Let X be a B_0-space. We say that a pseudonorm $\| \|$ defined in X is *infinite* (*finite*) *dimensional* if the quotient space $X/\{x: \|x\| = 0\}$ is infinite (finite) dimensional.

The following proposition, communicated to the author by Dr. C. Bessaga, is strictly connected with problem 5.

PROPOSITION 2. *There are infinite-dimensional B_0-space X and a continuous linear operator A acting in X which is not continuous in any infinite-dimensional pseudonorm.*

Proof. Let $X = M(n^m)$ be a space of all sequences $x = \{x_n\}$ such that

$$\|x\|_n = \sup n^m |x_n|.$$

$M(n^m)$ is a B_0-space with topology induced by pseudonorms $\|x\|_m$. Let A be defined by the formula

$$A(\{x_n\}) = \{nx_n\}.$$

The basis vectors $e_n = (0, 0, \ldots \underset{n\text{-th place}}{0, 1, 0}, \ldots)$ are eigenvectors of A respective to the eigenvalues $\lambda_n = n$. This implies that A is not continuous on any infinite-dimensional pseudonorm.

Remark. The example given in proposition 2 can be slightly extended. Namely, putting $T(s)\ (x_n) = n^s x_n$, we obtain in $M(n^m)$ a continuous group of continuous operators such that, for $s > 0$, $T(s)$ is not continuous in any infinite-dimensional pseudonorm.

The author wishes to express his warmest thanks to Dr. C. Bessaga for his keen remarks and his help in the preparation of this paper.

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An analytic approach to semiclassical potential theory

by

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§ 0. Introduction. The aim of this paper is to give a new non-probabilistic approach to the semiclassical potential theory. The method used here is, may be, less interesting but much simpler. The semiclassical potential theory was started in 1950 by M. Kac who, using probabilistic methods, derived an analytic formula for the capacitory potential. Then it was systematically developed by Z. Ciesielski who indicated analogies between classical and semiclassical potential theories. Such notions as balayage, thinness, Dirichlet problem and barrier have their corresponding ones in the semiclassical theory. The sets of Lebesgue measure zero play essentially the same role as the polar sets. A brief, non-probabilistic account of this theory is given in § 2. For detailed treatment of this subject the reader is refered to [2] and [3]. Improving Kac's technique Stroock [7] has generalized the Kac formula on the strong balayage of an arbitrary superharmonic function. He has also obtained an analytic formula for the solution of the semiclassical Dirichlet problem. The method used in this article leads to the same formulas. We deal with this topic in § 3. § 4 is mainly devoted to non-probabilistic proofs of some Stroock's results (cf. [9]). In it a new method of solving the classical Dirichlet problem is estabilished. The solution is obtained as a limit of solutions of some integral equations (cf. Corollaries 4.5 and 4.6). We finish this paper by suggesting some possible generalizations.

The author wishes to thank Docent Z. Ciesielski for his guidance in the topic and much help and advice.

§ 1. Some basic lemmas. In the following U denotes a Greenian domain in the k-dimensional Euclidean space R^k and $G(x,y)$ the Green function for this domain. It will be convenient to employ the following notations:

$H_+^\uparrow(U)$ — the class of all positive and superharmonic functions on U;
$BH_+^\uparrow(U)$ — the class of all bounded $f \in H_+^\uparrow(U)$;
$CH_+^\uparrow(U)$ — the class of all continuous $f \in BH_+^\uparrow(U)$;

$B(U)$ — Banach space of all bounded Borel functions on U with the norm $\|f\| = \sup_{x\in U} |f(x)|$;

$C(U)$ — Banach space of all bounded continuous functions on U with the same norm.

Let E be a Borel bounded set such that $\bar{E} \subset U$ and let $f \in B(U)$. $G_E f$ denotes the function on U given by

$$G_E f(x) = \int_E G(x, y) f(y)\, dy.$$

Since

$$\sup_{x\in U} \int_E G(x, y)\, dy \leqslant M,$$

$G_E f$ is well defined. Moreover, we have

LEMMA 1.1. *G_E is a compact linear operator from $B(U)$ into $C(U)$ with $\|G_E\| \leqslant M$.*

The proof of this lemma is omitted (see the end of § 4).

PROPOSITION 1.2 (1). *If $\lambda \geqslant 0$, $f \in B(U)$, then there is exactly one function $\varphi_\lambda \in B(U)$ such that $\varphi_\lambda(x) + \lambda G_E \varphi_\lambda(x) = f(x)$ for $x \in U$.*

Moreover, $\|\varphi_\lambda\| \leqslant N_\lambda \|f\|$, where N_λ is a constant independent of f.

Proof. By the spectral theory of compact linear operators, it is enough to prove that $-\lambda$ is not an eigenvalue of G_E, i.e., if, for any $\varphi_\lambda \in B(U)$, $\varphi_\lambda + \lambda G_E \varphi_\lambda = 0$, then $\varphi_\lambda = 0$. But this follows from the following proposition:

PROPOSITION 1.3. *If $f \in BH_+^{\updownarrow}(U)$, $\lambda \geqslant 0$ and φ_λ is a bounded solution of the equation $\varphi_\lambda + \lambda G_E \varphi_\lambda = f$ on U, then φ_λ is positive on U.*

Proof. Let $A = \{x : x \in U, \varphi_\lambda(x) \geqslant 0\}$. Then

$$\varphi_\lambda \chi_A + \varphi_\lambda \chi_{U-A} + \lambda G_E(\varphi_\lambda \chi_A) = f + \lambda G_E(-\varphi_\lambda \chi_{U-A}),$$

where χ_A is the characteristic function of A. Thus for $x \in A$

$$\lambda G_E(\varphi_\lambda \chi_A) \leqslant f + \lambda G_E(-\varphi_\lambda \chi_{U-A}).$$

But the right-hand side of this inequality is a function from $H_+^{\updownarrow}(U)$ and the left-hand side is a potential of a function which vanishes outside A. Thus applying the domination principle of H. Cartan we get

$$\lambda G_E(\varphi_\lambda \chi_A) \leqslant f + \lambda G_E(-\varphi_\lambda \chi_{U-A})$$

for each x in U or, which is the same, $\lambda G_E \varphi_\lambda \leqslant f$ on U and hence $\varphi_\lambda \geqslant 0$.

(1) During the printing of this note the author learned that this proposition and some others of this section were proved by P. Meyer (cf. P. Meyer, *Probability and potentials*, Section III).

PROPOSITION 1.4. *Let $f \in B(U)$ and let φ_λ for $\lambda \geqslant 0$ be the bounded solutions of the equations*

$$\varphi_\lambda + \lambda G_E \varphi_\lambda = \lambda G_E f.$$

Then

(a) $\varphi_\lambda \in C(U)$ and $\|\varphi_\lambda\| \leqslant \sup_{x\in E} |f(x)|$;

(b) if $f \geqslant 0$, then $\varphi_\lambda \geqslant 0$;

(c) if $f \in BH_+^{\updownarrow}(U)$, then $\varphi_\lambda \leqslant f$, $\varphi_{\lambda'} \leqslant \varphi_\lambda$ for $\lambda' \leqslant \lambda$ and $\lim_{\lambda\to\infty} \varphi_\lambda = f$ a.e. on E.

Proof. (b) is a consequence of Proposition 1.3.

(a) φ_λ is continuous because $\varphi_\lambda = \lambda G_E(f - \varphi_\lambda)$ and G_E inverts $B(U)$ into $C(U)$. Let

$$D = \sup_{x\in E} |f(x)|;$$

then

$$D - \varphi_\lambda + \lambda G_E(D - \varphi_\lambda) = \lambda G_E(D - f) + D \quad \text{on } U.$$

The right-hand side function is in $BH_+^{\updownarrow}(U)$, so, by Proposition 1.3, $D - \varphi_\lambda \geqslant 0$. Analogously, we can show that $D + \varphi_\lambda \geqslant 0$. Hence $\|\varphi_\lambda\| \leqslant D$.

(c) If $f \in BH_+^{\updownarrow}(U)$, then $f - \varphi_\lambda$ satisfies the equation

$$f - \varphi_\lambda + \lambda G_E(f - \varphi_\lambda) = f$$

and again Proposition 1.3 gives $f \geqslant \varphi_\lambda$. If $\lambda' \leqslant \lambda$, then

$$(\varphi_\lambda - \varphi_{\lambda'}) + \lambda G_E(\varphi_\lambda - \varphi_{\lambda'}) = (\lambda - \lambda') G_E(f - \varphi_{\lambda'}).$$

Since $f \geqslant \varphi_{\lambda'}$, we get $\varphi_\lambda - \varphi_{\lambda'} \geqslant 0$. It remains to prove that $\lim_{\lambda\to\infty} \varphi_\lambda = f$ a.e. on E. From the equation we have

$$G_E(f - \varphi_\lambda) = \frac{\varphi_\lambda}{\lambda} \leqslant \frac{\sup_{x\in E} |f(x)|}{\lambda}$$

and, moreover, $f - \varphi_\lambda$ decreases as $\lambda \to \infty$. The Lebesgue theorem implies

$$G_E\big(\lim_{\lambda\to\infty}(f - \varphi_\lambda)\big) = 0.$$

Hence $\lim_{\lambda\to\infty} \varphi_\lambda = f$ a.e. on E.

PROPOSITION 1.5. (a) *Let $f \in BH_+^{\updownarrow}(U)$ and let φ_λ for $\lambda > 0$ be the bounded solutions of the equations $\varphi_\lambda + \lambda G_E \varphi_\lambda = f$. Then φ_λ is completely monotone in λ, i.e., φ_λ is infinitely many differentiable in λ and*

$$(-1)^n \frac{d^n \varphi_\lambda}{d\lambda^n} \geqslant 0.$$

(b) *Let $f \in B(U)$ and let φ_λ for $\lambda > 0$ be the bounded solutions of the equations*

$$\varphi_\lambda + \lambda G_E \varphi_\lambda = G_E f.$$

Then φ_λ is infinitely many differentiable in λ and

$$-\frac{D}{\lambda} \leqslant (-1)^n \frac{\lambda^n}{n!} \frac{d^n \varphi_\lambda}{d\lambda^n} \leqslant \frac{D}{\lambda},$$

where $D = \sup_{x \in E} |f(x)|$.

Proof. (a) By Proposition 1.3, $\varphi_\lambda \geqslant 0$. Let $\lambda, \lambda_0 > 0$; then

$$\frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0} + \lambda_0 G_E \left(\frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0} \right) = G_E(-\varphi_\lambda).$$

Because

$$\|\varphi_\lambda\| \leqslant \frac{1}{\lambda} \|f\|$$

(by Proposition 1.4) we have

$$\left\| \frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0} \right\| \leqslant N_{\lambda_0} \frac{1}{\lambda} M \|f\|,$$

hence $\|\varphi_\lambda - \varphi_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0$. From Proposition 1.2 we infer that $(\varphi_\lambda - \varphi_{\lambda_0})/(\lambda - \lambda_0)$ for $\lambda \to \lambda_0$ uniformly on U approaches the solution of the equation

$$\varphi'_{\lambda_0} + \lambda_0 G_E \varphi'_{\lambda_0} = G_E(-\varphi_{\lambda_0}).$$

Thus

$$\frac{d\varphi_\lambda}{d\lambda} = \varphi'_\lambda \leqslant 0.$$

Now it is seen how to continue this procedure to obtain, for arbitrary n,

$$(-1)^n \frac{d^n \varphi_\lambda}{d\lambda^n} \geqslant 0.$$

(b) $D/\lambda - \varphi_\lambda$ is a solution of the equation

$$\left(\frac{D}{\lambda} - \varphi_\lambda \right) + \lambda G_E \left(\frac{D}{\lambda} - \varphi_\lambda \right) = \frac{D}{\lambda} + G_E(D - f);$$

hence $D/\lambda - \varphi_\lambda \geqslant 0$. Analogously, as before, it may be proved that

$$(-1)^n \frac{d^n (D/\lambda - \varphi_\lambda)}{d\lambda^n} \geqslant 0$$

and this is the same as

$$\frac{(-1)^n \lambda^n}{n!} \frac{d^n \varphi_\lambda}{d\lambda^n} \leqslant \frac{D}{\lambda}.$$

The case of the second inequality is similar.

§ 2. Semiclassical approach to the potential theory. Let E be a subset of U. The *strongly swept out* f, $f \in H_+^\uparrow(U)$, onto E (*strong balayage*) is defined as

$$S_f^E(x) = \inf\{g(x) : g \in H_+^\uparrow(U), g \geqslant f \text{ a.e. on } E\}.$$

The connection between strong and ordinary balayage B_f^E is seen from the following

Proposition 1.2. *Let $f \in H_+^\uparrow(U)$ and let E be a subset of U. Then there exists $E_0 \subset E$ such that $|E - E_0| = 0$ and $S_f^E(x) = B_f^{E_0}(x)$ on U.*

The proof may be found in [2] and here it is omitted. Using this proposition, the following list of properties can be easily established (we assume that $f, g, f_n \in H_+^\uparrow(U)$):

P.1. $S_f^E \in H_+^\uparrow(U)$.

P.2. $S_f^E \leqslant f$ on U.

P.3. $S_f^E = f$ a.e. on E.

P.4. $S_f^E \leqslant S_g^E$ on U if $f \leqslant g$ a.e. on E.

P.5. $S_{\alpha f + \beta g}^E = \alpha S_f^E + \beta S_g^E$ on U for $\alpha, \beta \geqslant 0$.

P.6. $S_{S_f^E}^E = S_f^E$ on U.

P.7. If $f_n \uparrow f$ a.e. on E, then $S_{f_n}^E \uparrow S_f^E$ on U.

The notion of thinness in the classical theory has its analogue in the semiclassical approach. It is s-thinness. The set E, $E \subset U$, is called *s-thin* at x_0, $x_0 \in U$, if either $|E \cap V| = 0$ for some neighbourhood V of x_0 or if there exists $f \in H_+^\uparrow(U)$ such that

$$f(x_0) < \varliminf_{x \to x_0, x \in E} \operatorname{ess} f(x).$$

Thinness and s-thinness are connected as follows:

Proposition 2.2. *E is s-thin at x_0 if and only if there is $E^0 \subset E$ such that $|E - E^0| = 0$ and E^0 is thin at x_0.*

The proof can be easily derived from the definitions.

As a consequence of this proposition and the corresponding propositions in the classical potential theory, a new characterization of s-thinness is obtained.

Corollary 2.3. *E is not s-thin at x_0 if, and only if $S_f^E(x_0) = f(x_0)$ for each $f \in H_+^\uparrow(U)$.*

Corollary 2.4. *E is not s-thin at x_0 if it has positive Lebesgue upper density at x_0.*

For a given set E we denote by E^* the set of all points at which E is not s-thin.

The set E is said to be *s-regular* if $E \subset E^*$.

Corollary 2.5. *If $f \in H_{+}^{\uparrow}(U)$ is continuous at x_0 and $x_0 \in E^*$, then S_f^E is continuous at x.*

This is consequence of lower semicontinuity of S_f^E and equality $S_f^E(x_0) = f(x_0)$.

Theorem 2.6. *Let E be a bounded Borel set such that $\bar{E} \subset U$ and let $f \in H_{+}^{\uparrow}(U)$. Then there is a sequence q_n of positive functions in $C(U)$ such that*

$$G_E q_n \uparrow S_f^E \quad \text{on } U.$$

Proof. First we assume that $f \in CH_{+}^{\uparrow}(U)$. For $\lambda > 0$ let φ_λ be the bounded solution of the equation

$$\varphi_\lambda + \lambda G_E \varphi_\lambda = \lambda G_E f.$$

By Proposition 1.4, φ_λ have the following properties: $\varphi_\lambda \leqslant \varphi_{\lambda'}$ for $\lambda \leqslant \lambda'$, $\varphi_\lambda = G_E\big(\lambda(f-\varphi_\lambda)\big)$ with $f-\varphi_\lambda$ continuous and positive and, moreover, $\varphi_\lambda \uparrow f$ a.e. on E. Property P.7 implies $S_{\varphi_\lambda}^E \uparrow S_f^E$. Thus to end the proof it suffices to show that $S_{\varphi_\lambda}^E = \varphi_\lambda$. But if, for any $g \in H_{+}^{\uparrow}(U)$, $g \geqslant \varphi_\lambda = G_E\big(\lambda(f-\varphi_\lambda)\big)$ a.e. on E, then by Cartan domination principle $g \geqslant \varphi_\lambda$ on U and this shows that $S_{\varphi_\lambda}^E \geqslant \varphi_\lambda$. This together with property P.2 ends the proof of this case.

Now let $f \in H_{+}^{\uparrow}(U)$ and let f^n be non-decreasing sequence from $CH_{+}^{\uparrow}(U)$ convergent to f on U. Denote by φ_λ^n the bounded solution of the equation

$$\varphi_\lambda^n + \lambda G_E \varphi_\lambda^n = \lambda G_E f^n.$$

The sequence $q_n = n(f^n - \varphi_n^n)$ satisfies all demands. By Proposition 1.4, q_n are continuous positive functions:

$$G_E q_n = \varphi_n^n \leqslant \varphi_{n+1}^n \leqslant \varphi_{n+1}^{n+1} = G_E q_{n+1}$$

(because $f^n \leqslant f^{n+1}$). Furthermore,

$$\underline{\lim}_{n\to\infty} G_E q_n \geqslant S_{f^m}^E \quad \text{and} \quad \overline{\lim}_{n\to\infty} G_E q_n \leqslant S_f^E$$

and since $S_{f^m}^E \uparrow S_f^E$, we get

$$\lim_{n\to\infty} G_E q_n = S_f^E.$$

This completes the proof.

Remark. If E is compact and S_f^E is continuous on E, then by Dini's theorem the convergence

$$\lim_{n\to\infty} G_E q_n = S_f^E$$

is uniform on E.

Now we are going to discuss the Dirichlet problem in the semiclassical potential theory. Let K be a compact subset of U. Let us denote by $C(K)$ the set of all continuous functions on K and let

$$\|f\|_K = \max_{x\in K} |f(x)|.$$

Moreover, let

$$C_0(K) = \{(u)_K : u = u_1 - u_2;\ u_1, u_2 \in CH_{+}^{\uparrow}(U)\},$$

where $(u)_K$ is the restriction of u to K.

Simple Stone-Weierstrass argument shows that $C_0(K)$ is a dense linear manifold in $C(K)$ with respect to the norm $\|\cdot\|_K$.

For given $x \in U$ we define on $C_0(K)$ a functional as follows:

$$D_f(x) = D_f^K(x) = S_{u_1}^K(x) - S_{u_2}^K(x)$$

if $f = (u_1 - u_2)_K$, $u_1, u_2 \in CH_{+}^{\uparrow}(U)$.

Since $S_{u_2}^K = u_2$ a.e. on K, we have

$$S_{u_2}^K + \|f\|_K \geqslant u_1$$

a.e. on K; and this implies

$$S_{u_2}^K + \|f\|_K \geqslant S_{u_1}^K$$

on U. Analogously,

$$S_{u_1}^K + \|f\|_K \geqslant S_{u_2}^K$$

on U. Thus we have

$$|D_f^K(x)| \leqslant \|f\|_K.$$

From this and property P.5 it follows that $D_f^K(x)$ is well defined and linear in f. Thus $D_f(x)$ has a unique extension to $C(K)$ which will be also denoted by $D_f(x)$. For extended functional we have again

$$|D_f(x)| \leqslant \|f\|_K, \qquad x \in U,\ f \in C(K).$$

It has been pointed out in Corollary 2.5 that $D_f(x)$ is continuous at each $x_0 \in K^*$ for $f = (u)_K$ with $u \in CH_{+}^{\uparrow}(U)$. It is also clear that D_f for $f \in C_0(K)$ is harmonic on $U - K$. Thus we have arrived at the following result:

Proposition 2.7. *For each $f \in C(K)$ the function D_f is defined on U, it is harmonic on $U - K$, equal to f and continuous at each $x \in K^*$. Moreover,*

$$|D_f(x)| \leqslant \|f\|_K.$$

There is another method of defining D_f^K. It is established in the following theorem:

THEOREM 2.8. *Let K be a compact subset of U, $f \in C(K)$. Let φ_λ for $\lambda \geqslant 0$ denote the solution of the equation $\varphi_\lambda + \lambda G_K \varphi_\lambda = \lambda G_K f$. Then*

$$\lim_{\lambda\to\infty} \varphi_\lambda(x) = D_f^K(x) \quad \textit{for each } x \textit{ in } U.$$

If K is s-regular, then the convergence is uniform on U.

Proof. By Proposition 1.4, $\|\varphi_\lambda\| \leqslant \|f\|_K$ and from the proof of Theorem 2.6 it is seen that

$$\lim_{\lambda\to\infty} \varphi_\lambda(x) = D_f(x)$$

on U if $f \in C_0(K)$. Since $C_0(K)$ is dense in $C(K)$, the standard arguments show that

$$\lim_{\lambda\to\infty} \varphi_\lambda(x) = D_f(x) \quad \text{for each } f \in C(K).$$

If K is s-regular and $f \in CH_+^\ddagger(U)$, then S_f^K is continuous on K. This and the remark after Theorem 2.6 proves that for such f the convergence is uniform on K, which, as before, permits us to state that the convergence is uniform on K for arbitrary f in $C(K)$. Now the uniform convergence on U is a consequence of such convergence on K because, for $\lambda > \lambda'$,

$$(\varphi_\lambda - \varphi_{\lambda'}) + \lambda G_K(\varphi_\lambda - \varphi_{\lambda'}) = \lambda G_K\left(\frac{\lambda-\lambda'}{\lambda}(f-\varphi_{\lambda'})\right)$$

and by Proposition 1.4

$$\|\varphi_\lambda - \varphi_{\lambda'}\| \leqslant \frac{\lambda-\lambda'}{\lambda}\|f-\varphi_{\lambda'}\|_K \leqslant \|f-\varphi_{\lambda'}\|_K.$$

COROLLARY 2.9. *Let K be a compact subset of U. Then K is s-regular if and only if for each $f \in C(K)$ and arbitrary $\varepsilon > 0$ there exists $g \in C(K)$ such that*

$$\|f - G_K g\|_K < \varepsilon.$$

This corollary was proved by Ciesielski [2] and the proof is omitted.

We end this paragraph with characterizations of the sets E for which $S_f^E = B_f^E$ on U for all $f \in H_+^\ddagger(U)$. The set E, $E \subset U$, is said to be *quasi s-regular* if $E - E^*$ is a polar set.

PROPOSITION 2.10. *The set $E \subset U$ is quasi s-regular if and only if $S_f^E = B_f^E$ for each $f \in H_+^\ddagger(U)$.*

The proof of this proposition may be found in [3].

Let H_f^K denote the classical solution of the generalized Dirichlet problem in $U - K$ with the boundary values 0 on ∂U and f on K. As a corollary of Proposition 2.10 we get

COROLLARY 2.11. *Let K be a compact subset of U. Then $H_f^K = D_f^K$ if and only if K is quasi s-regular.*

§ 3. Kac-Stroock formulas. In this paragraph E denotes a bounded Borel set such that $\bar{E} \subset U$, $|E| > 0$. We shall use the following notation:

$L^2(E)$ — Hilbert space of all functions f on E for which

$$\|f\|_2 = \Big(\int_E f^2(y)\,dy\Big)^{1/2} < +\infty;$$

(f, g) the inner product $\int_E f(y)g(y)\,dy$; $f, g \in L^2(E)$;

$G_E^1(x, y) = G(x, y)$, $x, y \in U$;

$G_E^{m+1}(x, y) = \int_E G_E^m(x, z)G(z, y)\,dz$, $x, y \in U$;

$G_E^m f(x) = \int_E G_E^m(x, y)f(y)\,dy$, $x \in U$.

Since $G(x, y)$ is a weakly singular kernel on $U \times U$, the following lemma is true:

LEMMA 3.1. *There are a constant C and an integer m_0 such that for $m \geqslant m_0$ and $x, y \in U$*

$$G^m(x, y) \leqslant C, \qquad G(x, y) = G(y, x)$$

and hence

LEMMA 3.2. *The linear operator $G_E: L_2^2(E) \to L_2^2(E)$ is compact, self-adjoint and positive definite.*

(Positiveness is concluded from the energy principle.)

Let $\{\lambda_j, \varphi_j\}$ denote the complete orthonormal system of eigenfunctions of G_E with their corresponding eigenvalues.

LEMMA 3.3.

(a) $\sum_{j=1}^{\infty} \lambda_j^{2m_0} < +\infty$.

(b) $|\varphi_j(x)| \leqslant L(1/\lambda_j)^{m_0}$ *on E for some constant L independent of j and x.*

The proofs of Lemmas 3.3 and 3.4 are standard (cf. [7]) and here they are omitted.

The following notation will simplify formulation of theorems. Let f be an integrable function on E, i.e.

$$\int_E |f(y)|\,dy = \|f\|_1 < +\infty,$$

let m be an integer and $t > 0$. Then $S_t^m f$ is defined as a function on U given by the series:

$$S_t^m f(x) = \sum_{j=1}^{\infty} e^{-t/\lambda_j}\lambda_j^{m-2}(f, \varphi_j)\int_E G(x, y)\varphi_j(y)\,dy.$$

The series is convergent which is seen from

LEMMA 3.4. *Let f be integrable on E, $t>0$. Then*

(a) *the series*

$$\sum_{j=1}^{\infty} e^{-t/\lambda_j}\lambda_j^{m-2}\cdot(f,\varphi_j)\int_E G(x,y)\varphi_j(y)\,dy$$

is uniformly and absolutely convergent in x on U;

(b) *if $m\geqslant 4m_0+1$, then*

$$\sum_{j=1}^{\infty}\int_0^{+\infty}\Big|e^{-t/\lambda_j}\lambda_j^{m-2}(f,\varphi_j)\int_E G(x,y)\varphi_j(y)\,dy\Big|\,dt$$

is uniformly in x convergent on U.

We shall write $S_t f$ instead of $S_t^1 f$. For the proof of the main theorems of this paragraph we shall need the lemma which can be easily derived from the S. Bernstein theorem on completely monotones functions.

LEMMA 3.5. (a) *If $D-\lambda\psi(\lambda)$ is completely monotone function of λ for $\lambda>0$,*

$$\lim_{\lambda\to\infty}\big(D-\lambda\psi(\lambda)\big)=0,\qquad \lim_{\lambda\to 0}\big(D-\lambda\psi(\lambda)\big)=D,$$

then

$$\psi(\lambda)=\int_0^{+\infty}e^{-\lambda t}g(t)\,dt$$

and $g(t)$ is a non-increasing, positive and right-continuous function of t with

$$\lim_{t\to 0+}g(t)=D.$$

(b) *If $\psi(\lambda)$ is infinitely many differentiable for $\lambda>0$ and*

$$-\frac{D}{\lambda}\leqslant\frac{(-1)^n\lambda^n}{n!}\,\frac{d^n\psi(\lambda)}{d\lambda^n}\leqslant\frac{D}{\lambda},$$

then

$$\psi(\lambda)=\int_0^{+\infty}e^{-\lambda t}g(t)\,dt$$

and $|g(t)|\leqslant D$ for $t>0$.

Now we are ready to prove the main result.

THEOREM 3.6. *Let $f\in H_+^\uparrow(U)$. Then*

(a) $S_f^E(x)=\lim\limits_{t\to 0+}S_t f(x)=\lim\limits_{t\to 0+}\sum\limits_{j=1}^{\infty}e^{-t/\lambda_j}(f,\varphi_j)\int_E G(x,y)\varphi_j(y)\,dy$ *for $x\in U$;*

(b) $S_t f\leqslant S_{t'}f$ *on U for $t\geqslant t'>0$;*

(c) $S_t f=G_E v_t$ *on U where v_t, $t>0$, are bounded and positive functions on E.*

Proof. First we assume that $f\in BH_+^\uparrow(U)$. Let ψ_λ be for $\lambda>0$ the bounded solution of the equation

$$\psi_\lambda+\lambda G_E\psi_\lambda=G_E f$$

on U. Then $\varphi_\lambda=\lambda\psi_\lambda$ fulfils the equation

$$\varphi_\lambda+\lambda G_E\varphi_\lambda=\lambda G_E f.$$

Thus from the proof of Theorem 2.6 we get

$$\lim_{\lambda\to\infty}(S_f^E-\lambda\psi_\lambda)=0.$$

Because $\|\psi_\lambda\|\leqslant M\|f\|$, we have

$$\lim_{\lambda\to 0}(S_f^E-\lambda\psi_\lambda)=S_f^E.$$

Moreover, $S_f^E-\lambda\psi_\lambda$ is a solution of the equation

$$S_f^E-\lambda\psi_\lambda+\lambda G_E(S_f^E-\lambda\psi_\lambda)=S_f^E+\lambda G_E(S_f^E-f).$$

Since $S_f^E=f$ a.e. on E, $G_E(S_f^E-f)=0$ and hence

$$S_f^E-\lambda\psi_\lambda+\lambda G_E(S_f^E-\lambda\psi_\lambda)=S_f^E.$$

Now Proposition 1.5 (a) implies that $S_f^E-\lambda\psi_\lambda$ is completely monotone function of λ. Thus $S_f^E-\lambda\psi_\lambda$ fulfils all the assumptions of Lemma 3.5 (a). Hence

$$\psi_\lambda(x)=\int_0^{+\infty}e^{-\lambda t}g_t(x)\,dt,$$

where $g_t(x)$ is for every $x\in U$ right-continuous, non-increasing, positive for $t>0$ and

$$\lim_{t\to 0+}g_t(x)=S_f^E(x).$$

Integrating by parts, we obtain

$$\psi_\lambda(x)=\frac{\lambda^{m+1}}{m!}\int_0^{+\infty}e^{-\lambda t}\Big(\int_0^t(t-s)^m g_t(x)\,ds\Big)\,dt.$$

Now let $m\geqslant 4m_0+1$ and let

$$h_\lambda(x)=\sum_{i=0}^{m}(-\lambda)^i G_E^{i+1}f(x)+\sum_{j=1}^{\infty}\frac{(-\lambda\lambda_j)^{m+1}}{1+\lambda\lambda_j}(f,\varphi_j)G_E\varphi_j(x),\qquad x\in U.$$

Inequality $m \geqslant 4m_0+1$ provides uniform and absolute convergence on U of this series (analogously as in Lemma 3.4). Putting h_λ into equation we check that h_λ satisfies $h_\lambda + \lambda G_E h_\lambda = G_E f$ on U. Hence $h_\lambda = \psi_\lambda$ on U. But

$$h_\lambda(x) = \sum_{i=0}^{m}(-\lambda)^i G_E^{i+1} f(x) + \sum_{j=0}^{\infty}\frac{(-\lambda\lambda_j)^{m+1}}{1+\lambda\lambda_j}(f,\varphi_j)G_E\varphi_j(x)$$

$$= \lambda^{m+1}\int_0^{+\infty} e^{-\lambda t}\Big[\sum_{i=0}^{m}(-1)^i\frac{t^{m-i}}{(m-i)!}G_E^{i+1}f(x) + $$

$$+(-1)^{m+1}\sum_{i=1}^{\infty} e^{-t/\lambda_j}\lambda_j^m(f,\varphi_j)G_E\varphi_j(x)\Big]dt.$$

The uniform convergence of this series and integration term by term is provided by Lemma 3.4. From the uniqueness theorem on Laplace transform we have for $t>0$

$$\frac{1}{m!}\int_0^t (t-s)^m g_t(x)\,dt = \sum_{i=0}^{m}(-1)^i\frac{t^{m-i}}{(m-i)!}G_E^{i+1}f(x) + (-1)^{m+1}S_t^{m+2}f(x).$$

Differentiating this equality $m+1$ times with respect to t we obtain

$$g_t(x) = S_t f(x) \text{ a.e. in } t, \quad t \geqslant 0 \text{ for each } x \in U.$$

But $g_t(x)$ is right-continuous, and $S_t f$ is continuous in t, so $g_t(x) = S_t f(x)$ for each $t>0$, $x \in U$. Already proved properties of $g_t(x)$ imply that $S_t f$ fulfill (a) and (b). Since

$$S_t f = G_E\Big(\sum_{j=1}^{\infty}\frac{1}{\lambda_j}e^{-t/\lambda_j}(f,\varphi_j)\varphi_j\Big),$$

to prove (c) it is enough to show that

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j}e^{-t/\lambda_j}(f,\varphi_j)\varphi_j$$

is bounded and positive on E. Estimates like those in Lemma 3.4 prove boundedness. $S_t f$ is non-decreasing, so

$$0 \geqslant \frac{d}{dt}S_t f = -\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}e^{-t/\lambda_j}(f,\varphi_j)G_E\varphi_j \quad \text{on } U.$$

In particular, for $x \in E$ we get

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j}e^{-t/\lambda_j}(f,\varphi_j)\varphi_j \geqslant 0$$

and this completes the proof of the first case.

Now let $f \in H_+^\downarrow(U)$. Let $f^n = \min(f, n)$. Then $f^n \in BH_+^\downarrow(U)$ and $f^n \uparrow f$ on U.

It is apparent that

$$\lim_{n\to\infty} S_t f^n = S_t f.$$

From the proof of the first case it is seen that Laplace transform ϱ_λ^n of $S_t(f^{n+1}-f^n)$ satisfies

$$\varrho_\lambda^n + \lambda G_E \varrho_\lambda^n = G_E(f^{n+1}-f^n).$$

By Proposition 1.5 (a), ϱ_λ^n is completely monotone, hence by Bernstein theorem, $S_t(f^{n+1}-f^n) \geqslant 0$. Thus the last convergence is monotonic, i.e. $S_t f^n \uparrow S_t f$. This implies (a) and (b). (c) is proved in the same way as in the first case.

The obtained analytic formula may be used in order to get a formula of the same type for the solution of the semiclassical Dirichlet problem.

THEOREM 3.7. *Let $E = K$ be a compact subset of U and let $f \in C(K)$. Then*

$$\lim_{t\to 0+} S_t f(x) = D_f^K(x), \quad x \in U.$$

Proof. From the proof of Theorem 3.6 we know that if $f \in C_0(K)$, then the Laplace transform ψ_λ of $S_t f$ fulfils the equation $\psi_\lambda + \lambda G_E \psi_\lambda = G_E f$ on U. By Proposition 1.5 (b)

$$-\frac{\|f\|_K}{\lambda} \leqslant \frac{(-1)^n\lambda^n}{n!}\frac{d^n\psi_\lambda}{d\lambda^n} \leqslant \frac{\|f\|_K}{\lambda}$$

and hence by Lemma 3.5 (b)

$$|S_t f(x)| \leqslant \|f\|_K \quad \text{for } t>0 \ (x \in U).$$

Furthermore, for such f by Theorem 3.7

$$\lim_{t\to 0+} S_t f(x) = D_f^K(x) \quad \text{for all } x \in U.$$

Since $C_0(K)$ is dense in $C(K)$, the last assertion holds for all $f \in C(K)$.

Remark. If K is s-regular, then like as in Theorem 2.8 it may be proved that the convergence is uniform on U.

§ 4. Generalizations. Let U, as in the preceding paragraphs, denote Greenian domain in R^k. If μ is a Radon measure on U which vanishes on polar sets, then the *strong balayage* of f, $f \in H_+^\downarrow$, relatively to μ is defined as

$$B_f^\mu(x) = \inf\{g(x) : g \in H_+^\downarrow(U),\ g \geqslant f\ \ \mu\ \text{a.e.}\}.$$

Analogously as for the Lebesgue measure it may be proved that B_f^μ fulfils Proposition 2.1 and has all Properties P.1-P.7 (with obvious changes in formulation).

A positive Radon measure μ on U is said to be a *W-measure* if

$$G\mu(x) = \int_U G(x, y)\, d\mu(y)$$

is a bounded function of x on U.

For a given function f on U, let us write

$$G_\mu f(x) = \int_U G(x, y) f(y)\, d\mu(y), \qquad x \in U.$$

Lemma 4.1. *Let μ be a W-measure, $f \in B(U)$ and $\lambda > 0$. Then there is exactly one $\varphi_\lambda \in B(U)$ such that*

$$\varphi_\lambda(x) + \lambda G_\mu \varphi_\lambda(x) = \lambda G_\mu f(x) \quad \text{on } U.$$

Proof. By Riesz theorem G_μ is a bounded linear operator from $L^2(\mu, U)$ into itself. The energy principle implies that G_μ is positive definite, hence the above equation has a unique solution φ_λ in $L^2(\mu, U)$. Now to complete the proof of the lemma it is sufficient to demonstrate that φ_λ is positive each time f is positive. But it can be proved exactly in the same way as Proposition 1.3.

For a W-measure μ, Propositions 1.3, 1.4 (except the continuity of φ_λ in the item (a)) and 1.5 with their proofs will remain valid if we put G_μ instead G_E everywhere in their formulations. Proposition 1.4 reformulated in this way implies

Proposition 4.2. *Let μ be a W-measure, $f \in BH_+^\downarrow(U)$ and let, for $\lambda > 0$, φ_λ be the bounded solution of the equation*

$$\varphi_\lambda + \lambda G_\mu \varphi_\lambda = \lambda G_\mu f \quad \text{on } U.$$

Then $\varphi_\lambda \leqslant \varphi_{\lambda'}$ for $\lambda \leqslant \lambda'$, $0 \leqslant \varphi_\lambda \leqslant f$ and

$$\lim_{\lambda \to \infty} \varphi_\lambda = B_f^\mu \quad \text{on } U.$$

This allows us to prove an analogue of Theorem 2.6.

Theorem 4.3. *Let μ be a positive Radon measure on U which vanishes on the polar sets, and let $f \in H_+^\downarrow(U)$. Then there is a non-decreasing sequence μ_n of W-measures and a sequence q_n of positive functions in $B(U)$ such that $G_{\mu_n} q_n \uparrow B_f^\mu$ ($\mu_n \leqslant \mu_{n+1}$ if $\mu_n(E) \leqslant \mu_{n+1}(E)$ for each Borel set E).*

Proof. Let μ_n be a restriction of μ to the set $A_n = \{x \in U : G\mu(x) \leqslant n\}$. Since

$$U \setminus \bigcup_{n=1}^{\infty} A_n$$

is a polar set we get $\lim\limits_{n\to\infty} \mu_n = \mu$. Moreover, it is seen that μ_n are W-measures and the sequence is non-decreasing. Denote $\min(f, n)$ by f^n. For each n let φ_n be the bounded solution of the equation

$$\varphi_n + nG_{\mu_n}\varphi_n = nG_{\mu_n} f^n.$$

Define q_n as $q_n = n(f_n - \varphi_n)$. Then Proposition 4.2 and the arguments like those used in the proof of Theorem 2.6 give $\varphi_n \leqslant \varphi_{n+1}$ and

$$\lim_{n\to\infty} \varphi_n = B_f^\mu \quad \text{on } U.$$

This completes the proof.

Using probabilistic methods, Stroock [3] has constructed for every bounded set E a W-measure μ such that $B_f^E(x) = B_f^\mu(x)$ on U for all f in $H_{+\infty}^\downarrow(U)$. Now we are going to construct such a measure in a simple way.

Theorem 4.4. *Let E be a bounded subset of U such that $\bar{E} \subset U$. Then there exists a W-measure ξ^E such that $B_f^E(x) = B_f^{\xi^E}(x)$ on U for all $f \in H_+^\downarrow(U)$.*

Proof. Let U_0 be an open bounded set such that $\bar{E} \subset U_0$, and $\bar{U}_0 \subset U$, and let ξ denote the Lebesgue measure restricted to U_0. Then there is unique W-measure ξ^E on U such that $B_{G\xi}^E = G\xi^E$. (It is the sweeping out of ξ onto E.) Since ξ^E is concentrated on the set of regular points of E and it vanishes on the polar sets, by domination principle it is $B_f^E \geqslant B_f^{\xi^E}$ on U. It remains to prove that $B_f^E \leqslant B_f^{\xi^E}$ on U, but it is enough to prove this for $f = G_\xi g$, where g is a positive bounded function on U_0. For such f, $B_f^E = G_\gamma$ where γ is a W-measure absolutely continuous with respect to ξ^E (it is seen from the equalities $aB_{G\xi}^E = B_f^E + B_{G\xi}^E(a-g)$ and hence $a\xi^E = \gamma + \gamma_1$, where $a = \sup g(x)$). Now from the domination principle we conclude that $B_{B_f^E}^{\xi^E} = B_f^E$. Because $B_{B_f^E}^{\xi^E} \leqslant B_f^{\xi^E}$, this closes the proof.

Remark. If E is an analytical set, then a slight modification of ξ^E may be done so that the essential support of ξ^E is contained in E.

Corollary 4.5. *Let E, ξ^E be as in the proof of Theorem 4.4, and*

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let $f \in B(U)$. Denote for $\lambda > 0$ by φ_λ the bounded solution of the equation $\varphi_\lambda + \lambda G_{\xi^E}\varphi_\lambda = \lambda G_{\xi^E} f$. Then

(a) *if $f \in BH_+^\dagger(U)$, then $\lim\limits_{\lambda\to\infty} \varphi_\lambda = B_f^E$ on U;*

(b) *if E is compact and $f \in C(E)$, then $\lim\limits_{\lambda\to\infty} \varphi_\lambda = H_f^E$ on U.*

This corollary is an immediate consequence of Proposition 4.2 and Theorem 4.4.

Let V be a domain, $\overline{V} \subset U$, with the boundary ∂V in C^1 (or piecewise in C^1) and let σ denote the surface measure on ∂V. Then it is not difficult to prove that σ is W-measure and $B_f^\sigma = B_f^{\partial V}$ on U for all $f \in H_+^\dagger(U)$. For the same reasons as in the case of Corollary 4.5 the following is true:

COROLLARY 4.6. *Let V, σ be as above, $f \in C(\partial V)$. Denote by φ_λ for $\lambda > 0$ the bounded solution of the equation $\varphi_\lambda + \lambda G_\sigma \varphi_\lambda = \lambda G_\sigma f$. Then*

$$\lim_{\lambda\to\infty} \varphi_\lambda = H_f^{\partial V} \quad \text{on } U.$$

For such σ and V we can prove that for sufficiently large m

$$G_\sigma^m(x, y) = \int_{\partial V} \ldots \int_{\partial V} G(x, y_1) G(y_1, y_2) \ldots G(y_{m-1}, y)\, d\sigma(y_1)\, d\sigma(y_2) \ldots d\sigma(y_{m-1})$$

is a bounded function of x, y on $U \times U$. Hence exactly in the same way as in § 3 Kac-Stroock formulas for $B_f^{\partial V}$, $H_f^{\partial V}$ may be established.

The semiclassical potential theory like that in § 1 and § 2 may be developed without essential changes for much more general kernels. For instance it may be done for potential kernel U of a Markov process which fulfils Hunt's [4] hypotheses A, F, G and for which Lemma 1.1 is valid. The Lebesgue measure should be replaced by ξ measure from the hypothesis F of § 17. Lemma 1.1 holds if $h(\alpha_n, x, y)$ are bounded functions of x, y on $U \times U$ for some fundamental system α_n (all notations are taken from § 17 of [5], Hunt). This is the case of the Newton, the M. Riesz and the heat potentials.

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