3) $$\sum_{s \in Z} w_s(x) f^{(\mu_p e_p)}(x_s) = 0,$$

where Z is the set of all s satisfying $0 \leqslant s \leqslant m_{p-1}$. Since the polynomials $w_s(x)$ do not depend on ξ_p, we obtain, integrating (3) μ_p times with respect to ξ_p,

(4) $$\sum_{s \in Z} w_s(x) f(x_s) - \xi_p^{\mu_p - 1} g_{\mu_p - 1} - \dots - g_0 = 0,$$

where g_i are unknown functions of x which do not depend on ξ_p. We can determine g_i in the following way. Substituting in (4), $\xi_p = \xi_{pi}$ with $i = 1, \dots, \mu_p$, we obtain

(5) $$S_i - \xi_{pi}^{\mu_p - 1} g_{\mu_p - 1} - \dots - g_0 = 0,$$

where

$$S_i = \sum_{s \in Z} w_s(x) f(x_s + i e_p).$$

We can solve the system (5) in g_k, because in its determinant $|\xi_{pi}^j|$, which is a Vandermondean, all $\xi_{p1}, \dots, \xi_{p\mu_p}$ are different. If a_{ij} denotes the quotient of the subdeterminant adjoint to the element ξ_{pi}^j by the determinant itself, then we obtain

$$g_j = a_{1j} S_1 + \dots + a_{\mu_p j} S_{\mu_p}.$$

Hence we get

$$\xi_p^{\mu_p - 1} g_{\mu_p - 1} + \dots + g_0 = \sum_{i=1}^{\mu_p} (\xi_p^{\mu_p - 1} a_{i,\mu_p - 1} + \dots + a_{i,0}) S_i$$

$$= -\sum_{i=1}^{\mu_p} \sum_{s \in Z} w_{s + ie_p}(x) f(x_{s + ie_p}),$$

where

$$w_{s + ie_p}(x) = -(\xi_p^{\mu_p - 1} a_{i,\mu_p - 1} + \dots + a_{i,0}) w_s(x).$$

Replacing $s + ie_p$ by s, we obtain

(6) $$\xi_p^{\mu_p - 1} g_{\mu_p - 1} + \dots + g_0 = -\sum_{\lambda=1}^{\mu_p} \sum_{s \in Z_i} w_s(x) f(x_s),$$

where Z_i is the set of all s satisfying $ie_p \leqslant s \leqslant m_{p-1} + ie_p$.

Now, we observe that the sets $Z, Z_1, \dots, Z_{\mu_p}$ are disjoint and their union is the set of all s satisfying $0 \leqslant s \leqslant m_p$. Thus, substituting (6) into (4), we obtain the equality

$$\sum_{0 \leqslant s \leqslant m_p} w_s(x) f(x_s) = 0.$$

By induction we obtain the required equality (2).

It remains to generalize the result to arbitrary continuous functions. Let δ_n be a delta-sequence (see [2]) and let $f_n = f * \delta_n$. Then $f_n^{(m)} = 0 * \delta_n = 0$. Since the functions f_n are smooth, we may apply the just proved case and write

$$\sum_{0 \leqslant s \leqslant m} w_s(x) f_n(x_s) = 0.$$

But the sequence $f_n(x)$ converges almost uniformly to $f(x)$, thus also $f_n(x_s)$ converges almost uniformly to $f(x_s)$. Thus, letting $n \to \infty$, we obtain (2).

Remark. In the sum (2), there is an element with $s = 0$; this element is $w_0(x) f(x_0)$, i.e., $f(x)$. We therefore may write

$$f(x) = -\sum_{s \in Z'} w_s(x) f(x_s),$$

where Z' is the set of all s satisfying $0 \leqslant s \leqslant m$, $s \neq 0$. Equality (2) says that the values of the function $f(x)$ are expressed, in the whole interval I, by its values on U. It is important to remember that the polynomials w_s do not depend on the function f; they depend only on the choice of hyperplanes $\xi_j = \xi_{jk}$. This remark implies

COROLLARY. *If $f_n (n = 1, 2, \ldots)$ are continuous functions in I, such that $f_n^{(m)} = 0$ and the sequence f_n converges uniformly on the set U, then it converges uniformly in I to a function f satisfying $f^{(m)} = 0$ in I.*

2. In this section we are going to state an existence theorem and a unicity theorem for $f^{(m)} = 0$.

THEOREM 2. *Let $I_1, \ldots, I_p$ be open intervals such that $I = I_1 \cap \ldots \cap I_p \neq 0$ and let $K = I_1 \cup \ldots \cup I_p$. If f is a continuous function on the closure $\bar{K}$ of K such that $f^{(m)} = 0$ in K, then f can be extended over the whole space $\boldsymbol{R}^q$ so as to satisfy $f^{(m)} = 0$ in $\boldsymbol{R}^q$.*

Proof. It is known that f can be extended to some continuous function h in $\boldsymbol{R}^q$. That function will satisfy $h^{(m)} = 0$ in K, but not necessarily outside K. Let us consider hyperplanes $\xi_j = \xi_{jk}$, cutting I, as in section 1. We put

$$g(x) = -\sum_{s \in Z'} w_s(x) h(x_s) \quad \text{in } \boldsymbol{R}^q.$$

Since h is continuous in $\boldsymbol{R}^q$, so is $g(x)$. It is easy to see that g is the required function. In fact, we have $g^{(m)} = 0$ in $\boldsymbol{R}^q$. If $y \in I$, then y belongs to some interval I_i. Since $g = h$ on the intersection of I_i with the union of the hyperplanes $\xi_j = \xi_{jk}$, and $h^{(m)} = 0$ in I_i, we have $g = h$ in I_i, which follows from Theorem 1. Hence $g = f$ in I_i and in particular $g(y) = f(y)$. Since the point y is arbitrary in K, we have $g = f$ in K, which completes the proof.

The extension of f, ensured by Theorem 2, is not unique in general. However, it is possible to determine a domain of unicity. Let us denote by $P_j(x)$ the straight line parallel to the axis of ξ_j and passing through x. Given any set G, we shall denote by G^* the set of all points x such that $G \cap P_j(x) \neq 0$ for $j = 1, \ldots, q$. Evidently, $G \subset G^*$. If G is open, so is G^*.

THEOREM 3. *Under hypotheses in Theorem 2, the extension of f is unique in K^*.*

Proof. Let y be any point in K^* and I' an open interval such that $y \in I'$ and $I' \in K^*$. Finally, let L be the least open interval including I' and I. If W is the intersection of L with the union of the hyperplanes $\xi_j = \xi_{jk}$, then $W \subset K$; this purely geometrical property can be easily deduced from the definition of K^*. Since the values of f are given on W, f is uniquely determined in L, by Theorem 1. Thus, in particular, f is uniquely determined at y. Since y is an arbitrary point of K, theorem 3 is proved.

3. So far, we have considered functions, only the differentiation was understood in the distributional sense. Now, we are going to state some theorems on distributions. The theorem in this section will be of type "sticking distributions" and will have nothing in common with the equation $f^{(m)} = 0$. However, its proof will be based on the results of the preceding sections.

THEOREM 4. *Let $I = I_1 \cup \ldots \cup I_p$, $J = J_1 \cup \ldots \cup J_r$, where I_i and J_i are open intervals such that $I_1 \cap \ldots \cap I_p \cap J_1 \cap \ldots \cap J_r \neq 0$. If f and g are distributions in $\boldsymbol{R}^q$ such that $f = g$ in $I \cap J$, then there is a distribution h in $\boldsymbol{R}^q$ such that $h = f$ in I and $h = g$ in J.*

Proof. Assume first that I and J are bounded. Then there is an open bounded interval B including I and J, and there are continuous functions F and G on $\bar{B}$ such that $F^{(k)} = f$ and $G^{(k)} = g$ in B for some order k. Then the difference $P = F - G$ satisfies $P^{(k)} = 0$ in $I \cap J$. Since

$$I \cap J = \bigcup_i I_i \cap \bigcup_j J_j = \bigcup_{ij} (I_i \cap J_j)$$

and

$$\bigcap_{ij} (I_i \cap J_j) = \bigcap_i I_i \cap \bigcap_j J_j \neq 0,$$

we may apply Theorem 2 to the set $K = I \cap J$ and to the function P. Thus, there is a continuous function Q satisfying $Q^{(k)} = 0$ in $\boldsymbol{R}^q$ such that $Q = P$ in $I \cap J$. Let $H = G + Q$ on $\bar{J}$. Then $H = F$ in $I \cap J$. Thus we may put $H = F$ on I and we then obtain, as H, a continuous function on $\overline{I \cup J}$. This function can be extended continuously over the whole space $\boldsymbol{R}^q$. Then the distribution $h = H^{(k)}$ has the required properties. In fact, since $H = F$ on $\bar{I}$, we obtain $h = F^{(k)} = f$ in I. On the other hand, since $H = G + Q$ on $\bar{J}$ and $Q^{(k)} = 0$ in $\boldsymbol{R}^q$, we have $h = G^{(k)} + Q^{(k)} = g + 0$ in J.

Now, we drop the provisory assumption that I and J are bounded. Let $B_1, B_2, \ldots$ be a sequence of bounded open intervals such that $B_1 \cap I_1 \cap \ldots \cap I_p \cap J_1 \cap \ldots \cap J_r \neq 0$, $\bar{B}_n \subset B_{n+1}$ and $\lim B_n = \boldsymbol{R}^q$. Since

$$B_1 \cap I = (B_1 \cap I_1) \cup \ldots \cup (B_1 \cap I_p),$$

$$B_1 \cap J = (B_1 \cap J_1) \cup \ldots \cup (B_1 \cap J_r)$$

and

$$(B_1 \cap I_1) \cap \ldots \cap (B_1 \cap I_p) \cap (B_1 \cap J_1) \cap \ldots \cap (B_1 \cap J_r) \neq 0,$$

we may apply the case just proved and state the existence of a distribution h_1 in $\boldsymbol{R}^q$ such that $h_1 = f$ in $B_1 \cap I$ and $h_1 = g$ in $B_1 \cap J$.

Applying again the case just proved to the sets $B_2 \cap I$ and B_1, we state the existence of a distribution k_1 in $\boldsymbol{R}^q$ such that $k_1 = f$ in $B_2 \cap I$ and $k_1 = h_1$ in B_1.

Now we consider the sets $I' = (B_2 \cap I) \cup B_1$ and $J' = B_2 \cap J$. Evidently, $I' \cap J' = (B_2 \cap I \cap J) \cup (B_1 \cap J)$. We have $k_1 = g$ in $I' \cap J'$. In fact, $k_1 = h_1 = g$ in $B_1 \cap J$ and $k_1 = f = g$ in $B_2 \cap I \cap J$. Thus there is a distribution h_2 in $\boldsymbol{R}^q$ such that $h_2 = h_1$ in B_1, $h_2 = f$ in $B_2 \cap I$ and $h_2 = g$ in $B_2 \cap J$.

Assume that we have already defined $h_1, \ldots, h_n$ so that

$$(7) \qquad h_i = h_{i-1} \text{ in } B_{i-1}, \quad h_i = f \text{ in } B_i \cap I \quad \text{and} \quad h_i = g \text{ in } B_i \cap J.$$

Then we define h_{n+1} as follows. There is a distribution k_n such that $k_n = f$ in $B_{n+1} \cap I$ and $k_n = h_n$ in B_n. We consider the sets

$$I'' = (B_{n+1} \cap I) \cup B_n \quad \text{and} \quad J'' = B_{n+1} \cap J.$$

As before,

$$I'' \cap J'' = (B_{n+1} \cap I \cap J) \cup (B_n \cap J).$$

We have $k_n = g$ in $I'' \cap J''$ for $k_n = h_n = g$ in $B_n \cap J$ and $h_n = f = g$ in $B_{n+1} \cap I \cap J$. Thus there is a distribution h_{n+1} in $\boldsymbol{R}^q$ such that $h_{n+1} = k_n$ in I'' and $h_{n+1} = g$ in J''. This implies that $h_{n+1} = h_n$ in B_n, $h_{n+1} = f$ in $B_{n+1} \cap I$ and $h_{n+1} = g$ in $B_{n+1} \cap J$.

By induction it follows that there is an infinite sequence of distributions $h_1, h_2, \ldots$ in $\boldsymbol{R}^q$ satisfying (7). Since $\lim B_n = \boldsymbol{R}^q$, we obtain the required distribution, on letting $h = \lim h_n$.

4. The preceding "sticking distributions" theorem will now be used in order to prove the following existence theorem for $f^{(m)} = 0$:

THEOREM 5. *Let K be as in Theorem 2. If f is a distribution in $\boldsymbol{R}^q$ such that $f^{(m)} = 0$ in K, then there is a distribution g in $\boldsymbol{R}^q$ such that $g = f$ in K and $g^{(m)} = 0$ in $\boldsymbol{R}^q$.*

Proof. Assume first that K is bounded. There is a function F, continuous on $\bar{K}$, such that $F^{(k)} = f$ in K. Evidently, $F(k+m) = 0$ in K. Thus, by Theorem 2, there is a function G, continuous in $\boldsymbol{R}^q$, such that $G = F$ on $\bar{K}$ and $G^{(k+m)} = 0$ in $\boldsymbol{R}^q$. Letting $g = G^{(k)}$, we have already the required distribution.

Now, we drop the assumption that K is bounded. Let $B_1, B_2, \ldots$ be a sequence of bounded open intervals such that $B_1 \cap I_1 \cap \ldots \cap I_p \neq 0$, $\bar{B}_n \subset B_{n+1}$, $\lim B_n = \boldsymbol{R}^q$. Applying the case just proved to the set $B_1 \cap K$, we see that there is a distribution g_1 in $\boldsymbol{R}^q$ such that $g_1 = f$ in $B_1 \cap K$ and $g_1^{(m)} = 0$ in $\boldsymbol{R}^q$.

Assume that we have already defined distributions $g_1, \ldots, g_n$ so that

$$(8) \qquad g_i = g_{i-1} \text{ in } B_{i-1}, \quad g_i = f \text{ in } B_i \cap K, \quad g_i^{(m)} = 0 \text{ in } \boldsymbol{R}^q.$$

Then we define g_{n+1} as follows. We consider sets B_n and $B_{n+1} \cap K$. The distributions g_n and f are equal in the intersection $B_n \cap (B_{n+1} \cap K) = B_n \cap K$. Thus there is a distribution h in $\boldsymbol{R}^q$ such that $h = g_n$ in B_n and $h = f$ in $B_{n+1} \cap K$, by Theorem 4. Evidently, $h^{(m)} = 0$ in $K' = B_n \cup (B_{n+1} \cap K)$. Since the set K' is bounded, there is a distribution g_{n+1} in $\boldsymbol{R}^q$ such that $g_{n+1} = h$ in K' and $g_{n+1}^{(m)} = 0$ in $\boldsymbol{R}^q$. Evidently,

$$g_{n+1} = g_n \text{ in } B_n, \quad g_{n+1} = f \text{ in } B_{n+1} \cap K, \quad g_{n+1}^{(m)} = 0 \text{ in } \boldsymbol{R}^q.$$

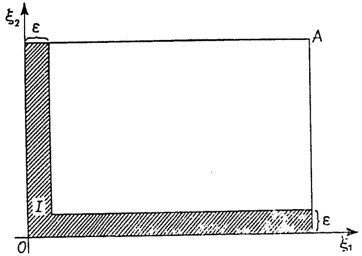
Thus, we have proved by induction that there is an infinite sequence of distributions $g_1, g_2, \ldots$ satisfying (8). The distribution $g = \lim g_n$ has the required properties.

The preceding existence theorem can be completed by the following unicity theorem whose proof is also based on the "sticking distributions" theorem:

THEOREM 6. *Under conditions of Theorem 5, the distribution g is determined uniquely in K^*.*

Proof. Assume that there are two distributions g_1 and g_2 such that $g_1 = g_2 = f$ in K and $g_1^{(m)} = g_2^{(m)} = 0$ in $\boldsymbol{R}^q$. Let B be any bounded interval such that $B \cap I_1 \cap \ldots \cap I_p \neq 0$. Evidently $(B \cap K)^* = B \cap K^*$. There are continuous functions G_1 and G_2 on $\bar{B}$ such that $G_1^{(k)} = g_1$ and $G_2^{(k)} = g_2$ for some order k. Let $G = G_1 - G_2$. Then $G^{(k)} = 0$ in $B \cap K$ and $G^{(k+m)} = 0$ in B. By theorem 2, there is a continuous function H in $\boldsymbol{R}^q$ such that $H = G$ in $B \cap K$ and $H^{(k)} = 0$ in $\boldsymbol{R}^q$. Of course, $H^{(k+m)} = 0$ in $\boldsymbol{R}^q$. This implies, by Theorem 3, that $H = G$ in $B \cap K^*$. Hence $G^{(k)} = 0$ in $B \cap K^*$, and, consequently, $g_1 - g_2 = G_1^{(k)} - G_2^{(k)} = 0$ in $B \cap K^*$. Thus the distribution g in Theorem 6 is determined uniquely in $B \cap K^*$. Since we may take B arbitrary large, g is determined uniquely in K^*.

5. In this section we shall discuss a simple example in the particular case $q = 2$. We assume that the set I has the form of the shadowed area on the enclosed figure. From Theorem 2 it follows that if a continuous



function $f(x) = f(\xi_1, \xi_2)$ satisfies the partial equation (1) in I, then this solution can be extended over the whole rectangle OA. Now, Theorem 3 says that this extension is unique. The situation recalls the well known boundary problem with Cauchy's initial conditions. In fact, that problem can be considered as the limit case of ours, viz., when the thickness ε of I tends to 0. Note that, if we wish to admit, as solutions, functions f which are not of class C^m, then the Cauchy conditions become inapplicable. Still our formulation allows to extend the problem onto solutions f which are arbitrary distributions (Theorems 5 and 6).

The problem of extending solutions of (1) was also considered by Łojasiewicz in [1], but the purpose of his extension lemmas was quite different. It can be noted that neither Theorems 2, 3, 5 and 6 of the present paper nor the particular case, considered in this section can be deduced from Łojasiewicz lemmas.

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STUDIA MATHEMATICA, T. XXXII. (1969)

On orbits of elements

by

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Let X be a linear metric space. Let A be a linear continuous operator mapping X into itself. Let $x \in X$. We shall write

$$\mathcal{O}(A:x) = \{A^n x : n = 0, 1, \ldots\}$$

and we shall call $\mathcal{O}(A:x)$ an *orbit* of x with respect to the operator A.

It is well known that if X is a space of finite dimension, then there are three possibilities:

1° $\lim A^n x = 0$,

2° $\lim \|A^n x\| = +\infty$,

3° the closure of the orbit $\mathcal{O}(A:x)$ is compact and 0 does not belong to this closure.

This follows for instance from [4], lemma 1, p. 270.

The purpose of the present paper is to show that in the infinite-dimensional case it is not so: it may happen that for some A and x, the orbit $\mathcal{O}(A:x)$ is dense in the whole space X. Examples of this situation in concrete spaces are given. It is not clear whether it may take place in an arbitrary infinite-dimensional separable Banach space (cf. Problem 1). Some related questions are also discussed.

The basic terminology and notation are the same as in Banach's book [1] and paper [3] of Mazur and Orlicz. In particular, by an *F-space* we mean any complete linear metric space and by a *B_0-space* we mean a locally-convex F-space. The norm in the sense of [1] (i.e. a subadditive, symmetric functional vanishing only at 0) is called in this paper an *F-norm*; norms and pseudonorms used here are always homogeneous and continuous.

THEOREM 1. *Let X be either l^p $(1 \leqslant p < +\infty)$ or c_0. For any arbitrary real $a > 1$, there are a linear continuos operator A and an element x_0 such that the orbit $\mathcal{O}(A:x_0)$ is dense in the whole of X.*