Now, we choose M such that $M|t_i|\,|a_i'\colon|\lambda_i||>1$. Hence, by (4.3), $|f(a_i)|>1$, and, by (4.2),

$$\big|a_i\colon|\lambda_i|\big| = |t_i|\,\big|a_i'\colon|\lambda_i|\big| \leqslant \frac{1}{|\lambda_i|}.$$

This implies, by definition of $\big|a_i\colon|\lambda_i|\big|$ in (3.1) that

$$|a_{i0}|\leqslant\frac{1}{|\lambda_i|},\qquad |a_{in}|^{1/n}\,|\lambda_i|\leqslant\frac{1}{|\lambda_i|^{1/n}},\qquad n\geqslant 1,$$

i.e.

$$(4.4)\qquad |a_{in}|^{1/n}|\lambda_i|\leqslant A\left(\frac{1}{|\lambda_i|}\right).$$

Since $|\lambda_i|\to\infty$, we can assume that $|\lambda_i|\geqslant 1$ for all i. Then (4.4) gives that $|a_{in}|^{1/n}|\lambda_i|\leqslant 1$, i.e. $|a_{in}|^{1/n}\leqslant 1/|\lambda_i|$ for each $n\geqslant 1$. Thus we have

$$|a_{i0}|\leqslant\frac{1}{|\lambda_i|},\qquad |a_{in}|^{1/n}\leqslant\frac{1}{|\lambda_i|},\qquad n\geqslant 1.$$

Therefore, by definition of $|a_i|$ in (2.2), $|a_i|\leqslant 1/|\lambda_i|$. But $1/|\lambda_i|\to 0$ as $i\to\infty$. Therefore $|a_i|\to 0$ as $i\to\infty$. On the other hand, $|f(a_i)|>1$, i.e. $f(a)$ is not continuous on S in the topology of $K\langle x\rangle$. This proves the result.

The author is indebted to the referee for correcting an error in an earlier version of the paper and for pointing out that Theorem 3.2 is equivalent to Theorem 3.1 of [5].

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An L^1-algebra for algebraically irreducible semigroups*

by

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1.1. Introduction. This paper is another chapter in the theory of L^1-algebras of linearly quasi-ordered semigroups. Algebraically irreducible commutative semigroups are known to be a special case of linearly quasi-ordered semigroups, but the structure of the semigroup over an idempotent arc in the decomposition space $S/\mathscr{L}$ is more amenable for the algebraically irreducible semigroups (see Theorem 1.3). Adapting the work of Lardy [4] on $L^1(a,b)$, where (a,b) is an idempotent commutative semigroup and using Lebesgue measure on (a,b) we introduce a measure M on the algebraically irreducible semigroups S for which $S/\mathscr{L}$ is an idempotent semigroup. We show that $L^1(S,M)$ is semisimple and that the multiplicative linear functionals (maximal ideal space) of this algebra is in one-to-one correspondence with the measurable semicharacters on S. We conclude the paper with some remarks as to the extension of the results here to a wider class of linearly quasi-ordered semigroups. Our work here was motivated by Lardy [4] and the remarks in Rothman [7] about assigning measure zero to idempotent arcs in $S/\mathscr{L}$. The methods of [6] and [7] are used. While the notation here is different, it is clear that it is in agreement with that of [6] and [7] when passing from functions in $L^1(S,M)$ to the corresponding measures in $M(S)$.

1.2. Definitions and basic theorems. In what follows, a *semigroup* S is a Hausdorff topological space together with a continuous associative multiplication. We shall use 1 to denote the identity element, K to denote the minimal ideal (which exists in S is compact [9]), and H to denote the maximal subgroup of S with identity 1.

A compact connected semigroup S is *algebraically irreducible about* $B\subset S$ if S contains no proper closed connected subsemigroup containing B. In particular, a compact connected abelian semigroup with an identity element, 1, algebraically irreducible about $K\cup H$ will be called an *A-I semigroup* [5]. We use the left equivalence of Green [2] defined for S

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by $x \equiv y(\mathscr{L})$ if and only if $\{x\} \cup Sx = \{y\} \cup Sy$, and L_x to denote the equivalence class of all p such that $p \equiv x(\mathscr{L})$. It is known that for S compact and normal, the quotient space, S modulo $\mathscr{L}$, is again a compact semigroup which we shall denote by S', and that the canonical mapping $\varphi\colon S \to S'$ is a continuous homomorphism. It is proved in [5] that S' is a standard thread if S is an A-I semigroup.

We make frequent use of the following canonical representation theorem:

THEOREM 1.3 [5]. *Let S be an A-I semigroup. If S' consists entirely of idempotent elements, then there is an arc subsemigroup P in S such that $\varphi|_P$ is an isomorphism onto S', and $\overline{S-K}$ is the union of the orbits of the elements of P under action by H.*

A *semicharacter* on a semigroup S is a bounded complex-valued function τ satisfying

$$\tau \not\equiv 0 \text{ and } \tau(ab) = \tau(a)\tau(b) \text{ for all } a, b \in S.$$

Let (S, m) be a semigroup with a measure m. If $f, g \in L^1(S, m)$, we will write $h = f * g$ if and only if there exists an element $h \in L^1(S, m)$ such that, for every $k \in L^\infty(S, m)$, we have

$$\int_S k(x)h(x)\,dm(x) = \int_S\int_S f(x)g(y)k(xy)\,dm(x)\,dm(y).$$

The function h is referred to as the *convolution of f and g*, and a measure m on a semigroup will be called *admissible* if $f, g \in L^1(S, m)$ imply that $f*g \in L^1(S, m)$ and that $\|f*g\|_1 \leqslant \|f\|_1\|g\|_1$. A non-negative regular Borel measure m on S will be termed *quasi-invariant* if for every Borel set E in S and $x \in S$ we have $m(xE) \geqslant m(E)$. We note that the measures fdm and gdm are in $M(S)$, the space of bounded complex regular Borel measures on S if S is locally compact, and that the convolution defined here is that of the convolution of measures, i.e. for $\nu, \mu \in M(S)$,

$$(\mu * \nu)(f) = \int\int f(xy)\,\mu(dx)\,\nu(dy) \quad \text{for } f \in C_0(S).$$

We consider only the class of A-I semigroups for which S' is an idempotent arc. The techniques which we employ can be used in the case where S' is a unit thread, but the resulting measure is quasi-invariant and the methods developed in [6] and [7] are directly applicable.

2.1. Existence of a measure M for S. In this section we consider an A-I semigroup S for which S' is an idempotent arc. From Theorem 1.3 it can be seen that $S \setminus K$ is the union of orbits of the elements of $P_0 = P \setminus \{0\}$ (where P is an isomorphic pre-image of S') under action by H. For $x \in P_0$ we let $T_x\colon H \to L_x$ denote the multiplication map from the group at the identity onto the orbits $L_x = Hx$. The orbits themselves are compact groups and we use μ_x to denote the normalized Haar measure on the orbit L_x. The normalized invariant measure on H will be denoted by μ, and λ will denote Lebesgue measure on P_0. In addition, we use $S \setminus A$ to denote the complement of the set A in S, and we use $\chi_A(t)$ to denote the characteristic function of the set A.

PROPOSITION 2.1. *Let S be an A-I semigroup with S' an idempotent arc and define the function $J_E(x)\colon P_0 \to R$ for $E \subset S$ by*

$$J_E(x) = \mu_x(E \cap L_x).$$

If $M(E) = \int_{P_0} J_E(x)\,d\lambda(x)$, where λ is Lebesgue measure for the additive reals, then M is a measure on the Borel sets of $S \setminus K$.

Proof. The proof that M is a measure is a routine exercise once it has been shown that the integral of $J_E(x)$ exists. The easiest way to prove the existence of this integral is to define the measure in another way.

Let $\pi\colon H \times (0, 1] \to S \setminus K$ by $\pi(h, x) = hx$ and for $E \subset S \setminus K$ let

$$\begin{aligned} M'(E) &= (\mu \times \lambda)\big(\pi^{-1}(E)\big) = \int \chi_{\pi^{-1}(E)}(t)\,d(\mu\times\lambda)(t) \\ &= \int\int \chi_{\pi^{-1}(E)}(h, x)\,d\mu(h)\,d\lambda(x) \\ &= \int\Big(\int \chi_{\pi_x^{-1}(E)}(h)\,d\mu(h)\Big)\,d\lambda(x) \\ &= \int\Big(\mu\big(\pi_x^{-1}(E \cap L_x)\big)\Big)\,d\lambda(x). \end{aligned}$$

But $\pi_x^{-1} \equiv T_x^{-1}$ and since $L_x = H/T_x^{-1}(x)$, from the theorems in [3] on quotient space measures we have

$$\mu\big(\pi_x^{-1}(E \cap L_x)\big) = \mu_x(E \cap L_x).$$

Thus we infer that

$$M'(E) = \int_{P_0}\big(\mu_x(E \cap L_x)\big)\,d\lambda(x) = \int_{P_0} J_E(x)\,d\lambda(x) = M(E),$$

and hence $M(E)$ exists for each Borel set $E \subset S \setminus K$.

Remark. There is no hope of finding a quasi-invariant measure on $S \setminus K$ since, for any interval $E = (a, b) \subset P_0$, we have $xE = \{x\}$ for any $x \in P_0$, $x < a$.

2.2. The admissibility of M. A careful look at the convolution of measures absolutely continuous with respect to a given measure shows that the following lemma is exactly what is required to make the convolution absolutely continuous with respect to the fixed measure:

LEMMA 2.2. *Let S be an A-I semigroup with S' an idempotent thread and M the measure of Proposition 2.1. Let $\chi_E(t) = 0$ a.e. (M) for $t \in S$. Then $\chi_E(s_0 t) = 0$ a.e. (M) as a function of t for fixed s_0 for almost all $s_0 \in S$.*

Proof. Since E is a set of measure zero in S, it follows from the definition of M that the set

$$Q = \{x\colon x \in P_0,\ \mu_x(L_x \cap E) \neq 0\}$$

must be a set of λ-measure zero and hence we can ignore any s_0 in the orbit of such an x. This leaves us with two cases to consider.

Case 1. Suppose that $s_0 < t$ and $t \in L_y$ (a fixed orbit) is such that $\chi_E(s_0 t) = 1$. If $A_y = \{t\colon t \in L_y,\ \chi_E(s_0 t) = 1\}$, then $\mu_y(A_y) = 0$.

For, let $s_0 = zh_1$, where $z \in P_0$ and $h_1 \in H$, and let $T_{ys_0}\colon L_y \to L_z$. We have $s_0 A_y \subseteq E \cap L_z$ since $\chi_E(s_0 t) = 1$ for $t \in A_y$, thus $A_y \subseteq T_{ys_0}^{-1}(L_z \cap E)$ and for simplicity we briefly adopt the notation $B_z = L_z \cap E$.

Then $zh_1 t \in B_z \Rightarrow zt \in h_1^{-1}(B_z)$, hence $A_y \subset T_{yz}^{-1}\big(h_1^{-1}(B_z)\big)$, where $T_{yz}\colon L_y \to L_z$. Now, however, $\mu_z(B_z) = \mu_z\big(h_1^{-1}(B_z)\big)$ by invariance of μ_z on L_z and since $T_z^{-1} = T_y^{-1} \circ T_{yz}^{-1}$, we have

$$\begin{aligned}\mu_z(B_z) &= \mu_z\big(h_1^{-1}(B_z)\big) = \mu\Big(T_z^{-1}\big(h_1^{-1}(B_z)\big)\Big)\\ &= \mu\Big(T_y^{-1} \circ T_{yz}^{-1}\big(h_1^{-1}(B_z)\big)\Big)\\ &\geqslant \mu\big(T_y^{-1}(A_y)\big) = \mu_y(A_y).\end{aligned}$$

But $\mu_z(B_z) = 0$, whence $\mu_y(A_y) = 0$.

Case 2. Suppose $t < s_0$ and $s_0 = z_1 h_1$. Note that if $t = z_2 h_2$, then $s_0 t = h_1 h_2 z_1 z_2 = h_1 h_2 z_2 = h_1 t$. Now $\chi_E(s_0 t) = \chi_E(h_1 t) = 1$ if and only if $t \in h_1^{-1}(E \cap L_{z_2})$, but $\mu_{z_2}(E \cap L_{z_2}) = 0$ by assumption so that by translation invariance we have $\mu_{z_2}\big(h_1^{-1}(E \cap L_{z_2})\big) = 0$. Hence the set of $t \in L_{z_2}$ for which $\chi_E(s_0 t) = 1$ is a set of measure zero.

PROPOSITION 2.3. *If S is an A-I semigroup with S' an idempotent thread and M the measure of Proposition 2.1, then M is admissible.*

Proof. (a) Suppose $f, g \in L^1(S, M)$ and E is a measurable set in $S \setminus K$. Define $\nu(E)$ as follows:

$$\begin{aligned}\nu(E) &= \int_S (f*g)(t)\chi_E(t)\,dM(t)\\ &= \int_S\int_S f(t)g(s)\chi_E(st)\,dM(s)\,dM(t).\end{aligned}$$

Now in particular suppose $M(E) = 0$, so that $\chi_E(t) = 0$ a.e. (M). By Lemma 2.2 the function $\chi_E(st) = 0$ a.e. (M) as a function of t for almost all s, whence

$$\nu(E) = \int g(s)\Big(\int f(t)\chi_E(st)\,dM(t)\Big)dM(s) = 0.$$

Moreover, by the Lebesgue Dominated Convergence Theorem and the corollary to Theorem 6 in Chapter II, Section 9 of [1], we infer that ν

is countably additive, and hence a measure. But then, since $\nu \ll M$, the Radon Nikodym Theorem implies the existence of a function $h \in L^1(S, M)$ with

$$\nu(E) = \int_S h(s)\chi_E(s)\,dM(s)$$

and we extend this to all simple functions. Thus for $k \in L^\infty(S, M)$ we have

$$\int\int (f*g)(t)k(t)\,dM(t) = \int h(t)k(t)\,dM(t),$$

and it is this h which we identify with $f*g$.

(b) To show that $\|f*g\|_1 \leqslant \|f\|_1\|g\|_1$, we embed L^1 isometrically into its second dual by means of the usual mapping $f \to f^{**}$ (where $f \in L^1$, $\varphi \in L^\infty$ imply $f^{**}(\varphi) = \varphi(f)$), and calculate

$$\sup_{\substack{k \in L^\infty\\ \|k\| \leqslant 1}} |\langle f*g, k\rangle|.$$

It is sufficient to consider elements k in L^∞ which are linear combinations of characteristic functions of sets of positive measure. But then

$$\begin{aligned}|\langle f*g, k\rangle| &= \Big|\int\int f(x)g(y)k(xy)\,dM(x)\,dM(y)\Big|\\ &\leqslant \int\int |f(x)|\,|g(y)k(xy)|\,dM(y)\,dM(x)\\ &\leqslant \int |f(x)|\Big[\int |g(y)k(xy)|\,dM(y)\Big]dM(x)\\ &\leqslant \|g\|_1\|k_x(y)\|_\infty \int |f(x)|\,dM(x)\\ &\leqslant \|g\|_1\cdot\|f\|_1\cdot\|k_x(y)\|_\infty.\end{aligned}$$

But for the class of functions $k \in L^\infty$ which we are considering it is true that

$$\|k_x(y)\|_\infty \leqslant \|k(y)\| \leqslant 1,$$

and hence

$$\|f*g\|_1 \leqslant \|f\|_1\|g\|_1.$$

Remark. In [4], where $S = S' =$ idempotent interval, the symbol $\langle a, b\rangle$ is used to denote the function on $S \times S$ defined for $a, b \in S$ by

$$\langle a, b\rangle = \begin{cases}1 & \text{if } a \leqslant b,\\ 0 & \text{if } a > b.\end{cases}$$

With this notation, if f is any function on S, it is true that

$$f(xy) = \langle x, y\rangle f(y) + \langle y, x\rangle f(x). \tag{3.2}$$

If λ denotes the Lebesgue measure of the additive reals and $f, g \in L^1(S, \lambda)$, it is shown in [4] that $f(*')g \in L^1$, where

$$\big(f(*')g\big)(x) = f(x)\int_a^x g(y)\,d\lambda(y) + g(x)\int_a^x f(y)\,d\lambda(y) \quad \text{a.e.}$$

LEMMA 2.4. *The convolution defined in this paper for S an A-I semigroup with S' an idempotent thread agrees with the convolution $(*')$ defined in* [4] *for the special case $H = \{1\}$.*

Proof. In this special case $M = \lambda$, so that the convolutions agree if

$$(3.3)\qquad \int\Big(f(x)\int_a^x g(y)\,d\lambda(y)+g(x)\int_a^x f(y)\,d\lambda(y)\Big)k(x)\,d\lambda(x) = \iint f(x)g(y)k(xy)\,d\lambda(x)\,d\lambda(y)$$

for an arbitrary $k\in L^\infty(S,\lambda)$; that is, $\langle f(*')g, k\rangle = \langle f*g, k\rangle$. However, by using (3.2) above we have

$$\iint f(x)g(y)k(xy)\,d\lambda(x)\,d\lambda(y) = \iint f(x)g(y)k(y)\langle x,y\rangle\,d\lambda(x)\,d\lambda(y)+\iint f(x)g(y)k(x)\langle y,x\rangle\,d\lambda(x)\,d\lambda(y).$$

But then we have

$$\iint f(x)g(y)k(x)\langle y,x\rangle\,d\lambda(x)\,d\lambda(y) = \int f(x)k(x)\Big(\int g(y)\langle y,x\rangle\,d\lambda(y)\Big)d\lambda(x) = \int_S f(x)k(x)\Big(\int_a^x g(y)\,d\lambda(y)\Big)d\lambda(x) = \text{1-st term of l.h.s. of (3.3)}.$$

Similarly, the term

$$\iint f(x)g(y)k(y)\langle x,y\rangle\,d\lambda(x)\,d\lambda(y)$$

can be shown to equal the remaining term in the l.h.s. of (3.3). Thus $\langle f(*')g, k\rangle = \langle f*g, k\rangle$ for all $k\in L^\infty$ and hence as functions on S, we have

$$f(*')g = f*g \quad \text{a.e. } (\lambda).$$

2.3. The measurable dual. We shall now discuss the bounded measurable semicharacters on S. Of crucial importance in what follows is the fact that if τ is measurable on (S, M), then the definition of M implies that $\tau|_{L_x}$ is measurable with respect to μ_x for almost all x. But L_x is a compact group, thus if $\tau|_{L_x}$ is measurable on L_x, it is also continuous on L_x. From this and the idempotent nature of P_0 it follows that if $\tau|_{L_{x_0}} \not\equiv 0$ for some $x_0\in P_0$, then $\tau|_{L_y}$ is continuous on L_y for all $y > x_0$ and, moreover, τ is a continuous function of $S\setminus Sx_0$.

THEOREM 2.5. *If S is an A-I semigroup for which S' is an idempotent interval and S^* denotes the bounded measurable semicharacters on S, then*

$$S^* = K^* \cap \{\bigcup_{x\in P_0}(\chi_{[x,1]}\times L_x^*)\}.$$

Proof. Since in the terminology of [8], K is a generating prime ideal we know that $S^* = K^*\cup S_0^*$, where $S_0^* = \{\tau\colon \tau\in S^*, \tau|_K\equiv 0\}$. By our preliminary remarks of this section, if $\tau\in S_0^*$, then $\tau|_{L_x}$ is continuous on almost every orbit L_x. Now $\tau(x) = 0$ or 1 for all $x\in P_0$. Thus if $\tau\not\equiv 0$, then there exists $y\in P_0$ such that $\tau(y) = 1$. By continuity on $S\setminus Sy$, and in particular on $P_0\setminus P_0y$, we have $\tau|_{P_0}\equiv 1$ for all $x > y$. Let x_0 denote the $\inf\{y\colon y\in P_0, \tau(y) = 1\}$; it follows that $\tau|_{L_y}$ is a non-zero element of $(L_y)^*$ for all $y > x_0$ and $\tau|_{L_y}\equiv 0$ for all $y < x_0$. By the continuity of τ on $S\setminus Sx_0$ we can define $\tau|_{L_{x_0}}\in(L_{x_0})^*$. From this and the fact that the non-zero values of τ on any orbit L_z uniquely determine the values of τ for all $s > z$, we can identify τ with a semicharacter of the form $\chi_{[x_0,1]}\times\varrho$, where $\varrho\in L_{x_0}^*$. Then for any $s_1 = h_1x_1\in S$, we have

$$(\chi_{[x_0,1]}\times\varrho)(s) = \begin{cases} 0 & \text{if } x_1 < x_0,\\ \varrho(x_0h_1) & \text{if } x_1\geqslant x_0.\end{cases}$$

Certainly every such function is a semicharacter which is continuous almost everywhere, whence measurable, so that

$$S_0^* = \bigcup_{x\in P_0}\{\chi_{[x,1]}\times(L_x)^*\}.$$

3.1. Maximal ideals and semicharacters. In this section we shall study the relationships between the measurable dual of semigroup S and $\mathrm{Hom}\big(L^1(S,M), C\big)$.

Definition 3.1. Let S be an A-I semigroup with S' an idempotent thread. If $f\in L^1(S,M)$ with $\mathrm{Supp} f\subseteq Sx_0$ and $t_0\in S$ with $t_0 > x_0$, then $f*\bar t_0$ is that element $h\in L^1(S,M)$ for which

$$\int f(u)k(t_0u)\,dM(u) = \int h(u)k(u)\,dM(u)$$

for all $k\in L^\infty(S,M)$.

LEMMA 3.2. *The function $f*\bar t_0$ defined above exists whenever $t_0\in S\setminus Sx_0$, where* $\mathrm{Supp} f\subseteq Sx_0$.

Proof. Given $t_0\in S$, $f\in L^1(S,M)$ and $k\in L^\infty(S)$ we can define the linear functional L_{f,t_0} on L^∞ as follows:

$$L_{f,t_0}(k) = \int f(u)k(t_0u)\,dM(u).$$

If, in particular, $k = \chi_E$ with $M(E) = 0$, then for $u < t_0$ we have $\chi_E(t_0u) = 0$ a.e. (M) by Lemma 2.2. On the other hand, since $\mathrm{Supp} f\subseteq Sx_0$, for $u > t_0$ we have $f(u) = 0$. Hence $L_{f,t_0}(\chi_E) = 0$ and again by the Radon-Nikodym theorem, as in Proposition 2.3, there exists a unique $h\in L^1(S,M)$ which we identify as $f*\bar t_0$.

LEMMA 3.3. *Let S be an A-I semigroup with S' an idempotent thread. If $s_0, t_0 \in S$ and $f \in L^1(S, M)$ with $\operatorname{Supp} f \subseteq Sx_0$, $x_0 < s_0$, and $x_0 < t_0$, then* $(f*\bar{s}_0)*(f*\bar{t}_0) = f*(f*\overline{s_0 t_0})$.

Proof. Let k be any element in $L^\infty(S, M)$. Then we have

$$\begin{aligned}
\langle (f*\bar{s}_0)*(f*\bar{t}_0), k\rangle &= \int \big((f*\bar{s}_0)*(f*\bar{t}_0)\big)(s)k(s)\,dM(s)\\
&= \int\int (f*\bar{s}_0)(s)(f*\bar{t}_0)(t)k(st)\,dM(s)\,dM(t)\\
&= \int (f*\bar{s}_0)(s)\Big(\int f(t)k(t_0st)\,dM(t)\Big)dM(s)\\
&= \int f(t)\Big(\int (f*\bar{s}_0)(s)k_{t_0t}(s)\,dM(s)\Big)dM(t)\\
&= \int f(t)\Big(\int f(s)k_{t_0t}(s_0s)\,dM(s)\Big)dM(t)\\
&= \int f(t)\Big(\int f(s)k_t(s_0t_0s)\,dM(s)\Big)dM(t)\\
&= \int f(t)\Big(\int (f*\overline{s_0t_0})(s)k_t(s)\,dM(s)\Big)dM(t)\\
&= \int\int f(t)(f*\overline{s_0t_0})(s)k(ts)\,dM(s)\,dM(t)\\
&= \int \big(f*(f*\overline{s_0t_0})\big)(s)k(s)\,dM(s)\\
&= \langle f*(f*\overline{s_0t_0}), k\rangle.
\end{aligned}$$

Hence we conclude that $(f*\bar{s}_0)*(f*\bar{t}_0) = f*(f*\overline{s_0t_0})$ a.e. (M).

PROPOSITION 3.4. *Suppose $f \in L^1(S, M)$, where S is an A-I semigroup with S' an idempotent thread. If $\operatorname{Supp} f \subseteq Sx_0$ for some $x_0 \in P_0$, and $s \in S\setminus K$ with $s > x_0$, and $\varphi \in \operatorname{Hom}(L^1(S, M), C)$ with $\varphi(f) = 1$, then the function τ defined on S by*

$$\tau(s) = \begin{cases} \varphi(f*\bar{s}) & \text{if } s > x_0,\\ 0 & \text{if } s < x_0\end{cases} .$$

is a semicharacter on $S\setminus L_{x_0}$.

Proof. From Lemma 3.3 for $s_0, t_0 > x_0$ we have $(f*\bar{s}_0)*(f*\bar{t}_0) = f*(f*\overline{s_0t_0})$; hence $\tau(s_0t_0) = \tau(s_0)\tau(t_0)$. If $s_0 < x_0$, then $\tau(s_0) = 0$ but $s_0t < x_0$ for all t; hence $\tau(s_0t) = 0 = \tau(s_0)\tau(t)$.

Remark. The case where $t \in P_0$ with $t > x_0$ is of special interest. In this case for any $k \in L^\infty$ we have

$$\begin{aligned}
\langle f*\bar{t}, k\rangle &= \int f(u)k(tu)\,dM(u)\\
&= \int_{Sx_0} f(u)k(tu)\,dM(u) + \int_{S\setminus Sx_0} f(u)k(tu)\,dM(u)\\
&= \int_{Sx_0} f(u)k(tu)\,dM(u) = \int_{Sx_0} f(u)k(u)\,dM(u)\\
&= \langle f, k\rangle.
\end{aligned}$$

Hence $f*\bar{t} = f$ a.e. (M) so that $\tau(t) = \varphi(f*\bar{t}) = \varphi(f) = 1$, that is,

$$\tau|_{P_0}(t) = \begin{cases} 0 & \text{if } t < x_0,\\ 1 & \text{if } t > x_0.\end{cases}$$

LEMMA 3.5. *Let $\varphi \in \operatorname{Hom}(L^1(S, M), C)$ and $f, g \in L^1(S, M)$ with $\varphi(f) = \varphi(g) = 1$, where $\operatorname{Supp} f \subseteq Sx_0$, $\operatorname{Supp} g \subseteq Sx_0$ and $s_0 > x_0$; then*

$$g*(f*\bar{s}_0) = f*(g*\bar{s}_0).$$

Proof. For any function $k \in L^\infty$, we have that

$$\begin{aligned}
\langle f*(g*\bar{s}_0), k\rangle &= \int \big(f*(g*\bar{s}_0)\big)(t)k(t)\,dM(t)\\
&= \int\int f(u)(g*\bar{s}_0)(t)k(tu)\,dM(t)\,dM(u)\\
&= \int f(u)\Big(\int g(t)k(s_0tu)\,dM(t)\Big)dM(u)\\
&= \int g(t)\Big(\int f(u)k(s_0tu)\,dM(u)\Big)dM(t)\\
&= \int g(t)\Big(\int (f*\bar{s}_0)(u)k(tu)\,dM(u)\Big)dM(t)\\
&= \int \big(g*(f*\bar{s}_0)\big)(u)k(u)\,dM(u)\\
&= \langle g*(f*\bar{s}_0), k\rangle.
\end{aligned}$$

LEMMA 3.6. *For a given fixed element $\varphi \in \operatorname{Hom}(L^1(S), C)$ the non-zero values in $S\setminus Sx_0$ of the semicharacter defined by Proposition 3.4 are independent of the choice of f for any f with $\operatorname{Supp} f \subset Sx_0$.*

Proof. Suppose that $\varphi \in \operatorname{Hom}(L^1(S), C)$ and that $f, h \in L^1(S, M)$ with $\varphi(f) = \varphi(h) = 1$ where $\operatorname{Supp} f \subseteq Sx_0$, $\operatorname{Supp} h \subseteq Sx_1$ for $x_0, x_1 \in P_0$ and $x_1 \leqslant x_0 < 1$. If $s_0 \in S$ and $s_0 > x_0$, then, from Lemma 3.5 we have $h*(f*\bar{s}_0) = f*(h*\bar{s}_0)$ and it follows that $\tau_h(s_0) = \tau_f(s_0)$. Hence the non-zero values of the semicharacters generated by the pairs $\{\varphi, f\}$ and $\{\varphi, h\}$ agree everywhere in $S\setminus Sx_0$.

Combining Proposition 3.4 and Lemma 3.6 we make the following definition:

Definition 3.7. Let S be an A-I semigroup with S' an idempotent thread. For $\varphi \in \operatorname{Hom}(L^1(S, M), C)$ let $x_0 = \sup\{x: x \in P_0, \varphi(f) = 0$ for all f with $\operatorname{Supp} f \subseteq Sx\}$. If $x_0 = 0$ (respectively 1), then take the *semicharacter associated with* φ to be the identically 1 (respectively 0) semicharacter. If $x_0 \in (0, 1)$, consider a net $\{x_\alpha\}$ in P_0 with $\{x_\alpha\}\downarrow x_0$ with corresponding functions $\{f_\alpha\}$ having support in Sx_α and $\varphi(f_\alpha) \neq 0$. Let $\lim_\alpha \tau_\alpha$ be the *semicharacter associated with* φ, where τ_α is the semicharacter of Proposition 3.4 and Lemma 3.6 for each f_α.

Having established above the existence of a map from the maximal ideals to the semicharacters, we now show that the Fourier transform is the inverse of this map.

LEMMA 3.8. *Let S be an A-I semigroup with S' an idempotent thread and φ a non-trivial element of* $\mathrm{Hom}(L^1(S), C)$. *Then the semicharacter τ associated with φ is measurable on $S\setminus K$.*

Proof. The proof is based on the one used for Theorem 3.3, [6]. However, slight modifications are necessary, since the measure M is not quasi-invariant.

Note first that the semicharacter τ associated with φ in the idempotent case is a monotone pointwise limit of the semicharacters $\{\tau_a\}$, and hence τ is certainly measurable if each τ_a is measurable. Also, while M is not quasi-invariant, it does satisfy the hypothesis of Theorem 2.3, [6], and hence $L^1(S, M)$ is a subalgebra of $M(S)$. This implies that the semicharacter τ_a associated with φ for any one fixed pair $\{f_a, x_a\}$ agrees for all $x \in S\setminus Sx_a$ with the τ defined in Theorem 3.3 using the measure μ_a corresponding to f_a. To simplify the notation, we let $S_0 = S\setminus Sx_0$, $x_0 \in P_0$.

Suppose that $\varphi \in \mathrm{Hom}(L^1(S, M), C)$, $x_0 = \sup\{x\colon\ x \in P_0, \varphi(f) = 0$ if $\mathrm{Supp}\, f \subset Sx\}$, and $x_a > x_0$, $x_a \in P_0$. Let f_a be any function such that $\mathrm{Supp}\, f_a \subset Sx_a$ and $\varphi(f_a) = 1$; hence $\tau_a(x) = \varphi(f_a * \bar{x})$ for $x \in S\setminus Sx_a$. Recall that since $\varphi \in (L^1)^*$, there exists a function $k \in L^\infty(S, M)$ such that

$$\varphi(f) = \int_S f(t)k(t)\,dM(t) \quad \text{for all } f \in L^1.$$

In particular, since $\varphi(f_a) = \varphi(f) = \varphi(f\chi_{S_0})$, we have

$$\varphi(f_a) = \int_{S_0} f_a(t)k(t)\,dM(t).$$

There exists a measure $|\mu_a| \in M(S)$ corresponding to f_a; namely

$$|\mu_a|(E) = \int \chi_E(t)\,|f_a(t)|\,dM(t).$$

Now suppose $S_0\setminus E$ has $|\mu_a|$-measure zero; thus

$$\int \chi_{S_0\setminus E}(t)\,|f_a(t)|\,dM(t) = 0.$$

Then

$$\begin{aligned}
\tau_a(x) = \varphi(f_a * \bar{x}) &= \int_{S_0} (f_a * \bar{x})(t)k(t)\,dM(t)\\
&= \int_{S_0} f_a(t)k(xt)\,dM(t)\\
&= \int_{S_0} \chi_{S_0\setminus E}(t)f_a(t)k(xt)\,dM(t) +\\
&\quad + \int_{S_0} \chi_E(t)f_a(t)k(xt)\,dM(t)\\
&= \int_{S_0} \chi_E(t)f_a(t)k(xt)\,dM(t)
\end{aligned}$$

for all $x \in S_0\setminus A$, where $M(A) = 0$. (The final equality follows since

$$\begin{aligned}
\int \chi_{S_0\setminus E}(t)f_a(t)k(xt)\,dM(t) &\leqslant \|k_x(t)\|_\infty \int \chi_{S_0\setminus E}(t)\,|f_a(t)|\,dM(t)\\
&\leqslant \|k_x(t)\|\,|\mu_a|(S_0\setminus E)
\end{aligned}$$

and $\|k_x(t)\|_\infty < \infty$ for almost all x; that is, for all $x \notin A$, where $M(A) = 0$.)

Let F be a compact subset of S_0 with $M(F) > 0$. The function $g\colon (x, y) \to \chi_F(x)\chi_E(y)k(xy)$ is Borel measurable on $S_0 \times S_0$ and vanishes outside $F \times E$ a σ-compact set. Moreover,

$$\begin{aligned}
&\Big|\int_{S_0}\int_{S_0} \chi_F(x)\chi_E(y)k(xy)\,dM(x)\,dM(y)\Big|\\
&\qquad \leqslant \iint \chi_F(x)\chi_E(y)\,|k(xy)|\,|f_a(y)|\,dM(y)\,dM(x)\\
&\qquad \leqslant \|k_x\|_\infty \|f_a\|_1 M(F) < \infty \quad \text{for all } x \notin A
\end{aligned}$$

and hence the function g is contained in $L^1(S_0 \times S_0, M \times |f_a|\,dM)$. This implies that the function

$$\begin{aligned}
x \to \int_{S_0} \chi_F(x)\chi_E(y)k(xy)f_a(y)\,dM(y) &= \chi_F(x)\int \chi_E(y)k(xy)f_a(y)\,dM(y)\\
&= \chi_F(x)\tau_a(x)
\end{aligned}$$

is M-measurable on S_0. But then τ_a is measurable for each compact $F \subset S_0$, hence τ_a is M-measurable on S_0. But $\tau_a \equiv 0$ on $Sx_a\setminus L_{x_a}$ and hence τ_a is M-measurable on $S\setminus K$.

THEOREM 3.9. *Let S be an A-I semigroup with S' an idempotent thread. Then there exists a one-to-one onto mapping between* $\mathrm{Hom}(L^1(S, M), C)$ *and $S^*\setminus K^*$.*

Proof. Given $\varphi \in \mathrm{Hom}(L^1(S, M), C)$, let $x_0 = \sup\{x\colon\ x \in P_0, \varphi(f) = 0$ for all f with $\mathrm{Supp}\, f \subset Sx\}$. Choose a sequence $\{x_n\}\downarrow x_0$ and, for each x_n, let f_n be any function such that $\varphi(f_n) \neq 0$ and $\mathrm{Supp}\, f_n \subset Sx_n$. Normalize $\varphi(f_n)$ and define $\tau_n(s) = \varphi(f_n * \bar{s})$ for all $s > x_n$; then the semicharacter associated with φ is the pointwise limit of these τ_n, and we write $\tau_\varphi = \lim_n \tau_n$. Now, for any $f \in L^1(S, M)$ with $\mathrm{Supp}\, f \subset \mathrm{Supp}\, \tau_n$ for some n, we define the homomorphism θ_n by

$$\theta_n(f) = \int \tau_n(s)f(s)\,dM(s) = \int \varphi(f_n * \bar{s})f(s)\,dM(s),$$

the integrals existing by virtue of Lemma 3.8. Since $\varphi \in (L^1)^*$, there exists a unique $k \in L^\infty(S, M)$ such that $\varphi(h) = \int h(s)k(s)\,dM(s)$ for any $h \in L^1(S, M)$. Since $f_n * \bar{s}$ is in L^1 for every $s \in \mathrm{Supp}\, f$, we have

$$\begin{aligned}\theta_n(f) &= \iint (f_n * \bar{s})(t)\, k(t) f(s)\, dM(t)\, dM(s)\\ &= \iint f_n(t) f(s)\, k(st)\, dM(s)\, dM(t)\\ &= \int (f_n * f)(u)\, k(u)\, dM(u)\\ &= \varphi(f_n * f) = \varphi(f).\end{aligned}$$

Thus we have the relation

$$(*) \qquad \theta_n(f) = \varphi(f) \quad \text{whenever } \operatorname{Supp} f \subset \operatorname{Supp} \tau_n.$$

But for arbitrary $h \in L^1(S)$ and $x \in P_0$ we have,

$$h = h\chi_{S_x} + h\chi_{S \setminus S_x}$$

and hence

$$\varphi(h) = \varphi(h\chi_{S_{x_n}}) + \varphi(h\chi_{S \setminus S_{x_n}}).$$

By the continuity of φ and the fact that $\varphi(h\chi_{S_x}) \equiv 0$ for all $x < x_0$, we conclude that

$$(1) \qquad \varphi(h) = \varphi(h\chi_{S \setminus S_{x_0}}) = \lim_n \varphi(h\chi_{S \setminus S_{x_n}}).$$

Moreover, since $\theta_n(h\chi_{S_{x_0}}) = 0$ for every h and all n, we can define $\theta(h)$ for arbitrary $h \in L^1(S, M)$ by

$$(2) \qquad \theta(h) = \theta(h\chi_{S \setminus S_{x_0}}) + \theta(h\chi_{S_{x_0}}) = \theta(h\chi_{S \setminus S_{x_0}}) = \lim_n \theta_n(h\chi_{S \setminus S_{x_n}}).$$

We call θ the homomorphism associated with the semicharacter τ_φ.

Finally, we know by $(*)$ that $\theta_n(h\chi_{S \setminus S_{x_n}}) = \varphi(h\chi_{S \setminus S_{x_n}})$. Combining this with (1) and (2), we have

$$\varphi(h) = \lim \varphi(h\chi_{S \setminus S_{x_n}}) = \lim_n \theta_n(h\chi_{S \setminus S_{x_n}}) = \theta(h).$$

Hence, for $\varphi \in \operatorname{Hom}(L^1(S, M), C)$, there is a unique semicharacter τ_φ associated with φ. Moreover, the unique homomorphism θ associated with τ_φ agrees with φ, so the mapping from $\operatorname{Hom}(L^1(S, M), C)$ to $S^* \setminus K^*$ is one-to-one and onto.

3.2. The semisimplicity of $L^1(S, M)$.

PROPOSITION 3.10. *Let I denote the idempotent unit interval with Lebesgue measure λ and let H be a compact abelian group with Haar measure μ. Then $L^1(H \times I, \mu \times \lambda)$ is a semisimple algebra.*

Proof. By an argument similar to that used to establish the nature of S^*, one can show that the measurable dual of $H \times I$ is isomorphic to $H^* \times I^*$ in the sense that $\tau \in (H \times I)^*$ implies $\tau(h, x) = (\chi_{[r,1]}\varrho)(h, x) = \chi_{[r,1]}(x)\varrho(h)$ for some $\varrho \in H^*$ and some r with $0 < r < 1$.

Now suppose that for all $\tau \in (H \times I)^*$ and some $f \in L^1(H \times I)$ we have

$$\int_{H \times I} \tau(h, x) f(h, x)\, d(\mu \times \lambda) = 0.$$

Let I_h denote the cross section $I \times \{h\}$, $h \in H$, and define the function $g_1(h)$ on H by

$$g_1(h_0) = \int_{I_{h_0}} f(h_0, x) \chi_{r_1}(x)\, d\lambda(x),$$

where $\chi_{r_1} = \chi_{[r_1, 1]}(x)$ for some rational r_1, $0 < r_1 < 1$. For arbitrary $\varrho \in H^*$, let $\tau = \chi_r \cdot \varrho$; it follows that

$$\begin{aligned}\int_{H \times I} \tau_1(h, x) f(h, x)\, d(\mu \times \lambda) &= \int_H \varrho(h) \Big(\int_{I_h} \chi_{r_1}(x) f(h, x)\, d\lambda(x) \Big) d\mu(h)\\ &= \int_H \varrho(h) g_1(h)\, d\mu(h) = 0.\end{aligned}$$

Since this is true for all $\varrho \in H^*$, then $g_1(h) = 0$ a.e. (μ) as a function on H; i.e., there exists a set $A_1 \subset H$ such that $g_1|_{A_1} \equiv 0$ and $\mu(H \setminus A_1) = 0$. Now let $\{r_n\}$ be the set of all rationals in $[0, 1]$ and let $\{A_n, g_n\}$ be as above and let $A = \bigcap_{n=1}^{\infty} A_n$. For all n and any $h_0 \in A$ we have

$$g_n(h_0) = \int_{I_{h_0}} f(h_0, x) \chi_{r_n}(x)\, d\lambda(x) = 0.$$

Now consider $f(h_0, x)$ as a function in $L^1(I)$ for fixed $h_0 \in A$. Since $L^1(I)$ is semisimple [4] and the set $\{\chi_{r_n}\}$ is dense in I^*, we have $f(h_0, x) = 0$ a.e. (λ) (as a function of x). Thus $f(h, x) = 0$ a.e. (λ) on I_h for almost all $h \in H$ and hence $f(h, x) = 0$ a.e. $(\mu \times \lambda)$. Therefore $L^1(H \times I, \mu \times \lambda)$ is semisimple.

The proof that $L^1(S, M)$ is semisimple if S is an A-I semigroup with S' an idempotent thread follows from Proposition 3.10 and the existence of a norm-preserving homomorphic embedding of $L^1(S, M)$ as an ideal into $L^1(H \times I, \mu \times \lambda)$. As a first step we define the map $\varphi\colon H \times I \to S$ by $\varphi(h, x) = hx$ and use this map to define the map Φ, $\Phi\colon L^1(S) \to L^1(H \times I)$, where

$$\Phi f(h, x) = (f \cdot \varphi)(h, x) = f(hx) \qquad \text{for } f \in L^1(S, M).$$

LEMMA 3.11. *If $\varphi\colon H \times I \to S$ is as defined above and $M(S)$, $M(H \times I)$ denote the measure algebras for S and $H \times I$ respectively, then the map $Q\colon M(H \times I) \to M(S)$ defined for $m \in M(H \times I)$ and $g \in C_0(S)$ by $\langle Q(m), g\rangle = \int_{H \times I} (g \circ \varphi)\, dm$ is a homomorphism.*

Proof. Let $\nu, \mu \in M(H\times I)$ and $f \in C_0(S)$. Then we have

$$\begin{aligned}\langle (Q(\nu)*Q(\mu)), f\rangle &= \int f d(Q(\mu)*Q(\nu)) \\ &= \iint f(st)\, d(Q\mu_s)\, d(Q\nu_t) \\ &= \int_S \int_{H\times I} (f_t\circ\varphi)(a)\, d\mu_a\, d(Q\nu_t) \\ &= \int_{H\times I} \Big(\int_S f_{\varphi(a)}(b)\, d(Q\nu_b)\Big) d\mu_a \\ &= \int_{H\times I}\int_{H\times I} f_{\varphi(a)}(\varphi(b))\, d\nu_b\, d\mu_a \\ &= \iint f(\varphi(a)\varphi(b))\, d\mu(a)\, d\nu(b) \\ &= \iint (f\circ\varphi)(ab)\, d\mu(a)\, d\nu(b) \\ &= (f\circ\varphi)(y)\, d(\mu*\nu)(y) = \langle Q(\mu*\nu), f\rangle .\end{aligned}$$

PROPOSITION 3.12. *Let Φ: $L^1(S)\to L^1(H\times I)$ be as above. Then Φ is a norm-preserving homomorphism whose range is an ideal.*

Proof. (i) Consider the diagram

$$\begin{array}{ccc} M(S) & \xleftarrow{Q} & M(H\times I) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ L^1(S) & \xrightarrow{\Phi} & L^1(H\times I) \end{array}$$

where i and j are the ordinary injection maps and Q is the mapping of Lemma 3.11. The statement that Φ is a homomorphism follows from the fact that i, j, and Q are homomorphisms, once it is shown that for any $f \in L^1(S, M)$,

$$j(f) = Q\big(i(\Phi(f))\big).$$

But, for any $g \in C_0(S)$, we have

$$\begin{aligned}\langle j(f), g\rangle &= \int (f\cdot g)(s)\, dM(s) \\ &= \int_{H\times I} (f\cdot g\circ\varphi)(t)\, d(\lambda\times\mu)(t) \\ &= \int_{H\times I} (g\circ\varphi)(t)(f\circ\varphi)(t)\, d(\lambda\times\mu)(t).\end{aligned}$$

Since $f\circ\varphi = \Phi f$ and $\Phi(f)(t)\, d(\lambda\times\mu)(t) = i(\Phi(f))$, we have

$$\langle j(f), g\rangle = \int (g\circ\varphi)(t)\, d\big(i(\Phi(f))\big) = \Big\langle Q\big(i(\Phi(f))\big), g\Big\rangle .$$

Moreover, we conclude that Φ is norm preserving, since

$$\|f\|_{L^1(S)} = \int_S |f|\, dm = \int_{H\times I} |f\circ\varphi|\, d(\mu\times\lambda) = \|f\circ\varphi\|_{L^1(H\times I)} .$$

(ii) To prove that the image of $L^1(S)$ is actually an ideal in $L^1(H\times I)$, consider $\Phi(L^1(S))$ as a subset of $L^1(H\times I) \subset M(H\times I)$ and let $\alpha \in L^1(S)$, $\beta \in L^1(H\times I)$, and $M(E) = 0$ for $E\subset S$. Then we have

$$\begin{aligned}\big((\alpha\circ\varphi)*\beta\big)(\varphi^{-1}(E)) &= \int \chi_{\varphi^{-1}(E)}(t)\, d\big((\alpha\circ\varphi)*\beta\big)(t) \\ &= \iint \chi_{\varphi^{-1}(E)}(xy)\, d(\alpha\circ\varphi)_x\, d(\beta)_y \\ &= \int (\alpha\circ\varphi)\big(y^{-1}\varphi^{-1}(E)\big)\, d(\beta)_y .\end{aligned}$$

But (as in the Lemma on admissibility) since $M(E) = 0$, the set $y^{-1}\varphi^{-1}(E)$ has $\alpha\circ\varphi$ measure zero for almost all $y(\beta)$. Hence $\int (\alpha\circ\varphi)\times\times\big(y^{-1}\varphi^{-1}(E)\big)\, d\beta_y = 0$ and

$$(\alpha\circ\varphi)*\beta \in \Phi\big(L^1(S, M)\big).$$

Now, combining Lemma 3.11 with Propositions 3.10 and 3.12 we have

THEOREM 3.13. *Let S be an A-I semigroup for which S' is an idempotent thread.*

Then $L^1(S, M)$ is semisimple.

Remark. In any A-I semigroup for which S' is an idempotent thread, $x^2 = y^2 = xy$ implies that $x = y$.

4.1. Remarks. All the results here depend on the form of $S\setminus K$ *via* the structure Theorem 1.3. It is clear that no other property of algebraically irreducible semigroups is being used and that nothing would be lost by assuming that the semigroup has a structure as given in 1.3. In fact, it is easily seen that if S is a compact abelian semigroup which is linearly quasi-ordered [7] and S' is an idempotent semigroup and if S is a disjoint union of locally compact subsemigroups S_α each of which is of the form $H_\alpha P_\alpha$, where H_α is the maximal group at the identity of S_α and P_α is an idempotent semigroup in S_α topologically and algebraically isomorphic to S'_α, then using the techniques of [6] and [7] all the results of this paper hold for such an S. That is, that $L^1(S, M)$ is semisimple and the M-measurable semicharacters on S are in one-to-one correspondence with the multiplicative linear functionals on $L^1(S, M)$.

Then, there also follows a general theorem for linearly quasi-ordered compact abelian semigroups; that, for such an S with $\varphi^{-1}(ES')$ being as in the preceeding paragraph, there is a measure m on S such that $L^1(S, m)$ is semisimple if and only if $x^2 = y^2 = xy$ implies $x = y$. Further, the m-measurable semicharacters on S are in one-to-one correspondence with the multiplicative linear functionals on $L^1(S, m)$.

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On Abel summability of multiple Laguerre series

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INTRODUCTION

The purpose of the present paper is to extend the results in [1] concerning Abel Summability of Multiple Hermite Series to the case of Multiple Laguerre Series. The 1-dimensional case has been studied in [3]-[7]. The novelty of our method in the 1-dimensional case is the statement of weighted maximal theorems.

1. NOTATION AND DEFINITIONS

1.1. $L^p_{e,m}(a)$ denotes the family of Lebesgue measurable functions defined on $R^m_+ = R_+ \times \ldots \times R_+$ such that

$$(1.1.1)\qquad \int\limits_{R^m_+} |f|^p e^{-\sum\limits_1^m x_j} \prod_{j=1}^m x_j^{a_j}\, dx_1 \ldots dx_m = \int\limits_{R^m_+} |f|^p e^{-X} X^a dX < \infty,$$

where $1 \leqslant p < \infty$ and the a_j $(j = 1, \ldots, m)$ are such that $-\frac{1}{2} < a_j < +\infty$. The $L^p_{e,m}(a)$-norm is defined in the following way:

$$(1.1.2)\qquad \|f\|_p(e, a) \overset{\text{def}}{=} \Big(\int\limits_{R^m_+} |f|^p e^{-X} X^a dX\Big)^{1/p}, \quad 1 \leqslant p < \infty.$$

1.2. $\tilde{L}^{(a)}_{(n)}(X)$ denotes a family of m-dimensional polynomials defined as follows:

Let $n = (n_1, \ldots, n_m)$, where each n_j $(j = 1, \ldots, m)$ is a non-negative integer, and let $a = (a_1, \ldots, a_m)$, where each a_j $(j = 1, \ldots, m)$ is a real parameter such that $-\frac{1}{2} < a_j < \infty$ (see footnote (1)). Now

$$(1.2.1)\qquad \tilde{L}^{(a)}_{(n)}(X) \overset{\text{def}}{=} \prod_{j=1}^m \Gamma(n_j+1)^{1/2} \Gamma^{-1/2}(n_j+a_j+1) L^{(a_j)}_{n_j}(x_j).$$