Inequality (6.3.1) can be readily verified taking into account that it is valid for μ absolutely continuous and considering that there exists a sequence μ_n of such measures converging weakly to μ.

From lemma 6.1 it follows that $\mu(r, X^2) \to 0$ a.e. on each Q_M, $M > 0$. This ends the proof of Theorem 2.

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STUDIA MATHEMATICA, T. XXXIII. (1969)

The comparison of an unconditionally converging operator *

by

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1. Preliminaries. In [3] A. Pełczyński shows that every weakly compact operator is an unconditionally converging operator. In the following we show that if an operator is strictly singular, almost weakly compact, or completely continuous (not the same as compact), then the operator is unconditionally converging; but not conversely.

Our notation and terminology will follow rather closely that used in [1]. Two common abbreviations used are uc for unconditionally converging or unconditionally convergent and wuc for weakly unconditionally convergent. All spaces are to be Banach spaces and all operators are to be linear and continuous. A linear operator $T\colon X \to Y$ is said to be *weakly compact* if it maps bounded sets in X into weakly sequentially compact sets.

Definition 1.1. (a) A series $\sum_n x_n$ of elements from a Banach space X is *uc* if for every bounded real sequence $\{t_n\}$ the series $\sum_n t_n x_n$ is convergent.

(b) A series $\sum_n x_n$ is *wuc* if for every real sequence $\{t_n\}$ with $\lim_n t_n = 0$ the series $\sum_n t_n x_n$ is convergent.

Definition 1.2. Let X and Y be Banach spaces. A linear operator $T\colon X \to Y$ is said to be *unconditionally converging* (*uc operator*) if it sends every wuc series in X into uc series in Y.

LEMMA 1.3. *Let $T\colon X \to Y$. Then T is a uc operator if and only if T has no bounded inverse on a subspace E of X isomorphic to c_0.*

Proof. Assume T is not a uc operator. Then T has a bounded inverse on a subspace isomorphic to c_0 by Lemma 1 of [4].

The converse implication is an obvious consequence of the fact that in the space c_0 the series consisting of unit vectors $e_n = (0, 0, \ldots, 1, 0, \ldots)$ is wuc but not uc.

* This paper is taken from Chapter II of the author's doctoral dissertation and was done while the author was a PSL Fellow at New Mexico State University.

Definition 1.4. A linear operator $T: X \to Y$ is said to be *strictly singular* if it does not have a bounded inverse on any infinite-dimensional subspace contained in X.

PROPOSITION 1.5. *Let $T: X \to Y$. Then if T is a strictly singular operator, T is also a uc operator.*

Proof. Assume T is not a uc operator. Then by Lemma 1.3, T has a bounded inverse on a subspace E of X isomorphic to c_0. Therefore T is not strictly singular since E is infinite-dimensional.

Definition 1.6. A linear operator $T: X \to Y$ is said to be *almost weakly compact* if, whenever T has a bounded inverse on a closed subspace M of X, then M is reflexive.

PROPOSITION 1.7. *Let $T: X \to Y$. Then if T is an almost weakly compact operator, T is also a uc operator.*

Proof. Assume T is not a uc operator: Then by Lemma 1.3, T has a bounded inverse on a subspace E of X isomorphic to c_0. Therefore T is not almost weakly compact since E is not reflexive.

Definition 1.8. A linear operator $T: X \to Y$ is said to be *completely continuous* if it maps weak Cauchy sequences in X into norm convergent sequences in Y.

Remark. An equivalent condition is: A linear operator $T: X \to Y$ is a *completely continuous operator* if $\lim_n \|Tx_n\| \to 0$ for every sequence $\{x_n\}$ in X which converges weakly to 0.

PROPOSITION 1.9. *Let $T: X \to Y$. Then if T is a completely continuous operator, T is also a uc operator.*

Proof. Assume T is not a uc operator. Then by Lemma 1.3, T has a bounded inverse on a subspace E of X isomorphic to c_0. Let $a_1, a_2, \ldots$ be the elements of E which correspond to the unit vectors of c_0 under the isomorphism. Then $\{a_n\}$ converges weakly to 0.

Now assume T is a completely continuous operator. Then $\{T(a_n)\}$ converges in norm to $T(0) = 0$ since $\{a_n\}$ converges weakly to 0. Now T^{-1} is continuous on $T(E)$, therefore $\{T^{-1}(T(a_n))\} = \{a_n\}$ converges in norm to 0. This is a contradiction since the unit vectors of c_0 do not converge in norm. Therefore T is not a completely continuous operator.

Example 1.10. If T is a uc operator, then T is not necessarily an almost weakly compact or a weakly compact operator.

Proof. Let I be the identity operator on l_1. Clearly I is a uc operator but not an almost weakly compact operator. Since I is not almost weakly compact, it cannot be weakly compact for every weakly compact operator is almost weakly compact.

Example 1.11. If T is a uc operator, then T, is not necessarily a strictly singular or a completely continuous operator.

Indeed, let I be the identity operator on an infinite-dimensional reflexive space. Since I is a weakly compact operator, it is a uc operator. But clearly I is not strictly singular. Also, I is not completely continuous, for it it were, it would be compact since every completely continuous operator with reflexive domain is compact.

2. Properties V and Dunford-Pettis. In [1] several spaces are proved to have the Dunford-Pettis property.

Definition 2.1. Let X be a Banach space. If for every Banach space Y, every weakly compact operator $T: X \to Y$ is completely continuous, then S is said to have the *Dunford-Pettis property*.

A property of similar nature was given by Pełczyński in [3]. It is the following:

Definition 2.2. If for every Banach space Y, every uc operator $T: X \to Y$ is a weakly compact operator, then X is said to have *property V*.

THEOREM 2.3. *Let X have properties V and Dunford-Pettis and let $T: X \to Y$. Then the following are equivalent:*

(a) *T is strictly singular.*

(b) *T is almost weakly compact.*

(c) *T is uc.*

(d) *T is weakly compact.*

(e) *T is completely continuous.*

Proof. (a) implies (b): This is clear from the definitions of strictly singular and almost weakly compact operators.

(b) implies (c): This follows from proposition 1.7.

(c) implies (d): X has property V.

(d) implies (a): X has the Dunford-Pettis property, hence if T is weakly compact, then T is strictly singular by proposition 4 (a) of [4].

Hence, (a), (b), (c), and (d) are all equivalent. The proof will be complete if we show (d) implies (e) and (e) implies (c).

(d) implies (e): X has the Dunford-Pettis property.

(e) implies (c): This follows from proposition 1.9.

Remark. Examples of spaces that have both property V and Dunford-Pettis property are: $B(S)$, $C(S)$, c, c_0, l_∞, and $L_\infty(S, \Sigma, \mu)$.

Definition 2.4. A Banach space X is *almost reflexive* if every bounded sequence in X contains a weak Cauchy subsequence.

COROLLARY 2.5. *Let $T: X \to Y$ and let X be almost reflexive with properties V and Dunford-Pettis. Then if T is a uc operator, T is a compact operator.*

Proof. Let T be a uc operator. By Theorem 2.3, T is a completely continuous operator. Now since X is almost reflexive, by Theorem 5 of [2], T is a compact operator.

Remark. Examples of almost reflexive spaces that have properties V and Dunford-Pettis are c, c_0, and $C(S)$, where S is a compact Hausdorff dispersed space [5]. Hence we see that any uc operator $T: c_0 \to Y$ is compact. Note that T is a uc operator and hence compact if Y contains no subspace isomorphic to c_0.

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Principal ideals which are maximal ideals in Banach algebras

by

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1. Introduction. Let A be a commutative semisimple Banach algebra with identity. If for some $f \in A$, the principal ideal Af is a maximal ideal, then in a natural way there is associated with each element $g \in A$, a formal power series $\sum_{n=0}^{\infty} a_n f^n$ with complex coefficients. Indeed, as shown in Theorem 3 below, if Af is not in the Šilov boundary Γ, then for each $g \in A$, the Gelfand transform $\hat{g}$ is given by the power series

$$\hat{g}(y) = \sum_{n=0}^{\infty} a_n \hat{f}^n(y)$$

for all y in the maximal ideal space satisfying $|\hat{f}(y)| < \min\{|\hat{f}(t)|: t \in \Gamma\}$.

The phenomenon of a principal ideal being a maximal ideal occurs in the familiar "disk algebra" consisting of the continuous complex-valued functions on the plane disk $\{z: |z| \leqslant 1\}$ which are analytic in the interior. The ideal Az is maximal, and each $g \in A$ has a power series expansion $\sum_{n=0}^{\infty} a_n z^n$ holding in the interior of the disk.

We wish to acknowledge the work of Phillip E. Parker(1) concerning the relation of the norm and Gelfand topologies on the maximal ideal space when Af is a maximal ideal, and present a result of his in section 3.

2. Let us suppose throughout this section that A is a sup norm function algebra on a compact Hausdorff space X. This means that A is a closed subalgebra of the algebra $C(X)$, that

(i) A separates points in X, and

(ii) $1 \in A$.

We wish also to impose the condition that

(iii) the maximal ideal space of A is X.

By saying that the maximal ideal space of A is X, we mean that

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