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On the area function of Lusin*

by

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Introduction. Let $f(z)$ be an analytic function of one complex variable defined on the upper half plane of the complex numbers with positive imaginary part and belonging to an H^p-class, $p > 0$. In 1965, Calderón [2] proved that the area function of Lusin

$$S_a(f)(x) = \Big[\int_{|x-u|\leqslant at} |f'(u+it)|^2 du dt\Big]^{1/2}$$

corresponding to $f(z)$ satisfies the inequalities

$$c_1 \int_{-\infty}^{+\infty} |f(x)|^p dx \leqslant \int_{-\infty}^{\infty} S_a(f)^p(x) dx \leqslant c_2 \int_{-\infty}^{+\infty} |f(x)|^p dx,$$

where c_1 and c_2 are two positive constants depending on a and p only. Partial results of this theorem were known. In the present paper we extend Calderón's theorem to the case of systems of conjugate harmonic functions; see [5] and [8]. The area function that we will use is essentially that given by E. M. Stein for harmonic functions of several variables in [7] and [6]. With the intention of avoiding innecessary repetitions of similar arguments, we present our results with the generality that the application which will be the subject of a second part to this paper will require.

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By E_n we shall denote the n-dimensional Euclidean space of the n-tuples $x = (x_1, \ldots, x_n)$ of real numbers. The ball in E_n with center at x_0 and radius r will be denoted by $B(x_0, r)$. We shall refer to the set

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$E_{n+1}^{+} = \{(t, x): t > 0, x \in E_n\} \subset E_{n+1}$ as the upper half space and its boundary will be identified with E_n. In some cases it will prove to be convenient to denote the point $(t, x_1, \ldots, x_n)$ by $(x_0, \ldots, x_n)$; consequently $\partial/\partial x_0$ and $\partial/\partial t$ will denote the same partial derivative. By $P_n(t, x)$ we shall denote the n-dimensional Poisson kernel:

$$P_n(t, x) = \Gamma(n+1/2)\pi^{-(n+1)/2} t(|x|^2+t^2)^{-(n+1)/2}.$$

The convolution of a function $f(x) \in L^p(E_n)$ with the Poisson kernel will be called the *Poisson Integral* of $f(x)$. Let $x \in E_n$ and $\alpha > 0$. For a function $B(t, x)$ defined on E_{n+1}^{+}, we shall denote by $m_\alpha(B)(x_0)$ the least upper bound of the values of the function over the cone $\Gamma_\alpha(x_0) = \{x: |x-x_0| < \alpha t\}$. We shall say that a set P contained in an open set $A \subset E_n$ is a *polar set in* A if there exists a super-harmonic function defined on A and such that it takes the value $+\infty$ at every point of P. Every subset of a polar set is a polar set and a countable union of polar sets is a polar set (for these and more on polar sets see [1]).

CHAPTER I

In this chapter we present some basic results of the theory of H^p-spaces which will be needed later in this paper. Let $B(t, x)$ be a real-valued function defined on E_{n+1}^{+} and such that it satisfies:

(1.1) The function $B(t, x)$ is non-negative, continuous and sub-harmonic on E_{n+1}^{+}.

(1.1') There exists a constant K such that for every $t > 0$ we have

$$\int_{E_n} B(t, x)^q dx \leqslant K^q,$$

where q is a real number greater than 1.

(1.2) PROPOSITION. *The function* $B(t, x)$ *satisfies the inequality*

$$B(t, x)^q \leqslant K^q t^{-n}$$

for every $t > 0$ *and* $x \in E_n$. *Moreover, if* $0 < \varepsilon < t < 1/\varepsilon$, $\varepsilon < 1$, *we have*

$$\lim_{|x|\to\infty} B(t, x) = 0$$

uniformly in t.

Proof. See [8], lemma (3.2), p. 37.

(1.3) PROPOSITION. *There exists a function* $f(x) \in L^q(E_n)$ *with norm less than or equal to* K *such that* $B(t, x) \leqslant U(t, x)$, *where* $U(t, x)$ *denotes the Poisson integral of* $f(x)$.

Proof. See [8], lemma (3.8), p. 40.

(1.4) PROPOSITION. *The function* $m_\alpha(B)(x)$ *belongs to* $L^q(E_n)$ *and its norm is less than or equal to a constant times* K. *The constant depends on* α, n *and* q *only*.

Proof. See [8], lemma (3.14), p. 42.

(1.5) PROPOSITION. *Let* $\Phi(t)$ *be the function defined as*

$$\Phi(t) = \Big[\int_{E_n} B(t, x)^q dx\Big]^{1/q}.$$

We have

(i) *The function* $\Phi(t)$ *is non-increasing and convex.*

(ii) *The limit of* $\Phi(t)$ *for* t *tending to infinity is equal to zero.*

Proof. See [8], Theorem C, p. 47.

(1.6) PROPOSITION. *Let us assume that the limit* $\lim_{t\to 0} B(t, x)$ *exists for almost every* $x \in E_n$ *and denote the limit by* $B(0, x)$. *We have*

$$\int_{E_n} B(0, x)^q dx = \lim_{t\to 0} \int_{E_n} B(t, x)^q dx \leqslant K^q.$$

Moreover, if for $q' > 1$ *and* $K' > 0$ *we have*

$$\int_{E_n} B(0, x)^{q'} dx \leqslant K'^{q'},$$

then the inequality

$$\int_{E_n} B(t, x)^{q'} dx \leqslant K'^{q'}$$

holds for every $t \geqslant 0$.

Proof. See [8], Theorem D, p. 49.

(1.7) We consider now a harmonic function $F(t, x)$ defined on E_{n+1}^{+} with values in a real Hilbert space $\mathscr{H}$ and satisfying:

(A) There exists a constant $K > 0$ such that

$$\int_{E_n} \|F(t, x)\|^p dx \leqslant K^p$$

for every $t > 0$. Here p denotes a positive number.

(B) For almost every $x \in E_n$ the limit

$$\lim_{t\to 0} F(t, x) = F(0, x)$$

exists.

(C) For every $(t, x) \in E_{n+1}^{+}$ we have

$$(\delta_0 - 2)\sum_0^n \Big\langle \frac{\partial F(t, x)}{\partial x_i}, F(t, x)\Big\rangle^2 + \|F(t, x)\|^2 \sum_0^n \Big\| \frac{\partial F(t, x)}{\partial x_i}\Big\|^2 \geqslant 0,$$

where δ_0 is a positive number less than p.

The following lemma shows that condition (C) is equivalent to the subharmonicity of the function $\|F(t,x)\|^\delta$, $\delta \geqslant \delta_0$, on E_{n+1}^+.

(1.8) LEMMA. *Let $F(x)$ be a harmonic function defined on an open set $D \subset E_n$ and with values in a real Hilbert space $\mathscr{H}$. The function $\|F(x)\|^\delta$ is subharmonic on D if and only if for every $x \in D$ the inequality*

$$(1.9)\qquad (\delta-2)\sum_1^n \left\langle \frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2 + \|F(x)\|^2 \sum_1^n \left\|\frac{\partial F(x)}{\partial x_i}\right\|^2 \geqslant 0$$

holds.

Proof. The function $\|F(x)\|^\delta$ is non-negative and continuous on D. Moreover, this function is infinitely differentiable at every point $x \in D$ where $F(x)$ is different from zero. Then, it suffices to show that the laplacian of $\|F(x)\|^\delta$ is non-negative at every point where the function $F(x)$ is distinct from zero. We have

$$\frac{\partial}{\partial x_i}(\|F(x)\|^\delta) = \frac{\partial}{\partial x_i}\langle F(x), F(x)\rangle^{\delta/2} = \delta\|F(x)\|^{\delta-2}\left\langle \frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle$$

and

$$\frac{\partial^2}{\partial x_i^2}(\|F(x)\|^\delta) = \delta\|F(x)\|^{\delta-4}\left[(\delta-2)\left\langle \frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2 + \right.$$
$$\left. + \|F(x)\|^2\left(\left\langle \frac{\partial^2 F(x)}{\partial x_i^2}, F(x)\right\rangle + \left\|\frac{\partial F(x)}{\partial x_i}\right\|^2\right)\right]$$

Hence, considering that $F(x)$ is a harmonic function, we obtain for the sum of the second partial derivatives the expression:

$$\sum_1^n \frac{\partial^2}{\partial x_i}(\|F(x)\|^\delta)$$
$$= \delta\|F(x)\|^{\delta-4}\left[(\delta-2)\sum_1^n\left\langle \frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2 + \|F(x)\|^2\sum_1^n\left\|\frac{\partial F(x)}{\partial x_i}\right\|^2\right]$$

and this expression is non-negative if and only if (1.9) holds.

(1.10) PROPOSITION. *Let $F(t,x)$ satisfy conditions* (A), (B) *and* (C) *of* (1.7). *Then we have*

$$\int_{E_n} \|F(0,x)\|^p dx \leqslant K^p \quad \text{and} \quad \lim_{t\to 0}\int_{E_n}\|F(t,x)-F(0,x)\|^p dx = 0.$$

Proof. By lemma (1.8) the function $B(t,x) = \|F(t,x)\|^{\delta_0}$ is subharmonic on E_{n+1}^+ and since F satisfies (A) it follows that $\int_{E_n} B(t,x)^r dx \leqslant K^p$, where $r = p/\delta_0 > 1$. This shows that we can apply proposition (1.4) to $B(t,x) = \|F(t,x)\|^{\delta_0}$ and therefore that $m_a(B)(x)$ belongs to $L^r(E_n)$.

From the definition of $m_a(B)(x)$ we get that $B(t,x) \leqslant m_a(B)(x)$, which implies that $\lim_{t\to 0} B(t,x) = B(0,x) \leqslant m_a(B)(x)$ and hence

$$\|F(t,x)-F(0,x)\|^p \leqslant (\|F(t,x)\| + \|F(0,x)\|)^p \leqslant 2^p m_a(B)^r(x).$$

This shows that the first member of the inequality is majorized by an integrable function for every $t > 0$. Then, since $\lim_{t\to 0}\|F(t,x)-F(0,x)\|^p$ is equal to zero for almost every $x \in E_n$, it follows from the Lebesgue's bounded convergence theorem that

$$\lim_{t\to 0}\int_{E_n}\|F(t,x)-F(0,x)\|^p dx = 0.$$

The inequality

$$\int_{E_n}\|F(0,x)\|^p dx \leqslant K^p$$

is a straightforward consequence of Fatou's lemma.

(1.11) PROPOSITION. *Let $F(t,x)$ satisfy conditions* (A), (B) *and* (C) *of* (1.7). *The function*

$$\Phi(t) = \left[\int_{E_n}\|F(t,x)\|^p dx\right]^{\delta_0/p}$$

is convex, non-increasing and $\lim_{t\to\infty}\Phi(t) = 0$.

Proof. As in the preceding proposition take $B(t,x)$ and apply proposition (1.5).

(1.12) PROPOSITION. *Let $F(t,x)$ satisfy conditions* (A), (B) *and* (C) *of* (1.7) *and let us suppose that for a number $p' > \delta_0$ and a constant K' the inequality*

$$\int_{E_n}\|F(0,x)\|^{p'} dx \leqslant K'^{p'}$$

holds. Then, for every $t > 0$ we have

$$\int_{E_n}\|F(t,x)\|^{p'} dx \leqslant K'^{p'}.$$

Proof. Apply proposition (1.6) to $B(t,x) = \|F(t,x)\|^{\delta_0}$.

CHAPTER II

THE GREEN'S FORMULA

Let us denote by $g_k(t,x)$, $k = 1, 2, \ldots$, the function

$$g_k(t,x) = k\sin(k^{-1}t)\cos(k^{-1}x_1)\ldots\cos(k^{-1}x_n)$$

and by V_k the set

$$V_k = \{(t,x): 0 \leqslant t \leqslant k\pi,\ |x_i| \leqslant k\pi/2,\ i = 1, \ldots, n\}.$$

The purpose of this chapter is to prove the following proposition which will play an essential role in the proof of theorem (4.2):

(2.1) PROPOSITION. *Let $F(t,x)$ be a harmonic function defined on E_{n+1}^+ with values in a real Hilbert space $\mathscr{H}$. We assume that $F(t,x)$ satisfies* (A), (C) *of* (1.7) *and*

$$\text{(2.2)} \qquad \text{(D) The set of zeros of } F(t,x) \text{ is a polar set in } E_{n+1}^+.$$

Let $h(x)$ be non-negative and continuous function with compact support defined on E_n. For $\varrho > 0$ and $\eta > 0$ we denote by $H(t,x)$ and $G(t,x)$ the functions

$$H(t,x) = \int_{E_n} P_n(t+\varrho, x-u)h(u)\,du \quad \text{and} \quad G(t,x) = F(t+\eta, x)$$

respectively. Then, for $\delta > \delta_0$ we have:

$$\text{(2.3)} \qquad \lim_{k\to\infty}\int_{V_k} g_k(t,x)\,\Delta\left(\|G(t,x)\|^\delta H(t,x)\right)dx\,dt = \int_{E_n}\|G(0,x)\|^\delta H(0,x)\,dx.$$

Moreover, if $\delta \geqslant p$,

$$\text{(2.4)} \qquad \int_0^\infty\int_{E_n} t\Delta\left(\|G(t,x)\|^\delta\right)dx\,dt = \int_{E_n}\|G(0,x)\|^\delta\,dx.$$

We begin with some preparatory lemmas.

(2.5) LEMMA. *Let $f(x)$ be a continuous and subharmonic function defined on an open set $D \subset E_n$. Let us assume that $f(x)$ is infinitely differentiable in the difference $D \sim P$, where P is a closed set in D with Lebesgue measure equal to zero. Then, the Laplacian of $f(x)$ is integrable on compact subsets contained in D.*

Proof. Let x_0 be a point of D and $B(x_0, r)$ a ball with center at x_0 and radius r such that $\bar{B}(x_0, r) \subset D$. We denote by $\Psi(x)$ an infinitely differentiable function defined on E_n which takes the values 1 and 0 for $x \in \bar{B}(x_0, r/4)$ and $x \in \complement B(x_0, r/2)$ respectively. Let $g(x)$ be the function $g(x) = \Psi(x)f(x)$. This function $g(x)$ is continuous on E_n, has a compact support and is infinitely differentiable on $E_n \sim P_1$, where P_1 is the set $P \cap \bar{B}(x_0, r/2)$.

Let $\Phi(x)$ be a non-negative, infinitely differentiable function with support contained in the unit ball $B(0,1)$ and such that $\int \Phi(x)\,dx = 1$. The sequence $\{g_m(x)\}$, m a positive integer, given by

$$g_m(x) = m^n \int_{E_n} g(y)\,\Phi\left(m(x-y)\right)dy$$

has the following properties:

(i) $\sup_{x\in E_n} |g_m(x)| \leqslant \sup_{x\in E_n} |g(x)|$.

(ii) For every m, the function $g_m(x)$ is infinitely differentiable and its support is contained in the ball $\bar{B}(x_0, r/2+1/m)$.

(iii) The sequence $\{g_m(x)\}$ converges uniformly to $g(x)$.

(iv) If $m > 8/r$, the Laplacian $\Delta g_m(x)$ is non-negative for $x \in B(x_0, r/8)$.

(v) The sequence $\Delta g_m(x)$ converges to $\Delta g(x)$ at every point $x \in E_n \sim P_1$.

(i) and (ii) follow immediately from the definition of $g_m(x)$. (iii) is a consequence of the uniform continuity of $g(x)$. Let us consider (iv): the Laplacian $\Delta g_m(x)$ will be non negative on $B(x_0, r/8)$ if and only if for every function $\varphi(x)$, infinitely differentiable and with support contained in $B(x_0, r/8)$, the integral $\int_{E_n}\varphi(x)\,\Delta g_m(x)\,dx$ is non-negative. We have

$$\int_{E_n}\varphi(x)\,\Delta g_m(x)\,dx = \int_{E_n} g(x)\left[\Delta\int m^n\Phi(my)\varphi(x+y)\,dy\right]dx,$$

where the function $\int m^n\Phi(my)\varphi(x+y)\,dy$ is non-negative, infinitely differentiable and with support contained in $\bar{B}(x_0, r/8+1/m) \subset \bar{B}(x_0, r/4)$. By definition of $g(x)$ we have $g(x) = f(x)$ for $x \in \bar{B}(x_0, r/4)$, therefore,

$$\int_{E_n}\varphi(x)\,\Delta g_m(x)\,dx = \int_{E_n} f(x)\left[\Delta\int m^n\Phi(my)\varphi(x+y)\,dy\right]dx$$

and since $f(x)$ is a subharmonic function on D, we see that the second member is non-negative.

Let us consider (v): Since P_1 is a closed set, we have $E_n \sim P_1$ open and therefore, if $x \in E_n \sim P_1$, there exists an $\varepsilon > 0$ such that $B(x,\varepsilon) \subset E_n \sim P_1$. Let $m > 2/\varepsilon$; then

$$\Delta g_m(x) = \int_{|x-y|\leqslant\varepsilon/2} g(y)\,\Delta m^n\Phi\left(m(x-y)\right)dy.$$

Since $g(y)$ is an infinitely differentiable function, the expression above can be written as

$$\Delta g_m(x) = \int_{|x-y|\leqslant\varepsilon/2} m^n\Phi\left(m(x-y)\right)\Delta g(y)\,dy$$

and now (v) follows from the continuity of $\Delta g(y)$ on $B(x,\varepsilon)$.

Let $\varrho(x)$ be a non-negative and infinitely differentiable function defined on E_n such that $\varrho(x) = 1$ for $x \in B(x_0, r/16)$ and $\varrho(x) = 0$ for $x \in B(x_0, r/8)$. From (iv) we obtain that if $m > 8/r$, then

$$\int_{B(x_0,r/16)}\Delta g_m(x)\,dx \leqslant \int_{B(x_0,r/8)}\varrho(x)\,\Delta g_m(x)\,dx = \int_{B(x_0,r/8)} g_m(x)\,\Delta\varrho(x)\,dx$$

and applying (i) we obtain

$$\int_{B(x,r/16)} \Delta g_m(x)\,dx \leqslant \sup_{x\in E_n}|g(x)|\cdot\sup_{x\in E_n}|\Delta\varrho(x)|\cdot|B(x_0,r/8)|\leqslant c,$$

where the constant c does not depend on m. Hence, by Fatou's lemma, we have

$$\int_{B(x_0,r/16)} \liminf \Delta g_m(x)\,dx \leqslant c,$$

but using (v) we see that $\liminf \Delta g_m(x) = \lim \Delta g_m(x) = \Delta g(x)$ for almost every point $x\in B(x_0, r/16)$ and therefore

$$\int_{B(x_0,r/16)} \Delta g(x)\,dx = \int_{B(x_0,r/16)} \Delta f(x)\,dx \leqslant c.$$

This shows that for every point $x_0\in D$ there is a neighborhood $B(x_0, r/16)$ of x_0 on which $\Delta f(x)$ is integrable. The integrability of $\Delta f(x)$ on compact sets contained in D follows immediately.

(2.6) LEMMA. *Let $F(x)$ be a harmonic function defined on an open set $D\subset E_n$ with values in a real Hilbert space $\mathscr{H}$ and satisfying* (C) *of* (1.7). *Then, for every $x\in D$ and $\delta>\delta_0/2$ we have*

$$|\text{gradient}(\|F(x)\|^\delta)|^2 \leqslant \tfrac{1}{2}\frac{\delta}{2\delta-\delta_0}\Delta(\|F(x)\|^{2\delta}).$$

Proof. The partial derivatives of $\|F(x)\|^\delta$ and the Laplacian of $\|F(x)\|^{2\delta}$ are given by:

$$\frac{\partial}{\partial x_i}(\|F(x)\|^\delta) = \delta\|F(x)\|^{\delta-2}\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle$$

and

$$\Delta(\|F(x)\|^{2\delta})$$
$$= 2\delta\|F(x)\|^{2\delta-4}\left[(2\delta-2)\sum_1^n\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2 + \|F(x)\|^2\sum_1^n\left\|\frac{\partial F(x)}{\partial x_i}\right\|^2\right].$$

Therefore, for the gradient of $\|F(x)\|^\delta$ we have

$$|\text{grad}(\|F(x)\|^\delta)|^2 = \delta^2\|F(x)\|^{2\delta-4}\sum_1^n\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2$$

and our thesis can be written as

$$\frac{\delta^2}{2\delta-\delta_0}\|F(x)\|^{2\delta-4}\left[(2\delta-2)\sum_1^n\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2+\|F(x)\|^2\sum_1^n\left\|\frac{\partial F(x)}{\partial x_i}\right\|^2\right]$$
$$\geqslant \delta^2\|F(x)\|^{2\delta-4}\sum_1^n\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2$$

or

$$\delta^2\|F(x)\|^{2\delta-4}\left[(\delta_0-2)\sum_1^n\left\langle\frac{\partial F(x)}{\partial x_i}, F(x)\right\rangle^2+\|F(x)\|^2\sum_1^n\left\|\frac{\partial F(x)}{\partial x_i}\right\|^2\right]\geqslant 0,$$

but from (C) of (1.7) the expression in brackets is non-negative and the lemma is proved.

(2.7) LEMMA. *Let $F(x)$ satisfy the same hypotheses of lemma* (2.4). *Then, for $\delta>\delta_0/2$, the functions $(\partial/\partial x_i)(\|F(x)\|^\delta)$, $i=1,\ldots,n$, and $\Delta(\|F(x)\|^{2\delta})$ are integrable on compact subsets of D.*

Proof. Let K be a compact subset of D. Integrating the inequality $|(\partial/\partial x_i)(\|F(x)\|^\delta)|\leqslant|\text{grad}(\|F(x)\|^\delta)|$ over K and applying Schwarz's inequality we have

$$\int_K\left|\frac{\partial}{\partial x_i}(\|F(x)\|^\delta)\right|dx\leqslant|K|^{1/2}\left[\int_K|\text{grad}(\|F(x)\|^\delta)|^2\,dx\right]^{1/2},$$

where, by lemma (2.4), the second member is less than or equal to

$$\left(\frac{1}{2}\frac{\delta}{2\delta-\delta_0}\right)^{1/2}|K|^{1/2}\left[\int_K\Delta(\|F(x)\|^{2\delta})\,dx\right]^{1/2} \tag{2.8}$$

and since the function $\|F(x)\|^{2\delta}$ is subharmonic, continuous and its set of zeros has Lebesgue measure equal to zero, lemma (2.5) applied to $f(x)=\|F(x)\|^{2\delta}$ shows that (2.8) is finite and the lemma is proved.

(2.9) LEMMA. *Let $F(x)$ be a harmonic function defined on an open set $D\subset E_n$ with values in a real Hilbert space $\mathscr{H}$ and satisfying* (C) *of* (1.7) *and* (D) (2.2). *If $\delta>\delta_0/2$ and $\varphi(x)$ is an infinitely differentiable function with compact support contained in D, we have*

$$\int_{E_n}\varphi(x)\frac{\partial}{\partial x_i}(\|F(x)\|^\delta)\,dx = -\int_{E_n}\|F(x)\|^\delta\frac{\partial\varphi}{\partial x_i}(x)\,dx$$

and

$$\int_{E_n}\varphi(x)\Delta(\|F(x)\|^{2\delta})\,dx = \int_{E_n}\|F(x)\|^{2\delta}\Delta\varphi(x)\,dx.$$

In other words, for $\delta>\delta_0/2$ the pointwise partial derivatives of $\|F(x)\|^\delta$ and the pointwise Laplacian of $\|F(x)\|^{2\delta}$ coincide with the corresponding derivatives and Laplacian in the sense of the theory of distributions.

Proof. Let x_0 be a point in D and $C(x_0, r)$ a cube with center at x_0 and semi-amplitude r such that $C(x_0,r)\subset D$. If $x=(x_1,\ldots,x_n)$, we denote by $\hat{x}$ the point $(x_2,\ldots,x_n)\in E_{n-1}$ and $\hat{C}(x_0,r)$ denotes the set of all $\hat{x}$ such that $x\in C(x_0,r)$. Lemma (2.7) and Fubini's theorem imply that for almost every $\hat{x}\in\hat{C}(x_0, r)$ the function $\varphi(s,\hat{x})\partial(\|F(s,\hat{x})\|^{\delta})/\partial x_1$ is integrable in the variable s on the segment $L(\hat{x}) = \{s\colon (s,\hat{x})\in C(x_0, r)\}$. Let us assume that $F(s,\hat{x})$ is not identically zero in $L(\hat{x})$ and that the support of $\varphi(x)$ is contained in $C(x_0, r)$. Then there exists $h\in\mathscr{H}$ such that the real-valued harmonic function $\langle h, F(y)\rangle$ has a non-zero restriction to the segment $L(\hat{x})$. This implies that $\langle h, F(s,\hat{x})\rangle$ is a non-zero real analytic function of s defined in a neighborhood of the closure of $L(\hat{x})$ and therefore that the set of zeros of $F(s,\hat{x})$ contained in $L(\hat{x})$ is finite. Let $s_1 < \ldots < s_m$ be the zeros of $F(s,\hat{x})$ belonging to $L(\hat{x})$. For almost every $\hat{x}\in\hat{C}(x_0, r)$ we have

$$\int_{-\infty}^{+\infty} \varphi(s,\hat{x})\frac{\partial}{\partial x_i}(\|F(s,\hat{x})\|^{\delta})\,ds = \lim_{\varepsilon\to 0}\Big[\int_{-\infty}^{s_1-\varepsilon} + \sum_{1}^{m-1}\int_{s_i+\varepsilon}^{s_{i+1}-\varepsilon} + \int_{s_m+1}^{+\infty}\Big]$$

$$= -\int_{-\infty}^{+\infty}\|F(s,\hat{x})\|^{\delta}\frac{\partial\varphi(s,\hat{x})}{\partial x_1}\,ds + \lim_{\varepsilon\to 0}\sum_{1}^{m}\{\|F(s_i-\varepsilon,\hat{x})\|^{\delta}\varphi(s_i-\varepsilon,\hat{x}) - \\ - \|F(s_i+\varepsilon,\hat{x})\|^{\delta}\varphi(s_i+\varepsilon,\hat{x})\}$$

$$= -\int_{-\infty}^{+\infty}\|F(s,x)\|^{\delta}\frac{\partial\varphi(s,\hat{x})}{\partial x_1}\,ds,$$

therefore, integrating in the variables $x_2, \ldots, x_n$ we obtain

$$\int_{E_n}\varphi(x)\frac{\partial}{\partial x_1}(\|F(x)\|^{\delta})\,dx = -\int_{E_n}\|F(x)\|^{\delta}\frac{\partial\varphi(x)}{\partial x_1}\,dx$$

and analogously for the other derivatives.

We consider now the case of the Laplacian. Since $\|F(x)\|^{2\delta}$ is a sub-harmonic function of D, the Riesz's representation theorem gives that for every relatively compact open set A such that $\bar{A}\subset D$,

$$-\|F(x)\|^{2\delta} = a_n\frac{1}{|x|^{n-2}}*\mu_A + S_A(x), \quad \text{where} \quad a_n = \frac{1}{n-2}\,\frac{\Gamma(n/2)}{2\pi^{n-2}},$$

$S_A(x)$ is a harmonic function on A and μ_A is the restriction to A of the positive measure μ on D defined by

$$\int_D\varphi(x)\,d\mu(x) = \int_D\|F(x)\|^{\delta}\Delta\varphi\,dx,$$

that is to say, μ is the Laplacian of $\|F(x)\|^{2\delta}$ in the sense of the theory of distributions. Let $\mu = g(x)\,dx + \bar{\mu}$ be the Lebesgue decomposition of the measure μ with respect to the Lebesgue measure restricted to A

and let P be the set of zeros of $F(x)$. If $\varphi(x)$ is an infinitely differentiable function with compact support contained in $D\sim P$, then, since the function $\|F(x)\|^{2\delta}$ is infinitely differentiable in a neighborhood of the support of $\varphi(x)$, we have

$$\int_{E_n}\varphi(x)\,d\mu = \int_{E_n}\|F(x)\|^{2}\Delta\varphi(x)\,dx = \int_{E_n}\varphi(x)\,\Delta(\|F(x)\|^{2\delta})\,dx,$$

which shows that $g(x) = \Delta(\|F(x)\|^{2\delta})$ almost everywhere and that the support of $\bar{\mu}$ is contained in the set of zeros P.

In order to finish the proof, it remains to be shown that $\bar{\mu} = 0$. If $\bar{\mu}\neq 0$, then there exist three non-empty relatively compact and open sets A_1, A_2 and A_3 such that $\bar{A}_1\subset A_2$, $\bar{A}_2\subset A_3$, $\bar{A}_3\subset D$ and the restriction of $\bar{\mu}$ to A_1 is different from zero. Applying Riesz's representation theorem to A_3 we have

$$-\|F(x)\|^{2\delta} = a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_3} + S_{A_3}(x)$$

for every $x\in A_3$. Hence, since the function $\Delta(\|F(x)\|^{2\delta})$ is non-negative, the Lebesgue decomposition of μ gives the inequalities:

$$-\|F(x)\|^{2\delta} = a_n\frac{1}{|x|^{n-2}}*\mu_{A_3} + S_{A_3}(x)$$

$$= a_n\frac{1}{|x|^{n-2}}*\Delta(\|F(x)\|^{2\delta}\,dx)_{A_3} + a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_3} + S_{A_3}(x)$$

$$\geqslant a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_3} + S_{A_3}(x) \geqslant a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_1} + S_{A_3}(x).$$

The support of $\bar{\mu}_{A_1}$ is contained in the set $P\cap\bar{A}_1$ and since this set is a subset of the polar set P, we conclude that the support of $\bar{\mu}_{A_1}$ is polar and compact. Applying a result of Evans [4] and Choquet [3] which asserts that the Newtonian potential of a positive measure whose support is a compact polar set is equal to infinity at least at one point of the support, we see that the potential of μ_{A_1} is equal to infinity at a point $x_0\in A_1$. The lower semicontinuity of the potential implies that for an arbitrary positive number M there exists a neighborhood V of x_0 satisfying $V\subset\bar{A}_2$ and such that

$$a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_1} > M$$

for every point in V. Let us choose M equal to $\sup_{x\in A_2}|S_{A_3}(x)|$. Then, if $x\in V$, we have,

$$M = \sup_{x\in A_2}|S_{A_3}(x)| \geqslant -\|F(x)\|^{2} - |S_{A_3}(x)| \geqslant a_n\frac{1}{|x|^{n-2}}*\bar{\mu}_{A_1} > M,$$

which is a contradiction and the lemma is proved.

(2.10) LEMMA. *Let D be an open set of E_n and K a compact subset of D with boundary ∂K smooth enough to assure the validity of Green's formula. Let $H(x)$ and $\Phi(x)$ be two infinitely differentiable functions defined on D and such that $\Phi(x) = 0$ for $x \in \partial K$. Then, if $F(x)$ is a harmonic function with values in a real Hilbert space $\mathscr{H}$ and satisfies* (C) *of* (1.7) *and* (D) *of* (2.2), *we have for every $\delta > \delta_0$ the formula*

$$\int_K \Phi(x) \Delta\left(\|F(x)\|^\delta H(x)\right) dx - \int_K \|F(x)\|^\delta H(x) \Delta\Phi(x)\, dx = \int_{\partial K} \|F(x)\|^\delta H(x) \frac{\partial \Phi}{\partial n}\, d\sigma .$$

Here $d\sigma$ denotes the element of area of the boundary ∂K.

Proof. Let $\Psi(x)$ be an infinitely differentiable function with compact support contained in D and such that $\Psi(x) = 1$ in a neighborhood of K. We define $g(x)$ as

$$g(x) = \Psi(x) \|F(x)\|^\delta .$$

Let $g_m(x)$ be the convolution of $g(x)$ with $m^n \varphi(mx)$, where $\varphi(x)$ is an infinitely differentiable function with support in the closed unit ball $B(0,1)$ and $\int \varphi(x)\,dx = 1$. Since the functions $g_m(x)$, $H(x)$ and $\Phi(x)$ are infinitely differentiable, we can apply Green's formula and obtain

$$\int_K \Phi(x) \Delta\left(g_m(x) H(x)\right) dx - \int_K g_m(x) H(x) \Delta\Phi(x)\, dx = \int_{\partial K} g(x) H(x) \frac{\partial \Phi(x)}{\partial n}\, d\sigma .$$

To prove the lemma, it suffices to show that

(1) $$\lim_{m\to\infty} \int_K \Phi(x) \Delta\left(g_m(x) H(x)\right) dx = \int_K \Phi(x) \Delta\left(g(x) H(x)\right) dx,$$

(2) $$\lim_{m\to\infty} \int_K g_m(x) H(x) \Delta\Phi(x)\, dx = \int_K g_m(x) H(x) \Delta\Phi(x)\, dx,$$

and

(3) $$\lim_{m\to\infty} \int_{\partial K} g_m(x) H(x) \frac{\partial \Phi(x)}{\partial n}\, d\sigma = \int_{\partial K} g(x) H(x) \frac{\partial \Phi(x)}{\partial n}\, d\sigma .$$

For the first limit (1) we have

(2.11) $$\left|\int_K \Phi(x) \Delta[g_m(x) H(x) - g(x) H(x)]\, dx\right| \leqslant C \int_K |\Delta g_m(x) - \Delta g(x)|\, dx + C \int_K \left|\operatorname{grad}\left(g_m(x)\right) - \operatorname{grad}\left(g(x)\right)\right| dx + C \int_K |g_m(x) - g(x)|\, dx,$$

where C is a constant depending on Φ, H and K. Using lemma (2.7) it is easy to verify that $g(x)$, $\partial g/\partial x_i$ and Δg are functions in $L^1(E_n)$. Moreover, lemma (2.9) implies that

$$\Delta g_m(x) = \Delta(g * \varphi_m) = (\Delta g) * \varphi_m$$

and

$$\frac{\partial}{\partial x_i} g_m(x) = \frac{\partial}{\partial x_i}(g * \varphi_m) = \frac{\partial g}{\partial x_i} * \varphi_m .$$

Therefore, we have

$$\lim_{m\to\infty} \int_{E_n} |\Delta g_m(x) - \Delta g(x)|\, dx = 0,$$

$$\lim_{m\to\infty} \int_{E_n} \left|\frac{\partial g_m(x)}{\partial x_i} - \frac{\partial g(x)}{\partial x_i}\right| dx = 0$$

and

$$\lim_{m\to\infty} \int_{E_n} |g_m(x) - g(x)|\, dx = 0,$$

which shows that the second member of (2.11) is arbitrarily small for m large enough and therefore that (1) holds. The proofs of (2) and (3) are similar to that of (1) and even simpler.

Proof of Proposition (2.1). Let $M(t,x)$ be either the function $H(t,x)$ described in the formulation of this proposition or $M(t,x) \equiv 1$. Since $M(t,x)$ is bounded, we have

$$\int_{E_n} \|G(t,x)\|^\delta M(t,x)\, dx \leqslant C \int_{E_n} \|G(t,x)\|^\delta dx$$

and from (1.2) and (1.12) we conclude that if $\delta \geqslant p$, the second member above is a bounded function of t converging to zero for t tending to infinity. If $\delta < p$, we take $q = p/\delta > 1$, $q' = q/(q-1)$ and an application of Hölder's inequality gives

$$\int_{E_n} \|G(t,x)\|^\delta M(t,x)\, dx \leqslant \left(\int_{E_n} \|G(t,x)\|^p dx\right)^{1/q} \left(\int_{E_n} h(x)^{q'} dx\right)^{1/q'},$$

which shows that again the second member is a bounded function of t converging to zero for t tending to infinity. It follows then that in all the cases considered in this proposition we have

(2.12) $$\int_{E_n} \|G(t,x)\|^\delta M(t,x)\, dx \leqslant C \quad \text{and} \quad \lim_{t\to\infty} \int_{E_n} \|G(t,x)\|^\delta M(t,x)\, dx = 0 .$$

Since the functions $g_k(t,x)$ and $M(t,x)$ are infinitely differentiable for $t \geqslant 0$ and $g_k(t,x)$ vanishes on the boundary of V_k we can apply lemma (2.10) obtaining

$$\int_{V_k} g_k(t,x)\Delta\big(\|G(t,x)\|^\delta M(t,x)\big)\,dx\,dt$$
$$= \int_{V_k} \|G(t,x)\|^\delta M(t,x)\Delta g_k(t,x)\,dx\,dt - \int_{\partial V_k} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma.$$

Let us show that

$$\lim_{k\to\infty}\int_{V_k} \|G(t,x)\|^\delta M(t,x)\Delta g_k(t,x)\,dx\,dt = 0. \tag{2.13}$$

A simple computation gives $|\Delta g_k(t,x)| \leqslant (n+1)/k$ and therefore we have

$$\Big|\int_{V_k} \|G(t,x)\|^\delta M(t,x)\Delta g_k(t,x)\,dx\,dt\Big| \leqslant \frac{(n+1)}{k}\int_{V_k} \|G(t,x)\|^\delta M(t,x)\,dx\,dt.$$

From (2.12) we know that given $\varepsilon > 0$ there exists a number t_ε such that

$$\int_{E_n} \|G(t,x)\|^\delta M(t,x)\,dx < \varepsilon \quad \text{for } t > t_\varepsilon,$$

hence, if $k\pi > t_\varepsilon$ we have

$$\Big|\int_{V_k} \|G(t,x)\|^\delta M(t,x)\Delta g_k(t,x)\,dx\,dt\Big|$$
$$\leqslant \frac{(n+1)}{k}\int_0^{t_\varepsilon} dt\int_{E_n} \|G(t,x)\|^\delta M(t,x)\,dx + \frac{(n+1)}{k}\int_{t_\varepsilon}^{k\pi} dt\int_{E_n} \|G(t,x)\|^\delta M(t,x)\,dx$$
$$\leqslant Ct_\varepsilon\frac{1}{k} + \pi\varepsilon,$$

which shows the convergence to zero of (2.13).

Next, we define the sets B_k, T_k and $C_{k,j,e}$ ($j = 1,\ldots,n$; $e = 1$ or -1) as

$$B_k = \{(t,x): t = 0;\ |x_i| \leqslant k\pi/2,\ i = 1,\ldots,n\},$$
$$T_k = \{(t,x): t = k\pi;\ |x_i| \leqslant k\pi/2,\ i = 1,\ldots,n\}$$

and

$$C_{k,j,e} = \{(t,x): 0 \leqslant t \leqslant k\pi;\ |x_i| \leqslant k\pi/2 \text{ for } i \neq j,\ x_j = ek\pi/2\}.$$

With this notation, the boundary ∂V_k is given by the union:

$$\partial V_k = B_k \cup T_k \cup \Big(\bigcup_{j,e} C_{k,j,e}\Big). \tag{2.14}$$

We will show that

$$\lim_{k\to\infty}\int_{B_k} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma = \int_{E_n} \|G(0,x)\|^\delta M(0,x)\,dx, \tag{2.15}$$

$$\lim_{k\to\infty}\int_{T_k} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma = 0 \tag{2.16}$$

and

$$\lim_{k\to\infty}\int_{C_{k,j,e}} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma = 0 \tag{2.17}$$

for every j, e.

The decomposition (2.14) of the boundary of V_k and the limits above imply

$$\lim_{k\to\infty} - \int_{\partial V_k} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma = \int_{E_n} \|G(0,x)\|^\delta M(0,x)\,dx,$$

which, together with (2.13), proves (2.3).

Let us consider (2.15). On the hypersurface B_k the normal derivative and the element of area have the expressions

$$\frac{\partial}{\partial n} g_k = -\frac{\partial}{\partial t} g_k(0,x) = -\cos(k^{-1}x_1)\ldots\cos(k^{-1}x_n)$$

and

$$d\sigma = dx_1\ldots dx_n$$

respectively. Therefore, the integral over B_k is equal to

$$\int_{|x_i|\leqslant k\pi/2} \|G(0,x)\|^\delta M(0,x)\cos(k^{-1}x_1)\ldots\cos(k^{-1}x_n)\,dx_1\ldots dx_n$$

and since the integrand is an increasing sequence of non-negative functions, we obtain that the limit of the integral above is given by

$$\int_{E_n} \|G(0,x)\|^\delta M(0,x)\,dx.$$

Let us consider (2.16). On the hypersurface T_k the normal derivative of $g_k(t,x)$ is given by

$$\frac{\partial}{\partial n} g_k = \frac{\partial}{\partial t} g_k(k\pi,x) = \cos(k^{-1}x_1)\ldots\cos(k^{-1}x_n)$$

and $d\sigma = dx_1 \dots dx_n$; then, on T_k we have

$$\Big|-\int_{T_k} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma\Big|$$

$$\leqslant \int_{|x_i|\leqslant k\pi/2} \|G(k\pi,x)\| M(k\pi,x)\,dx \leqslant \int_{E_n} \|G(k\pi,x)\|^\delta M(k\pi,x)\,dx$$

and, by (2.12), the last integral converges to zero for k tending to infinity.

Finally, for limit (2.17) we see that on the hypersurface $C_{k,j,e}$ the normal derivative is

$$\frac{\partial}{\partial n} g_k(t,x) = -\sin(k^{-1}t)\cos(k^{-1}x_1)\dots\cos(k^{-1}x_{j-1})\cos(k^{-1}x_{j+1})\dots\cos(k^{-1}x_n)$$

and

$$d\sigma = dt\,dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n;$$

then,

$$(2.18)\qquad \Big|-\int_{C_{k,j,e}} \|G(t,x)\|^\delta M(t,x)\frac{\partial}{\partial n} g_k(t,x)\,d\sigma\Big|$$

$$\leqslant \int_0^{k\pi}\int_{-k\pi/2}^{k\pi/2}\dots\int_{-k\pi/2}^{k\pi/2} \|G\|^\delta M(t,x_1,\dots,x_{j-1},ek\pi/2,x_{j+1},\dots,x_n)\,dt\,dx_1\dots$$

$$\dots dx_{j-1}dx_{j+1}\dots dx_n.$$

The point $(t,x_1,\dots,x_{j-1},ek\pi/2,x_{j+1},\dots,x_n)$ belongs to the cone $\overline{\Gamma}_1(x_1,\dots,x_{j-1},ek\pi/2+et,x_{j+1},\dots,x_n)$ and therefore, the second member of (2.18) is less than or equal to

$$\int_0^{k\pi}\int_{-k\pi/2}^{k\pi/2}\dots\int_{-k\pi/2}^{k\pi/2} m_1(\|G\|^\delta)m_1(M)(x_1,\dots,x_{j-1},ek\pi/2+et,x_{j+1},\dots,x_n)\times$$

$$\times dt\,dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n.$$

Changing variables and enlarging the domain of integration we see that the integral above is majorized by

$$(2.19)\qquad \int_{|x|\geqslant k\pi/2} m_1(\|G\|^\delta)\cdot m_1(M)(x)\,dx.$$

If $\delta\geqslant p$, the integral $\int_{E_n} m_1(\|G\|^\delta)(x)\,dx$ is finite and since $m_1(M)(x)$ is bounded, it turns out that (2.19) can be made arbitrarily small by choosing k large enough. If $\delta<p$, we take $q=p/\delta>1$ and $q'=q/(q-1)$ and applying Hölder's inequality we obtain that (2.19) is bounded by

$$C.\quad \Big(\int_{|x|\geqslant k\pi/2} m_1(\|G\|)^p(x)\,dx\Big)^{1/q}\Big(\int_{E_n} h(x)^{q'}dx\Big)^{1/q'}$$

but since the integral $\int_{E_n} m_1(\|G\|)^p(x)\,dx$ is finite, it turns out again that (2.19) can be made arbitrarily small by choosing k large enough. This completes the proof of (2.3). In order to prove (2.4) let us observe that $\Delta(\|G(t,x)\|^\delta)$ is a non-negative function and that the sequence $g_k(t,x)$ converges increasingly to t for k tending to infinity. Then, we have

$$\lim_{k\to\infty}\int_{V_k} g_k(t,x)\,\Delta(\|G(t,x)\|^\delta)\,dx\,dt = \int_0^\infty\int_{E_n} t\Delta(\|G(t,x)\|^\delta)\,dt\,dx,$$

which completes the proof of the proposition.

CHAPTER III

SOME ALGEBRAIC LEMMAS

(3.1) LEMMA. *Let $F(t,x)$ be a harmonic function defined on E_{n+1}^+, with values in a real Hilbert space $\mathscr{H}$ and satisfying* (C) *of* (1.7). *Then, for almost every $(t,x)\in E_{n+1}^+$ and $\delta>0$ we have*

$$\Delta(\|F(t,x)\|^{2\delta})+2\delta_0\delta^{-1}|\mathrm{grad}(\|F(t,x)\|^\delta)|^2\geqslant 0.$$

Proof. From the expressions for $\Delta(\|F(t,x)\|^{2\delta})$ and $\partial(\|F(t,x)\|^\delta)/\partial x_i$ given in (1.8) we see that

$$\Delta(\|F(t,x)\|^{2\delta})+2\delta_0\delta^{-1}|\mathrm{grad}(\|F(t,x)\|^\delta)|^2$$

$$=2\delta\|F(t,x)\|^{2\delta-4}\Big[(2\delta-2+\delta_0)\sum_0^n\Big\langle F(t,x),\frac{\partial F(t,x)}{\partial x_i}\Big\rangle^2+$$

$$+\|F(t,x)\|^2\sum_0^n\Big\|\frac{F(t,x)}{x_i}\Big\|^2\Big]$$

$$\geqslant 2\delta\|F(t,x)\|^{2\delta-4}\Big[(\delta_0-2)\sum_0^n\Big\langle F(t,x),\frac{\partial F(t,x)}{\partial x_i}\Big\rangle^2+$$

$$+\|F(t,x)\|^2\sum_0^n\Big\|\frac{\partial F(t,x)}{\partial x_i}\Big\|^2\Big]$$

and since by (C) of (1.7) the expression in brackets is non-negative, we obtain the thesis.

(3.2) Definition. Let $F(t,x)$ be a harmonic function defined on E_{n+1}^+, with values in a real Hilbert space $\mathscr{H}$ and satisfying (C) of (1.7). Let

$\delta > \delta_0/2$ and $\alpha > 0$. We define the *area function* $S_\alpha(\|F\|^\delta)(x)$ of $\|F(t,x)\|^\delta$ as[1]

$$S_\alpha(\|F\|^\delta)(x) = \left[\int_0^\infty \int_{E_n} \frac{\chi_\alpha(t, x-u)}{t^{n-1}} \Delta\left(\|F(t,u)\|^{2\delta}\right) du\,dt\right]^{1/2}.$$

(3.3) Definition. Let $F(t,x)$ satisfy the assumptions of definition (3.2) and let $\delta > 0$, $\alpha > 0$. We define the *modified area function* $T_\alpha(\|F\|^\delta)(x)$ of $\|F(t,x)\|^\delta$ as

$$T_\alpha(\|F\|^\delta)(x)$$
$$= \left[\int_0^\infty \int_{E_n} \frac{\chi_\alpha(t,x-u)}{t^{n-1}} \left[\Delta\left(\|F(t,u)\|^{2\delta}\right) + 2\delta_0\delta^{-1}\left|\operatorname{grad}\left(\|F(t,u)\|^{\delta}\right)\right|^2\right] du\,dt\right]^{1/2}.$$

The two area functions just defined are related, at least for $\delta > \delta_0/2$, by the following lemma:

(3.4) Lemma. *The inequalities*

$$S_\delta(\|F\|^\delta)(x) \leqslant T_\alpha(\|F\|^\delta)(x) \leqslant \left(\frac{2\delta}{2\delta-\delta_0}\right)^{1/2} S_\alpha(\|F\|^\delta)(x)$$

hold for every $\delta > \delta_0/2$.

Proof. The first inequality is apparent. As for the second, lemma (2.4) implies that

$$\Delta\left(\|F(t,x)\|^{2\delta}\right) + 2\delta_0\delta^{-1}\left|\operatorname{grad}\left(\|F(t,x)\|^\delta\right)\right|^2 \leqslant \frac{2\delta}{2\delta-\delta_0}\cdot\Delta\left(\|F(t,x)\|^{2\delta}\right)$$

for every $\delta > \delta_0/2$ and the lemma follows by integrating this inequality.

(3.5) Lemma. *Let* $r \geqslant 1$ *and* $\delta > 0$. *Then*

$$T_\alpha(\|F\|^{\delta r})(x) \leqslant r m_\alpha(\|F\|^\delta)^{r-1}(x)\, T_\alpha(\|F\|^\delta)(x).$$

Proof. Since $r \geqslant 1$, we have

$$(r-1)\left[(\delta_0-2)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + \|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2\right] \geqslant 0,$$

which implies

$$(r\delta_0 - 2r - \delta_0 - 2)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + (r-1)\|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2 \geqslant 0.$$

(1) The function $\chi_\alpha(t,x)$ denotes the characteristic function of the cone $\Gamma_\alpha(0) = \{x: |x| < \alpha t\}$.

Adding and subtracting $2\delta r$ into the first parenthesis, we obtain

$$[(r2\delta - 2r + r\delta_0) - (2r\delta - 2 + \delta_0)]\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + (r-1)\|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2 \geqslant 0$$

and from this it follows that

$$r\left[(2\delta-2+\delta_0)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + \|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2\right]$$
$$\geqslant (2r\delta - 2 + \delta_0)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + \|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2.$$

Now, multiplying both members by $2\delta r\|F\|^{2\delta r-4}$, we have

$$2\delta r^2 \|F\|^{2\delta r-4}\left[(2\delta-2+\delta_0)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + \|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2\right]$$
$$\geqslant 2\delta r\|F\|^{2\delta r-4}\left[(2\delta r-2+\delta_0)\sum_0^n \left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2 + \|F\|^2 \sum_0^n \left\|\frac{\partial F}{\partial x_i}\right\|^2\right]$$

or, which is the same,

$$r^2\|F\|^{2\delta(r-1)}\left[\Delta(\|F\|^{2\delta}) + 2\delta_0\delta^{-1}|\operatorname{grad}(\|F\|^\delta)|^2\right]$$
$$\geqslant \Delta(\|F\|^{2\delta r}) + 2\delta_0\cdot\frac{1}{\delta r}\cdot|\operatorname{grad}(\|F\|^{\delta r})|^2$$

and integrating we obtain

$$T_\alpha^2(\|F\|^{\delta r})(x) = \int_0^\infty \int_{E_n} \frac{\chi_\alpha(t,x-u)}{t^{n-1}}\left[\Delta(\|F\|^{2\delta r}) + 2\delta_0\frac{1}{\delta r}\cdot|\operatorname{grad}(\|F\|^\delta)|^2\right] du\,dt$$
$$\leqslant r^2 \int_0^\infty \int_{E_n} \frac{\chi_\alpha(t,x-u)}{t^{n-1}} \|F\|^{2\delta(r-1)}\left[\Delta(\|F\|^{2\delta}) + 2\delta_0\delta^{-1}|\operatorname{grad}(\|F\|^\delta)|^2\right] du\,dt$$
$$\leqslant r^2 m_\alpha(\|F\|^\delta)^{2(r-1)}(x)\, T_\alpha^2(\|F\|^\delta)(x),$$

which proves the second inequality of the thesis.

(3.6) Lemma. *Let* $\beta_1 > 0$, $\beta_2 > 0$, $0 < \sigma < 1$ *and* $\delta = \sigma\beta_1 + (1-\sigma)\beta_2$. *Then, the modified area function* T_α *satisfies*

$$\frac{1}{\delta}T_\alpha(\|F\|^\delta)(x) \leqslant \left[\frac{1}{\beta_1}T_\alpha(\|F\|^{\beta_1})(x)\right]^\sigma \left[\frac{1}{\beta_2}T_\alpha(\|F\|^{\beta_2})(x)\right]^{1-\sigma}.$$

Proof. We shall study first the function $\varphi(s) = \lg(A+B/s)$, where $A, B \geqslant 0$, $A+B > 0$ and $s > 0$. The second derivative of this function,

$$\frac{d^2\varphi(s)}{ds^2} = \frac{B(2As+B)}{(As^2+Bs)},$$

is non-negative and therefore $\varphi(s)$ is a convex function of s for $s > 0$, which means that for $0 < \sigma < 1$, $\beta_1 > 0$ and $\beta_2 > 0$,

$$\varphi\big(\sigma\beta_1+(1-\sigma)\beta_2\big) \leqslant \sigma\varphi(\beta_1)+(1-\sigma)\varphi(\beta_2)$$

holds. This inequality implies

$$e^{\varphi(\sigma\beta_1+(1-\sigma)\beta_2)} \leqslant [e^{\varphi(\beta_1)}]^{\sigma}[e^{\varphi(\beta_2)}]^{1-\sigma}$$

and, replacing $\varphi(s)$ by its definition, we obtain

$$A+\frac{B}{\delta} \leqslant \left[A+\frac{B}{\beta_1}\right]^{\sigma}\left[A+\frac{B}{\beta_2}\right]^{1-\sigma}. \tag{3.7}$$

Let us consider now the expression

$$\frac{1}{\delta^2}[\Delta(\|F\|^{2\delta})+2\delta_0\delta^{-1}|\mathrm{grad}(\|F\|^{\delta})|^2]$$
$$= \|F\|^{2\delta-4}\left[4\sum_0^n\left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2+\left(2(\delta_0-2)\sum_0^n\left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2+\right.\right.$$
$$\left.\left.+2\|F\|^2\sum_0^n\left\|\frac{\partial F}{\partial x_i}\right\|^2\right)\frac{1}{\delta}\right].$$

If we take

$$A = 4\sum_0^n\left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2$$

and

$$B = 2(\delta_0-2)\sum_0^n\left\langle\frac{\partial F}{\partial x_i}, F\right\rangle^2+2\|F\|^2\sum_0^n\left\|\frac{\partial F}{\partial x_i}\right\|^2,$$

then (3.7) implies

$$\frac{1}{\delta^2}[\Delta(\|F\|^{2\delta})+2\delta_0\delta^{-1}|\mathrm{grad}(\|F\|^{\delta})|^2]$$
$$\leqslant\left\{\frac{1}{\beta_1^2}[\Delta(\|F\|^{2\beta_1})+2\delta_0\beta_1^{-1}|\mathrm{grad}(\|F\|^{\beta_1})|^2]\right\}^{\sigma}\times$$
$$\times\left\{\frac{1}{\beta_2^2}[\Delta(\|F\|^{2\beta_2})+2\delta_0\beta_2^{-1}|\mathrm{grad}(\|F\|^{\beta_2})|^2]\right\}^{1-\sigma}.$$

Finally, integrating this and applying Hölder's inequality to the second member, we obtain

$$\frac{1}{\delta^2}T_a^2(\|F\|^{\delta})(x) \leqslant \left[\frac{1}{\beta_1^2}T_a^2(\|F\|^{\beta_1})\right]^{\sigma}\left[\frac{1}{\beta_2^2}T_a^2(\|F\|^{\beta_2})\right]^{1-\sigma},$$

which proves the lemma.

CHAPTER IV

THE MAIN THEOREM

(4.1) LEMMA. *Let $F(t, x)$ be a harmonic function defined on E_{n+1}^+ with values in a real Hilbert space $\mathscr{H}$ and satisfying* (A) *and* (C) *of* (1.7). *Let us denote by $G(s, x)$ the function $F(t+s, x)$, $t > 0$. Then, if $\delta > 0$ and $q > 0$ satisfy $\delta q \geqslant p$, we have*

$$\int_{E_n} m_a(\|G\|^{\delta})^q(x)\,dx \leqslant C\int_{E_n}[\|G(0, x)\|^{\delta}]^q\,dx$$

where the constant C depends on n and $\delta q/\delta_0$ only.

Proof. The function $\|F(t, x)\|^{\delta_0}$ satisfies the hypotheses of proposition (1.2); then, if $t>0$, we infer that $\|G(t, x)\|^{\delta_0}$ is bounded and satisfies the same hypotheses. Let $\delta q = p'$. We have

$$\int_{E_n}[\|G(s, x)\|^{\delta_0}]^{p'/\delta_0}dx \leqslant \sup_{s\geqslant 0}\int_{E_n}\|G(s, x)\|^{p'}dx$$
$$= \int_{E_n}\|G(0, x)\|^{p'}dx \leqslant \big(\sup\|G(0, x)\|^{p'-p}\big)\int_{E_n}\|F(t, x)\|^{p}dx < \infty.$$

Hence, proposition (1.4) implies that

$$\int_{E_n} m_a(\|G\|^{\delta_0})^{p'/\delta_0}(x)\,dx \leqslant C\int_{E_n}\|G(0, x)\|^{p'}dx$$

or

$$\int_{E_n} m_a(\|G\|^{\delta})^q(x)\,dx \leqslant C\int_{E_n}[\|G(0, x)\|^{\delta}]^q\,dx.$$

(4.2) THEOREM. *Let $F(t, x)$ be a harmonic function defined on E_{n+1}^+ with values in a real Hilbert space $\mathscr{H}$. We shall assume that $F(t, x)$ satisfies conditions* (A), (C) *of* (1.7) *and* (D) *of* (2.2). *Then, there exist two positive* *constants* c_1 *and* c_2 *which depend on* α, n *and* p *only and such that*

$$c_1\Big[\int_{E_n}\|F(t,x)\|^p\,dx\Big]^{1/p}$$

$$\leqslant\Bigg\{\int_{E_n}\Bigg[\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}\Big[\sum_0^n\Big\|\frac{\partial F(t,u)}{\partial x_i}\Big\|^2\Big]du\,ds\Bigg]^{p/2}dx\Bigg\}^{1/p}$$

$$\leqslant c_2\Big[\int_{E_n}\|F(t,x)\|^p\,dx\Big]^{1/p}$$

for every $t>0$.

Proof. For the sake of simplicity, we shall denote by $M_p(\varphi)$, $p>0$, the "p-norm" of $\varphi(x)$; in other words,

$$M_p(\varphi)=\Big[\int_{E_n}|\varphi(x)|^p\,dx\Big]^{1/p}.$$

Let t be positive and fixed. As in lemma (4.1), $G(s,x)$ will denote the function $F(t+s,x)$. The function $G(s,x)$, $s>0$, satisfies all the assumptions made for $F(t,x)$ and from proposition (1.2) we obtain that $G(s,x)$ is a bounded function for $s\geqslant 0$. Therefore, if $q\geqslant p$,

$$\int_{E_n}\|G(s,x)\|^q\,dx\leqslant\sup\|G(s,x)\|^{q-p}\int_{E_n}\|G(s,x)\|^p\,dx$$

for every $s\geqslant 0$. This shows that $G(s,x)$ satisfies the assumptions of our theorem for every $q\geqslant p$.

Let $\delta\geqslant p/2$. We will prove the formula

$$CM_2(\|G\|^\delta)=M_2\big(S_\alpha(\|G\|^\delta)\big),\tag{4.3}$$

where C is a constant depending on n and α only.

We have

$$M_2^2\big(S_\alpha(\|G\|^\delta)\big)=\int_{E_n}dx\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}\varDelta\big(\|G(s,u)\|^{2\delta}\big)du\,ds$$

$$=C\int_0^\infty\int_{E_n}s\varDelta\|G(s,u)\|^{2\delta}du\,ds$$

and by proposition (2.1)

$$\int_0^\infty\int_{E_n}s\varDelta\big(\|G(s,u)\|^{2\delta}\big)du\,ds=\int_{E_n}\|G(0,u)\|^{2\delta}du=M_2^2(\|G\|),$$

hence (4.3) is proved.

The next step in the proof will be to show that if q and δ satisfy $q<2$, $\delta>\delta_0/2$ and $\delta q\geqslant p$, then,

$$M_q\big(S_\alpha(\|G\|^\delta)\big)\leqslant CM_q(\|G\|^\delta),\tag{4.4}$$

where the constant C depends on α, p, n and δ only. Let r denote the number $2/q$. Obviously, $r>1$. From lemmas (3.4) and (3.5) we have

$$S_\alpha(\|G\|^\delta)\leqslant T_\alpha(\|G\|^\delta)=T_\alpha(\|G\|^{(\delta/r)\cdot r})\leqslant rm_\alpha(\|G\|^\delta)^{(r-1)/r}T_\alpha(\|G\|^{\delta/r})$$

which, integrating and applying Hölder's inequality gives

$$M_q^q\big(S_\alpha(\|G\|^\delta)\big)\leqslant r^qM_1\big(m_\alpha(\|G\|^\delta)^{(r-1)q/r}T_\alpha(\|G\|^{\delta/r})^q\big)$$

$$\leqslant r^qM_{2/(2-q)}\big(m_\alpha(\|G\|^\delta)^{(r-1)q/r}\big)M_{2/q}\big(T_\alpha(\|G\|^{\delta/r})^q\big).$$

Now, since $r=2/q$, we have $\delta/r=\delta q/2\geqslant p/2>\delta_0/2$ and hence, by lemma (3.4), we obtain

$$M_{2/q}\big(T_\alpha(\|G\|^{\delta/r})^q\big)=M_2^p\big(T_\alpha(\|G\|^{\delta/r})\big)\leqslant CM_2^p\big(S_\alpha(\|G\|^{\delta/r})\big)$$

and by (4.3)

$$M_{2,q}\big(T_\alpha(\|G\|^{\delta/r})^q\big)\leqslant CM_2^q\big(S_\alpha(\|G\|^{\delta/r})\big)=CM_2^q(\|G\|^{\delta/r})\leqslant CM_q^{q/r}(\|G\|^\delta).$$

Applying lemma (4.1), we also have

$$M_{2/(2-q)}\big(m_\alpha(\|G\|^\delta)^{(r-1)q/r}\big)=M_q^{q(2-q)/2}\big(m_\alpha(\|G\|^\delta)\big)\leqslant CM_q^{q(2-q)/2}(\|G\|^\delta),$$

hence we can write the inequality

$$M_q^q\big(S_\alpha(\|G\|^\delta)\big)\leqslant CM_q^{q(2-q)/2}(\|G\|^\delta)M_q^{q/r}(\|G\|^\delta)=CM_q^q(\|G\|^\delta),$$

which proves (4.4).

Now, we will show that it q and δ satisfy $q>2$, $\delta>\delta_0/2$ and $\delta q\geqslant p$, then there exists a constant C which depends on δ, q, n and α such that

$$M_q\big(S_\alpha(\|G\|^\delta)\big)\leqslant CM_q(\|G\|^\delta).\tag{4.5}$$

In order to prove (4.5) we will assume that

$$M_{q/2}\big(S_\alpha(\|G\|^{2\delta})\big)<\infty\tag{4.6}$$

holds. This assumption will be removed later.

Let $k(x)$ be a non-negative and infinitely differentiable function with compact support. The functions $h(x)$ of the form

$$h(x)=\int_{E_n}P_n(\eta,x-u)k(u)\,du$$

obtained by varying $\eta>0$ and $k(x)$ are dense in the set of non-negative functions of $L^r(E_n)$ for every $r\geqslant 1$.

We have the identities

$$\int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx = \int_{E_n} h(x)\,dx\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}\Delta(\|G\|^{2\delta})\,du\,ds$$
$$= \int_0^\infty\int_{E_n}\Delta(\|G\|^{2\delta})\,du\,ds\int_{E_n} h(x)\frac{\chi_\alpha(s,x-u)\,dx}{s^{n-1}};$$

then, using the well-known inequality

$$\frac{\chi_\alpha(s,x)}{s^{n-1}} \leqslant CsP_n(s,x),$$

we obtain

$$\int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx \leqslant C\int_0^\infty\int_{E_n} s\Delta(\|G\|^{2\delta})\,du\,ds\int_{E_n} P_n(s,x-u)h(x)\,dx,$$

and denoting by $H(s,u)$ the Poisson integral of $h(x)$, the second member above reads:

$$(4.7)\qquad \int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx \leqslant C\int_0^\infty\int_{E_n} s\Delta(\|G\|^{2\delta})H(s,u)\,du\,ds$$
$$= C\lim_{k\to\infty}\int_{V_k} g_k(s,u)\Delta(\|G\|^{2\delta})H(s,u)\,du\,ds.$$

Since the function $H(s,u)$ is non-negative and harmonic, we have

$$\Delta(\|G\|^{2\delta}H) = H\Delta(\|G\|^{2\delta}) + 2\big(\operatorname{grad}(\|G\|^{2\delta})\operatorname{grad}(H)\big),$$

which implies

$$\Delta(\|G\|^{2\delta}H) \geqslant H\Delta(\|G\|^{2\delta}) - 2|\operatorname{grad}(\|G\|^{2\delta})|\cdot|\operatorname{grad}(H)|$$

or

$$H\Delta(\|G\|^{2\delta}) \leqslant \Delta(\|G\|^{2\delta}H) + 2|\operatorname{grad}(\|G\|^{2\delta})|\cdot|\operatorname{grad}H|.$$

Then, replacing $H\Delta(\|G\|^{2\delta})$ in (4.7) by the second member of the last inequality above, we obtain

$$\int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx$$
$$\leqslant C\lim_{k\to\infty}\int_{V_k} g_k(s,u)\Delta(\|G\|^{2\delta}H)\,du\,ds + C\int_0^\infty\int_{E_n} s|\operatorname{grad}(\|G\|^{2\delta})|\cdot|\operatorname{grad}H|\,du\,ds.$$

The last integral is equal to a constant times

$$(4.8)\qquad \int_{E_n} dx\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}\cdot|\operatorname{grad}(\|G\|^{2\delta})|\cdot|\operatorname{grad}H|\,du\,ds,$$

which, by Schwarz's inequality, is less than or equal to

$$\int_{E_n} dx\left[\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}|\operatorname{grad}(\|G\|^{2\delta})|^2\,du\,ds\right]^{1/2}\times$$
$$\times\left[\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}\cdot|\operatorname{grad}H|^2\,du\,ds\right]^{1/2}.$$

From lemma (2.4) we know that

$$|\operatorname{grad}(\|G\|^{2\delta})|^2 \leqslant C\Delta(\|G\|^{4\delta})$$

and, therefore, (4.8) is majorized by a constant times

$$\int_{E_n} S_\alpha(\|G\|^{2\delta})S_\alpha(H)\,dx \leqslant M_{q/2}\big(S_\alpha(\|G\|^{2\delta})\big)M_r\big(S_\alpha(H)\big),$$

where r is the conjugate exponent of $q/2>1$. Since G and H satisfy the hypotheses of proposition (2.1), the limit for k tending to infinity of the integral $\int_{V_k} g(s,u)\Delta(\|G\|^{2\delta}H)\,du\,ds$ is equal to

$$\int_{E_n}\|G(0,x)\|^{2\delta}h(x)\,dx \leqslant M_q^2(\|G\|^\delta)M_r(h).$$

Collecting results, we have

$$\int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx$$
$$\leqslant CM_{q/2}\big(S_\alpha(\|G\|^{2\delta})\big)M_r\big(S_\alpha(H)\big) + CM_q^2(\|G\|^\delta)M_r(h).$$

By theorem 4 in [6], we have $M_r\big(S_\alpha(H)\big) \leqslant CM_r(h)$ and therefore we can write

$$\int_{E_n} S_\alpha(\|G\|^\delta)^2(x)h(x)\,dx \leqslant C\{M_{q/2}\big(S_\alpha(\|G\|^{2\delta})\big) + M_q^2(\|G\|^\delta)\}\cdot M_r(h)$$

which, using assumption (4.6), implies

$$(4.9)\qquad M_q^2\big(S_\alpha(\|G\|^\delta)\big) = M_{q/2}\big(S_\alpha(\|G\|^\delta)^2\big) < \infty.$$

Going back to (4.8) and using $\operatorname{grad}(\|G\|^{2\delta}) = 2\|G\|^\delta\cdot\operatorname{grad}(\|G\|^\delta)$ we obtain

$$\int_0^\infty\int_{E_n} s|\operatorname{grad}(\|G\|^{2\delta})|\cdot|\operatorname{grad}H|\,du\,ds$$
$$\leqslant C\int_{E_n} m_\alpha(\|G\|^\delta)(x)\,dx\int_0^\infty\int_{E_n}\frac{\chi_\alpha(s,x-u)}{s^{n-1}}|\operatorname{grad}(\|G\|^\delta)|\cdot|\operatorname{grad}H|\,du\,ds$$

and applying lemma (2.4) and Schwarz's inequality as we did before we get

$$\int_0^\infty \int_{E_n} s\,|\mathrm{grad}(\|G\|^{2\delta})|\cdot|\mathrm{grad}\,H|\,du\,ds \leqslant C\int_{E_n} m_a(\|G\|^\delta)\,S_a(\|G\|^\delta)\,S_a(H)(x)\,dx$$

which implies

$$\int_0^\infty \int_{E_n} s\,|\mathrm{grad}(\|G\|^{2\delta})|\,|\mathrm{grad}\,H|\,du\,ds \leqslant CM_q\big(m_a(\|G\|^\delta)\big)M_q\big(S_a(\|G\|^\delta)\big)\,M_r\big(S_a(H)\big)$$

therefore we also have

$$\int_{E_n} S_a(\|G\|^\delta)^2(x)\,h(x)\,dx \leqslant C'\left\{M_q^2(\|G\|^\delta)+M_q(\|G\|^\delta)\,M_q\big(S_a(\|G\|^\delta)\big)\right\}M_r(h)$$

and so it follows that

$$M_q^2\big(S_a(\|G\|^\delta)\big) \leqslant C'M_q^2(\|G\|^\delta)+C'M_q(\|G\|^\delta)M_q\big(S_a(\|G\|^\delta)\big).$$

By (4.9), all the terms involved in this inequality are finite and since the constant C' is independent of G, we conclude the existence of a constant C, also independent of G, such that

$$M_q\big(S_a(\|G\|^\delta)\big) \leqslant CM_q(\|G\|^\delta).$$

Now, we want to remove assumption (4.6). This can be done by induction: If q is any number between 2 and 4, more precisely, $2<q\leqslant 4$, we have $1<q/2\leqslant 2$ and $(2\delta)(q/2)=\delta q\geqslant p$ and (4.6) follows from (4.3) or (4.4). In the same manner, if $4<q\leqslant 8$, (4.6) follows from (4.5) for $2<q\leqslant 4$, which has just been proved, an so on. Therefore, we have proved that if $\delta>\delta_0/2$ and $\delta q\geqslant p$, then there exists a constant depending on δ, q, n and a such that

$$M_q\big(S_a(\|G\|^\delta)\big) \leqslant CM_q(\|G\|^\delta). \tag{4.10}$$

In particular, taking $\delta=1$ and $q=p$, we have

$$M_p\big(S_a(\|G\|)\big) \leqslant c_2M_p(\|G\|)$$

which is the second inequality in the thesis.

Let us consider the first inequality in the thesis. We detine $\mu=(p+1)/2$ if $p>2$ and $\mu=(p+\delta_0)/(2p)$ if $p<2$. The number μ we have defined is greater than 1 for $p>2$ and smaller than 1 for $p<2$. Let σ, u, v and ν be the numbers given by

$$\sigma=\frac{p-2}{\mu p-2},\qquad u=\frac{\mu p-2}{\mu(p-2)},\qquad v=\frac{\mu p-2}{2(\mu-1)},\qquad \nu=\frac{2}{p}.$$

These numbers satisfy:

$$0<\sigma<1,\quad u>1,\quad \frac{1}{u}+\frac{1}{v}=1,\quad \nu>0,\quad 1=\sigma\mu+(1-\sigma)\nu.$$

From (4.3) we have

$$M_p^p(\|G\|)=M_2^2(\|G\|^{p/2})=CM_2^2\big(S_a(\|G\|^{p/2})\big)$$

and applying lemmas (3.4) and (3.6) we obtain the inequalities

$$S_a(\|G\|^{p/2})\leqslant T_a(\|G\|^{p/2})\leqslant C[T_a(\|G\|^{\mu p/2})]^\sigma[T_a(\|G\|)]^{1-\sigma};$$

squaring and integrating, we have

$$M_2^2\big(S_a(\|G\|^{p/2})\big)\leqslant CM_1\big(T_a(\|G\|^{\mu p/2})^{2\sigma}T_a(\|G\|)^{2(1-\sigma)}\big)$$

and, by Hölder's inequality

$$M_p^p(\|G\|)=CM_2^2\big(S_a(\|G\|^{p/2})\big)\leqslant CM_u\big(T_a(\|G\|^{\mu p/2})^{2\sigma}\big)M_v\big(T_a(\|G\|)^{2(1-\sigma)}\big).$$

From the identity $M_u\big(T_a(\|G\|^{\mu p/2})^{2\sigma}\big)=M_{2\sigma u}^{2\sigma}\big(T_a(\|G\|^{\mu p/2})\big)$ and cosidering that $\mu p/2>\delta_0/2$, $(2\sigma u)(\mu p/2)=p$, by lemma (3.4) and (4.10) we conclude that

$$M_u\big(T_a(\|G\|^{\mu p/2})^{2\sigma}\big)\leqslant CM_{2\sigma u}^{2\sigma}\big(S_a(\|G\|^{\mu p/2})\big)\leqslant CM_{2\sigma u}^{2\sigma}(\|G\|^{\mu p/2})=CM_p^{p/u}(\|G\|).$$

Also, $M_v\big(T_a(\|G\|)^{2(1-\sigma)}\big)\leqslant CM_v\big(S_a(\|G\|)^{2(1-\sigma)}\big)=CM_p^{p/v}\big(S_a(\|G\|)\big)$ and therefore,

$$M_p^p(\|G\|)\leqslant CM_p^{p/u}(\|G\|)M_p^{p/v}\big(S_a(\|G\|)\big),$$

which implies

$$M_p(\|G\|)\leqslant CM_p\big(S_a(\|G\|)\big)$$

and the theorem is proved.

CHAPTER V

APPLICATION TO H^p-SPACES OF HARMONIC FUNCTIONS

Let $U(t,x)$ be a harmonic function defined on E_{n+1}^+ and with values in a real Hilbert space $\mathscr{H}$. By definition, the gradient $\nabla U(t,x)$ of U is the set of $n+1$ functions $\partial U/\partial t, \partial U/\partial x_1, \ldots, \partial U/\partial x_n$. This gradient can be interpreted as a harmonic function $F(x,t)$ from E_{n+1}^+ to the Hilbert space $\mathscr{H}^{(n+1)}$ of all $(n+1)$-tuples of elements of $\mathscr{H}$. The scalar product of $h=(h_0,\ldots,h_n)$ and $k=(k_0,\ldots,k_n)$, two elements of $\mathscr{H}^{(n+1)}$, and the norm of h are given by

$$\langle h,k\rangle=\sum_0^n\langle h_i,k_i\rangle\qquad\text{and}\qquad \|h\|=\langle h,h\rangle^{1/2}=\Big[\sum_0^n\|h_i\|^2\Big]^{1/2}$$

respectively. Observe that we use the same notation for the scalar product and the norm in both $\mathscr{H}$ and $\mathscr{H}^{(n+1)}$.

(5.1) LEMMA. *The δ-power of the norm of $\nabla U(t,x)$ is a subharmonic function on E_{n+1}^+ for every $\delta\geqslant(n-1)/n$.*

Proof. The gradient $F(t,x)$ of $U(t,x)$ is a harmonic function from E_{n+1}^+ to $\mathscr{H}^{(n+1)}$, therefore, by lemma (1.8) we infer that $\|F(t,x)\|^\delta$ is subharmonic if and only if

$$(5.2)\qquad (\delta-2)\sum_0^n\Big\langle\frac{\partial F(t,x)}{\partial x_i},F(t,x)\Big\rangle^2+\|F(t,x)\|^2\sum_0^n\Big\|\frac{\partial F(t,x)}{\partial x_i}\Big\|^2\geqslant 0$$

holds for every $(t,x)\in E_{n+1}^+$. Let $\{h\}_{\lambda\in I}$ be a basis of $\mathscr{H}$. If we denote the scalar product $\langle U(t,x),h_\lambda\rangle$ by $U_\lambda(t,x)$, we have

$$U(t,x)=\sum_\lambda U_\lambda(t,x)\cdot h_\lambda\quad\text{and}\quad \nabla U(t,x)=\sum_\lambda \nabla U_\lambda(t,x)h_\lambda$$

or, writing $F_\lambda(t,x)=\nabla U_\lambda(t,x)$,

$$F(t,x)=\sum_\lambda F_\lambda(t,x)h_\lambda.$$

Since $U_\lambda(t,x)$ is a real-valued harmonic function on E_{n+1}^+, we know from [8] that the δ-power of the absolute value of the gradient $F_\lambda(t,x)$ of U_λ is a subharmonic function on E_{n+1}^+ provided that $\delta\geqslant(n-1)/n$ or, which is the same,

$$(5.3)\qquad (\delta-2)\sum_0^n\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)^2+\|F_\lambda\|^2\sum_0^n\Big\|\frac{\partial F_\lambda}{\partial x_i}\Big\|^2\geqslant 0$$

for $\delta\geqslant(n-1)/n$. We use this fact to prove our lemma.

For $\delta\geqslant 2$, inequality (5.2) is apparent, so we will assume in the sequel that $\delta<2$. We have

$$\Big\langle\frac{\partial F}{\partial x_i},F\Big\rangle^2=\Big(\sum_\lambda\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)\Big)^2=\sum_{\lambda,\mu}\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)\Big(\frac{\partial F_\mu}{\partial x_i},F_\mu\Big);$$

then

$$\sum_0^n\Big\langle\frac{\partial F}{\partial x_i},F\Big\rangle^2=\sum_{\lambda,\mu}\sum_0^n\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)\Big(\frac{\partial F_\mu}{\partial x_i},F_\mu\Big)$$

$$\leqslant\sum_{\lambda\mu}\Big[\sum_0^n\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)^2\Big]^{1/2}\Big[\sum_0^n\Big(\frac{\partial F_\mu}{\partial x_i},F_\mu\Big)^2\Big]^{1/2}=\Big\{\sum_\lambda\Big[\sum_0^n\Big(\frac{\partial F_\lambda}{\partial x_i},F_\lambda\Big)^2\Big]^{1/2}\Big\}^2.$$

Now, from (5.3) and using Schwarz's inequality, we obtain

$$\sum_0^n\Big\langle\frac{\partial F}{\partial x_i},F\Big\rangle^2\leqslant\frac{1}{2-\delta}\Big\{\sum_\lambda|F_\lambda|\Big(\sum_0^n\Big|\frac{\partial F_\lambda}{\partial x_i}\Big|^2\Big)^{1/2}\Big\}^2$$

$$\leqslant\frac{1}{2-\delta}\sum_\lambda|F_\lambda|^2\sum_\lambda\sum_0^n\Big|\frac{\partial F_\lambda}{\partial x_i}\Big|^2\leqslant\frac{1}{2-\delta}\|F\|^2\sum_0^n\Big\|\frac{\partial F}{\partial x_i}\Big\|^2,$$

which implies (5.2) and the proof of the lemma is complete.

(5.4) Lemma. *Let $U(x)$ be a harmonic function defined on an open and connected set $D\subset E_n$ with values in a real Hilbert space $\mathscr{H}$. We wil assume that the gradient of $U(x)$ is not identically zero on D. Then, the se*

$$\{x\colon\ \nabla U(x)=0\}$$

is a polar set in D.

Proof. Let $V(x)$ be a non-identically zero real harmonic function defined on D and let $\nabla(\nabla V(x))$ be the set of functions obtained by taking the gradient of all the components of ∇U. We will prove that the set

$$N=\{x\colon\ \nabla V(x)=0\ \text{and}\ \nabla(\nabla V(x))\neq 0\}$$

is a polar subset of D. Let $x\in N$ and $V_i=\partial V/\partial x_i$, $i=1,\ldots,n$. The Jacobian matrix

$$J(x)=\begin{bmatrix}\partial V_1/\partial x_1\ldots\partial V_1/\partial x_n\\ \ldots\ldots\ldots\ldots\\ \partial V_n/\partial x_1\ldots\partial V_n/\partial x_n\end{bmatrix}$$

is symmetric, different from zero and

$$\operatorname{trace}J(x)=\sum_1^n\frac{\partial V_i}{\partial x_i}=\sum_0^n\frac{\partial^2 V}{\partial x_i^2}=\nabla V=0.$$

Since $J(x)$ is symmetric, there exists an orthogonal matrix P such that

$$PJ(x)P^{-1}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}=\Lambda.$$

From the fact that $J(x)$ is not the null matrix, we conclude that at least one λ_i is different from zero. Let us suppose $\lambda_k\neq 0$. Then $0=\operatorname{trace}J(x)=\sum_1^n\lambda_i$ implies $0\neq\lambda_k=-\sum_{i\neq k}\lambda_i$ and therefore, there is an $h\neq k$ such that $\lambda_h\neq 0$. This proves that $m=\operatorname{rank}J(x)=\operatorname{rank}\Lambda\geqslant 2$. Without loss of generality, we may assume that the determinant

$$\begin{vmatrix}\dfrac{\partial V_1(x)}{\partial x_1}\cdots\dfrac{\partial V_1(x)}{\partial x_m}\\ \ldots\ldots\ldots\ldots\\ \dfrac{\partial V_m(x)}{\partial x_1}\cdots\dfrac{\partial V_m(x)}{\partial x_m}\end{vmatrix}$$

is different from zero. Therefore, by the implicit function theorem we have that there is a neighborhood S_x of the point x such that the set of points $y \in S_x$, where $V_i(y) = 0$, $i = 1, \ldots, m$, forms an infinitely differentiable manifold A_x whose dimension is $n-m \leqslant n-2$. So, for every $x \in N$ we have an open neighborhood S_x of x and a manifold A_x of dimension less than or equal to $n-2$ such that

$$N \cap S_x \subset A_x.$$

We can choose a sequence of points $x_i \in N$ such that

$$N \subset \bigcup_1^\infty (N \cap S_{x_i}) \subset \bigcup_1^\infty A_{x_i}$$

and since every submanifold of E_n with dimension less than or equal to $n-2$ is a polar set, the set N, which is a subset of the union of a countable family of polar sets, is a polar set itself.

Now we will consider the case when $U(x)$ is a real-valued function satisfying the hypotheses of the lemma. If $x \in M = \{x\colon \nabla U(x) = 0\}$, then there exists $k \geqslant 1$ such that $\nabla U(x) = \ldots = \nabla^k U(x) = 0$ and $\nabla^{k+1} U(x) \neq 0$. Otherwise, we would have $\nabla^k U(x)$ for every k which would imply $U(x) =$ constant and therefore $\nabla U(x) \equiv 0$ on D. Then, it immediately follows that

$$M \subset \bigcup_k M_k, \qquad M_k = \{x\colon \nabla^k U(x) = 0 \text{ and } \nabla^{k+1} U(x) \neq 0\}.$$

The first part of the proof shows that, for every k, the set M_k is polar and then, since M is a subset of a countable union of polar sets, it turns out that M is also a polar set.

Finally, let us consider the case of a harmonic function with values in a real Hilbert space $\mathscr{H}$. Since the gradient of $U(x)$ is not identically zero, there is $h \in \mathscr{H}$ such that the gradient of $\langle U(x), h\rangle$ is not identically zero and the lemma follows from the inclusion

$$\{x\colon \nabla U(x) = 0\} \subset \{x\colon \nabla(\langle U(x), h\rangle) = 0\}$$

(5.5) Definition. Let $U(t, x)$ be a harmonic function defined on E_{n+1}^+ with values in a real Hilbert space $\mathscr{H}$. We say that the gradient $\nabla U(t, x)$ belongs to the class $H^p(\mathscr{H})$, $p > 0$, of Hardy if the following conditions are satisfied:

(i) There exists a constant $K > 0$ such that

$$\int_{E_n} \|\nabla U(t, x)\|^p dx \leqslant K^p$$

for every $t > 0$.

(ii) The limit $\lim_{t\to 0} \nabla U(t, x)$ exists for almost every $x \in E_n$. This limit will be denoted by $\nabla U(0, x)$.

(5.6) THEOREM. *Let $\nabla U(t, x) \in H^p(\mathscr{H})$, $p > (n-1)/n$. The area function*

$$S_a(\nabla U)(x) = \left[\int_{E_{n+1}^+} \frac{\chi_a(s, x-u)}{s^{n-1}} \left[\sum_0^n \left\| \frac{\partial \nabla U(s, u)}{\partial x_i} \right\|^2\right] du\, ds\right]^{1/2}$$

satisfies the inequalities

$$c_1 \left[\int_{E_n} \|\nabla U(0, x)\|^p dx\right]^{1/p} \leqslant \left[\int_{E_n} S_a(\nabla U)^p(x)\, dx\right]^{1/p}$$
$$\leqslant c_2 \left[\int_{E_n} \|\nabla U(0, x)\|^p dx\right]^{1/p},$$

where c_1 and c_2 are two positive constants depending on a, p and n only.

Proof. Observe that conditions (i) and (ii) of definition (5.5) coincide with (A) and (B) of (1.7) for $\nabla U(t, x)$. Moreover, lemmas (5.1) and (5.4) show that the gradient $\nabla U(t, x)$ satisfies (C) of (1.7) and (D) of (2.2) with $\delta_0 = (n-1)/n$. Therefore, theorem (4.2) holds for $F(t, x) = \nabla U(t, x)$ and we have,

$$(5.7) \quad c_1 \left[\int_{E_n} \|\nabla U(t, x)\|^p dx\right]^{1/p}$$
$$\leqslant \left\{\int_{E_n} \left[\int_{E_{n-1}} \frac{\chi_a(s, x-u)}{s^{n-1}} \left[\sum_0^n \left\| \frac{\partial \nabla U(t+s, u)}{\partial x_i} \right\|^2\right] du\, ds\right]^p dx\right\}^{1/p}$$
$$\leqslant c_2 \left[\int_{E_n} \|\nabla U(t, x)\|^p dx\right]^{1/p}$$

for every $t > 0$. To prove the theorem, it suffices to show that the preceding inequalities still hold for $t = 0$. We introduce the following notations:

$$\sigma = (s, u), \qquad d\sigma = \frac{\chi_a(s, u)}{s^{n-1}} du\, ds,$$

$$f_m(\sigma, x) = \left(\frac{\partial \nabla U(s+1/m, u+x)}{\partial t}, \ldots, \frac{\partial \nabla U(s+1/m, u+x)}{\partial x_i}\right),$$

and

$$g_m(\sigma, x) = \|f_m(\sigma, x)\|^2 = \sum_0^n \left\| \frac{\partial \nabla U(s+1/m, u+x)}{\partial x_i} \right\|^2.$$

Now, we have

$$|g_m - g_{m'}| \leqslant \big|\|f_m\|^2 - \|f_{m'}\|^2\big| \leqslant \|f_m - f_{m'}\| (\|f_m\| + \|f_{m'}\|).$$

Integrating with respect to $d\sigma$ and applying Schwarz's inequality follows that

$$\int |g_m - g_{m'}|\, d\sigma \leqslant \int \|f_m - f_{m'}\| (\|f_m\| + \|f_{m'}\|)\, d\sigma \leqslant \Big[\int \|f_m - f_{m'}\|^2 d\sigma\Big]^{1/2} \Big[\Big(\int \|f_m\|^2 d\sigma\Big)^{1/2} + \Big(\int \|f_{m'}\|^2 d\sigma\Big)^{1/2}\Big]$$

and integrating the $p/2$-power of this inequality and applying Schwarz's inequality once more, we obtain

$$\int \Big[\int |g_m - g_{m'}|\, d\sigma\Big]^{p/2} dx \leqslant 2^{p/2} \Big(\int \Big[\int \|f_m - f_{m'}\|^2 d\sigma\Big]^{p/2} dx\Big)^{1/2} \leqslant \Big\{\Big(\int \Big[\int \|f_m\|^2 d\sigma\Big]^{p/2} dx\Big)^{1/2} + \Big(\int \Big[\int \|f_{m'}\|^2 d\sigma\Big]^{p/2} dx\Big)^{1/2}\Big\}$$

which, using (5.7), gives

$$\int \Big[\int |g_m - g_{m'}|\, d\sigma\Big]^{p/2} dx \leqslant 2^{p/2} \cdot 2 \cdot K^{p/2} \cdot c_2^p \Big[\int_{E_n} \|\nabla U(1/m, x) - \nabla U(1/m', x)\|^p dx\Big]^{1/2}.$$

By proposition (1.10), the second member of the inequality above tends to zero for m and m' tending to infinity. This shows that the sequence $\{g_m\}$ is a Cauchy sequence in the complete metric space of mixed norm $L^{(1,p/2)}$. Therefore, the sequence $\{g_m\}$ is convergent in this space and there is a subsequence which is pointwise convergent to the limit of $\{g_m\}$. Now, since $\lim_{m\to\infty} g_m(\sigma, x) = \|\nabla U(s, u+x)\|^2$ at every (σ, x), the limit of $\{g_m\}$ in the space $L^{(1,p/2)}$ must coincide with $\|\nabla U(s, u+x)\|^2$ and we obtain,

$$\lim_{m\to\infty} \int \Big[\int g_m(\sigma, x)\, d\sigma\Big]^{p/2} dx = \lim_{m\to\infty} \int_{E_n} \Big[\int_{E_{n+1}^+} \frac{\chi_a(s, x-u)}{s^{n-1}} \Big[\sum_0^n \Big\| \frac{\partial \nabla U(s+1/m, u)}{\partial x_i} \Big\|^2\Big] du\,ds\Big]^{p/2} dx = \int_{E_n} \Big[\int_{E_{n-1}} \frac{\chi_a(s, x-u)}{s^{n-1}} \Big[\sum_0^n \Big\| \frac{\partial \nabla U(s, u)}{\partial x_i} \Big\|^2\Big] du\,ds\Big]^{p/2} dx. \tag{5.8}$$

On the other hand, from proposition (1.6) we obtain

$$\lim_{m\to\infty} \int_{E_n} \|\nabla U(1/m, x)\|^p\, dx = \int_{E_n} \|\nabla U(0, x)\|^p dx; \tag{5.9}$$

therefore, taking $t = 1/m$ in (5.7), the limits (5.8) and (5.9) show that (5.7) holds for $t = 0$ and the proof of the theorem is complete.

References

[1] M. Brelot, *Sur la théorie autonome des fonctions sousharmoniques*, Bull. Sc. Math. 65 (1941), p. 72-98.

[2] A. P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U. S. A. 53 (1965), p. 1092-1099.

[3] G. Choquet, *Potentiels sur un ensemble de capacité nulle. Suites de potentiels*, C. R. Acad. Sci. Paris 244 (1957), p. 1707-1710.

[4] G. C. Evans, *On potentials of positive mass*, Part I, Trans. Amer. Math. Soc. 37 (1935), p. 226-253; Part II, ibidem 38 (1935), p. 201-236.

[5] J. Horvath, *Sur les fonctions conjuguées à plusieurs variables*, Kon. Med. Acad. Wet. 16 (1953), p. 17-29.

[6] E. M. Stein, *Functions of Littlewood-Paley, Lusin, Marcinkiewicz*, Trans. Amer. Math. Soc. 88 (1958), p. 430-460.

[7] — *On the theory of harmonic functions of several variables II. Behavior near the boundary*, Acta Math. 106 (1960), p. 137-174.

[8] — and G. Weiss, *On the theory of harmonic functions of several variables I. The theory of H^p-spaces*, ibidem 103 (1960), p. 25-62.

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