(here $N|\partial\Omega$ denotes the restriction of the bundle) satisfies $\Delta'(p) = 1$ — identity in $T_p(X)$ (of course $T_p(H\times N|\partial\Omega) = T_p(X)$). But $F_2(sv, s\varphi)(p) = \Delta_2^{-1}\cdot\Delta_1\big(sv(q)+s\varphi(q)\big)$, where $q := \beta(sv, s\varphi)^{-1}(p)\xrightarrow[s\to 0]{} p$. So the later proof goes as for $p\in\operatorname{int}\Omega$. We get

$$\lim_{s\to 0}\frac{1}{s}F_2(sv, s\varphi)(p) = v(p)+\varphi(p) \text{ modulo } T_p(\partial\Omega)$$

and so, for $p\in\partial\Omega$, $[(\varkappa_2\circ\varkappa_1^{-1})'(0)u](p) = u(p)$.

References

[1] A. Bastiani, *Applications différentiables et variétés différentiable de dimension infinie*, J. Analyse Math. 13 (1964), p. 1-114.

[2] F. E. Browder, *Infinite dimensional manifolds and nonlinear elliptic eigenvalue problems*, Ann. of Math. 82 (1965), p. 459-477.

[3] P. Dedecker, *Calcul des variations, formes différentielles et champs géodésiques*, Colloque International de Géométrie Différentielle, Strasbourg 1953.

[4] J. Eells, Jr., *On the geometry of function spaces*, Symp. Inter. de Topologia Alg., Mexico 1956-1958, p. 303-308.

[5] — *Analysis on manifolds*, Mimeographed notes, Cornell University, Ithaca, N. Y. 1964-1965.

[6] — *A setting for global analysis*, Bull. Amer. Math. Soc. 72 (1966), p. 751-807.

[7] A. Grothendieck, *Sur les espaces $\mathcal{F}$ et $\mathcal{D}\mathcal{F}$*, Summa Brasil. Math. 3 (1954), p. 57-123.

[8] J. Kijowski and J. Komorowski, *On the differentiable structure in the set of sections over compact sets of fiber bundle*, Studia Math. 32 (1968), p. 189-205.

[9] J. Kijowski and W. Szczyrba, *On differentiability in an important class of locally convex spaces*, ibidem 30 (1968), p. 247-257.

[10] Лаврентев и Люстерник, *Основы вариационного исчисления*, Москва 1935.

[11] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology 2 (1963), p. 299-340.

[12] — and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. 70 (1964), p. 165-172.

[13] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. 80 (1964), p. 382-396.

[14] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall 1964.

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On the class $L\log L$, martingales, and singular integrals*

by

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In a recent paper, Stein [9] has characterized the class of functions $f(x)$ such that

$$\int |f|\log^+|f|\,dx < +\infty,$$

the class $L\log L$, in terms of the Hardy-Littlewood maximal function as follows:

(S) *The Hardy-Littlewood maximal function Mf is integrable if and only if f belongs to $L\log L$.*

On the other hand, Burkholder [1] has characterized $L\log L$ as follows:

(B) *Let $f_1, f_2, \ldots$ be a sequence of stochastically independent, identically distributed random variables; let $A_n = \big(\sum_{k=1}^{n} f_k\big)/n$. Then $A^* = \sup_n |A_n|$ is integrable if and only if f_1 belongs to $L\log L$.*

Stein proves Theorem (S) by first obtaining the converse to a well-known inequality, due to Calderón and Zygmund, for the distribution function of Mf. The final result is then obtained by integrating both sides of this converse inequality. Burkholder's method is entirely different. He derives the final result without benefit of a converse inequality.

In the first section of this paper, we show that both theorems may be viewed as facts about special martingales. The converse inequality for Burkholder's problem is stated as Theorem 1. Theorem 2 extends Stein's converse inequality to a class of martingales, in which his result is a special case. While the martingale approach reveals that (S) and (B) are essentially the same theorem, there are differences. Theorem (S), in the martingale setting, holds for nonnegative functions only. (This fact is obscured in Stein's paper because the Hardy-Littlewood maximal function is always non-negative). Theorem (B), however, is stronger in the sense that the function f_1 is not assumed to be bounded below.

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The second section deals with martingales and singular integrals. We show that the norm inequalities for singular integrals are special cases of a more general theorem about mappings defined on martingales. The discussion here revolves around the Calderón-Zygmund lemma ([4], p. 91) in relation to a decomposition for L^1-bounded martingales [7], and uses results from the first section.

1. Converse maximal inequalities and $L\log L$. Given a sequence of random variables $X_1, X_2, \ldots$ let $S_n = \sum_{k=1}^{n} X_k$ and $A_n = S_n/n$. We define a sequence of maximal functions:

$$A_n^* = \max_{1\leqslant k\leqslant n} A_k, \qquad n \geqslant 1.$$

Kolmogorov's inequality for the Strong Law of Large Numbers has the following two-sided version:

THEOREM 1. *Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed non-negative, integrable random variables. For every fixed n and $\lambda > \int X_1$ we have*

$$\frac{1}{2\lambda}\int\limits_{\{A_n^*>\lambda\geqslant A_n\}} X_1 dP \leqslant P(A_n^*>\lambda) \leqslant \frac{1}{\lambda}\int\limits_{\{A_n^*>\lambda\}} X_1 dP.$$

The right-hand side inequality has been known for a long time. The proof is a standard argument applied to the martingale

$$A = (A_n, A_{n-1}, \ldots, X_1)$$

(See Doob, [6], p. 341-342). The proof of its converse, the left-hand side inequality, is achieved in virtually the same way by replacing the usual lower bounds by upper bounds as follows:

Proof of Theorem 1. Define the stopping time

$$t = \max_{1\leqslant k<n}\{k\colon A_k > \lambda\}.$$

This is well-defined on the set where there is a crossing, i.e., on the set where $A_n^* > \lambda > A_n$. This is all that matters, and we shall leave it undefined elsewhere. The stopped sequence

$$A^t = (A_n, A_{n-1}, \ldots, A_{t-1}, A_t, A_t \ldots)$$

is a martingale. Furthermore, on the set where $t = k$,

$$\frac{S_{k+1}}{k+1} = A_{k+1} \leqslant \lambda \quad \text{and} \quad \frac{S_k}{k} = A_k > \lambda.$$

By virtue of the fact that $X_{k+1} \geqslant 0$, these two inequalities may be combined as

$$\lambda < A_k = \frac{S_k}{k} < \frac{S_k + X_{k+1}}{k} = \frac{k+1}{k}\,\frac{S_{k+1}}{k+1} \leqslant 2\lambda$$

so that $\lambda < A_t < 2\lambda$. Therefore we find that

$$P(A_n^* > \lambda) \geqslant P(A_t > \lambda) \geqslant \frac{1}{2\lambda}\int\limits_{\{A_t>\lambda\}} A_t = \frac{1}{2\lambda}\int\limits_{\{A_t>\lambda\}} X_1,$$

where the final equality holds because A^t is a martingale obtained by stopping $A = (A_n, A_{n-1}, \ldots, X_1)$. This completes the proof of Theorem 1.

The crux of the preceding argument should be clear. The special structure of the non-negative martingale A is such that:

(1) When the martingale crosses level λ, the excursion is bounded above by a multiple of λ.

This property is also shared by a class of martingales discussed by Chow [5] and the present writer [8], the so-called regular martingales. We state a definition here in the spirit of Proposition 1, p. 727 of [8]. Let $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ be an increasing sequence of σ-fields, where $\mathscr{F}_1$ is the trivial field consisting of the entire space and the empty set. A martingale $\sum_{k=1}^{n} u_k$, $n \geqslant 1$, with respect to $\mathscr{F}_n$, $n \geqslant 1$ is said to be (L^∞) *regular* if it can be written as $\sum_{k=1}^{n} v_k d_k$, $n \geqslant 1$, where:

(i) The random variables d_k, $k \geqslant 1$, are an orthonormal system of uniformly bounded martingale differences, i.e. $d_1 = 1$, $E(d_k^2 \| \mathscr{F}_{k-1}) = 1$, $E(d_k \| \mathscr{F}_{k-1}) = 0$, $k \geqslant 2$, and $d = \sup_k \|d_k\|_\infty < +\infty$.

(ii) The multipliers v_k, $k \geqslant 1$, are integrable and have the property that v_k is measurable with respect to $\mathscr{F}_{k-1}$, $k > 1$, $v_1 =$ constant.

A given triple $(\Omega, \mathscr{F}_n, n \geqslant 1, P)$ is an *(L^∞)-regular probability space* if every martingale sequence on it is (L^∞)-regular where the bound d is uniform for all martingales.

An interesting (L^∞)-regular probability space has been introduced by Chow [5]. The sequence $\mathscr{F}_n$, $n \geqslant 1$, is generated by successive refinements of a partition of the space into disjoint "atoms" of positive measure. The regularity condition is satisfied by requiring that for any two atoms E_k belonging to $\mathscr{F}_k$, E_{k+1} belonging to $\mathscr{F}_{k+1}$ with $E_k \supseteq E_{k+1}$, we have $0 < \delta \leqslant P(E_{k+1})/P(E_k)$ for some $\delta > 0$ and all $k > 1$. The representation of any martingale on this space in terms of v_k, d_k, $k \geqslant 1$, is stated as Proposition 1 of [8]. In particular, if the underlying space is the unit cube of R^n, partitioned into congruent sub-cubes with sides parallel to the coordinate axes, the resulting sequence of σ-fields is regular with $\delta = 2^{-n}$.

THEOREM 2. *Let* $f_n = \sum_{k=1}^{n} v_k d_k$, $n > 1$, *be a non-negative* (L^∞)*-regular martingale and* $f_n^* = \max_{1<k<n} f_k$, $n > 1$, *the corresponding sequence of maximal functions. Then for any* $\lambda > \|f_1\|_1$, $n > 0$,

$$\frac{C}{\lambda}\int_{\{f_n^*>\lambda\}} f_n\,d\boldsymbol{P} < \boldsymbol{P}(f_n^* > \lambda) < \frac{1}{\lambda}\int_{\{f_n^*>\lambda\}} f_n\,d\boldsymbol{P},$$

where the constant C *depends only on the* L^∞*-bound* $d = \sup_k \|d_k\|_\infty$.

Remarks. The proof consists of a stopping time argument parallel to the one given in Theorem 1. When the underlying space is the unit cube in R^n, with the usual partitioning, an equivalent argument is given by Stein [9] using the Calderón-Zygmund lemma mentioned above. The probability viewpoint, however, seems to us to be both more general and natural since all restrictions are on the functions without involving the geometry of the underlying space.

Proof of Theorem 2. Since the stopping time part of the argument has been given, we only prove that (L^∞)-regular martingales satisfy (1). The following argument, suitably amplified, is basic to the results in [8]. The sequence d_k, $k > 2$, has the property

$$\boldsymbol{E}(d_k^\pm \| \mathscr{F}_{k-1}) = \boldsymbol{E}(|d_k| \| \mathscr{F}_{k-1})/2 \geqslant \boldsymbol{E}(d_k^2/d \| \mathscr{F}_{k-1}) \geqslant (2d)^{-1},$$

where $d = \sup_k \|d_k\|_\infty$. By a lemma of Paley and Zygmund (see [8], Lemma 1),

$$\boldsymbol{P}(d_k^\pm \geqslant (4d)^{-1} \| \mathscr{F}_{k-1}) \geqslant (8d)^{-2} > 0.$$

If a crossing at the level λ occurs at the index k,

$$f_{k-1} \leqslant \lambda, \quad f_k = f_{k-1} + v_k d_k > \lambda.$$

Since we assume that f_k is non-negative, it follows that

$$0 \leqslant (v_k d_k)^- \leqslant f_{k-1} \leqslant \lambda.$$

Furthermore, v_k and f_{k-1} are both measurable with respect to $\mathscr{F}_{k-1}$. These facts, together with the lemma of Paley and Zygmund, lead us to conclude that $|v_k| \leqslant 4d\lambda$, and finally

$$f_k \leqslant \lambda + 4d^2\lambda = (1+4d^2)\lambda$$

as stated in (1).

The following variant which is proved in the same way as Theorem 2, is stated for future reference:

THEOREM 2a. *Let* $f_n = \sum_{k=1}^{n} v_k d_k$ *be an* (L^∞)*-regular martingale that is bounded below by the constant* $-\lambda$, $\lambda > \|f_n\|_1$. *Then for any* $a \geqslant 1$,

$$\frac{C}{a\lambda}\int_{\{f_n^*\geqslant a\lambda\}} f_n\,d\boldsymbol{P} \leqslant \boldsymbol{P}(f_n^* \geqslant a\lambda) \leqslant \frac{1}{a\lambda}\int_{\{f_n^*\geqslant a\lambda\}} f_n\,d\boldsymbol{P},$$

where C *depends only on* $d = \sup_k \|d_k\|_\infty$.

Remark. The two-sided inequalities just given can be integrated to obtain two characterizations of $L\log L$ for non-negative random variables. The regular martingale statement is essentially Stein's theorem. The Strong Law statement is a weak form of Burkholder's theorem in that the variables are assumed to be nonnegative. This asymmetry is crucial. Cancellation of positive and negative values occurs with regular martingales as the following simple example shows. Let $r_n(x)$, $n \geqslant 0$, be the Rademacher functions, $I_n(x)$ the indicator function of $[1/2^{n+1}, 1/2^n)$ which is measurable with respect to the σ-field generated by $r_0, r_1, \ldots, r_n$, $n \geqslant 1$. The sequence

$$f_n(x) = \sum_{k=1}^{n} 2^n n^{-2} I_n(x) r_{n+1}(x)$$

is an (L^∞)-regular martingale, and $\sup_n |f_n(x)| = |f(x)|$ which belongs to L^1 but not to $L\log L$. On the other hand, such cancellation cannot take place for the Strong Law. In this case $X^\pm$ act almost independently so that if $\sup |A_n|$ is integrable, a version of Fubini's theorem may be applied to show that both X^+ and X^- belong to $L\log L$. We omit the details; an interested reader may wish to provide them himself or consult Burkholder's paper [1] where a stronger theorem is proved.

2. On singular integrals and martingales. Some of the real-variable techniques used by Calderón and Zygmund [4] in their study of singular integrals may be applied to certain problems in probability theory. In particular, their decomposition of a function into "good" and "bad" parts should be compared with a similar result for L^1-bounded martingales [7]. This martingale decomposition has been used by us to obtain the norm inequalities for martingale transforms, due to Burkholder [2], which in turn should be compared with the analogous inequalities for singular integrals, due to Calderón and Zygmund.

In the present section, we develop this analogy further [1]. The norm inequalities for singular integrals and martingale transforms are proved

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[1] It is interesting to note here that both the theory of singular integrals and martingale transforms have a common root in the papers of Paley, Marcinkiewicz, and Zygmund.

in a unified way by appealing to the general martingale decomposition theorem. The remark that the singular integral inequalities constitute a special case of a general fact is a theorem whose proof relies on results of the previous section.

In outline, we proceed as follows. A singular integral operator is viewed as a mapping from martingales to functions. We may then apply a general martingale theorem regarding a class of admissible mappings to obtain the weak type and norm inequalities, at least for $1 \leqslant p \leqslant 2$. There is only one awkward point: singular integrals visibly satisfy only three of the four conditions for an admissible map. However, the martingales involved are (L^∞)-regular. This fact, combined with Theorem 2 from the previous section, shows that for these martingales, the fourth admissibility condition follows automatically from the other three.

Since what follows is, to a large extent, a commentary on [7], we adopt the notation used there. Let $f = (f_1, f_2, \ldots)$ denote a sequence of random variables such that f_n is measurable on the subfield $\mathscr{F}_n$, $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$, $n \geqslant 1$. Let $\varphi = (\varphi_1, \varphi_2, \ldots)$ be the f-increment sequence, so that $f_n = \sum_{k=1}^{n} \varphi$, $n \geqslant 1$. The L^p-norm of a martingale sequence f is defined by $\|f\|_p = \sup_k \|f_k\|_p$. The letter C = constant, not always the same from line to line. Random variable sequences are added in the natural way: $f+g = (f_1+g_1, f_2+g_2, \ldots)$, and maximal functions are denoted as usual: $f^* = \sup_n |f_n|$. Simple martingales, by analogy with simple functions, are those for which $f_{n+k} = f_n$, $k \geqslant 1$, for some n. As usual, in probability theory, we assume that the total measure of the space is unity. This restriction is often unnecessary, but we shall adhere to it in order to simplify the discussion. In particular when we refer to the special case where the domain is R^n, we mean the *unit cube* in R^n.

The following analog of the Calderón-Zygmund decomposition of functions is proved in [7]:

THEOREM. *Let f be an L^1-bounded martingale. Corresponding to any $\lambda > 0$ the martingale f may be decomposed into three martingales a, b, d, so that $f = a+b+d$.*

(i) *The martingale $a = (a_1, a_2, \ldots)$, $a_n = \sum_{k=1}^{n} \alpha_k$ is L^1-bounded, $\|a\|_1 \leqslant C\|f\|_1$ and the increment sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ is such that $P(\alpha^* \neq 0) \leqslant C\|f\|_1/\lambda$.*

(ii) *The martingale $d = (d_1, d_2, \ldots)$, $d_n = \sum_{k=1}^{n} \delta_k$, is uniformly bounded, $\|d\|_\infty \leqslant C$, $\|d\|_1 \leqslant C\|f\|_1$, and $\|d\|_2^2 \leqslant C\lambda \|f\|_1$.*

(iii) *The martingale $b = (b_1, b_2, \ldots)$, $b_n = \sum_{k=1}^{n} \beta_k$, is absolutely convergent, $\|\sum_{k=1}^{\infty} |\beta_k|\|_1 \leqslant \|f\|_1$.*

Actually, the Calderón-Zygmund lemma corresponds to the special case where the underlying space is R^n, and the σ-fields are generated by successive refinements of a partition of the space into congruent cubes. Their *two-fold* decomposition, in which (iii) never appears, may be extended quite easily to the (L^∞)-regular case. One of our purposes in this section is to try to clarify the role of (iii) and its relation to the (L^∞)-regular case. To this end, we point out the following additional fact that is not mentioned in [7]: For the purpose of proving weak-type (1, 1) norm inequalities, it is clearly sufficient to restrict ourselves to the case $\lambda > \|f\|_1$. In this case, in the decomposition given above, the martingale b is *bounded below by* $-\lambda$. In fact, in the notation of [7],

$$b_n = \sum_{k=1}^{n} \varepsilon_k - E(\varepsilon_k \| \mathscr{F}_{k-1}) I(s \geqslant k)$$
$$\geqslant -\sum_{k=1}^{n} E(\varepsilon_k \| \mathscr{F}_{k-1}) I(s \geqslant k) > -\lambda, \quad n > 1,$$

and since $\lambda \geqslant \|f\|_1$, by definition $\varepsilon_1 = 0$.

We now revise the definition of class $\mathscr{B}$ of mappings, given in [7], to accommodate both probability and singular integral applications.

Definition. A mapping T is said to be *of class $\mathscr{B}$ relative to* $(\Omega, \mathscr{F}_n, n \geqslant 1, P)$ if its domain is the collection of simple martingales and its range is a collection of random variables such that:

1. T is quasi-linear: $|T(f+g)| \leqslant C(|Tf| + |Tg|)$,
2. T is local: $|Tf| \leqslant |(Tf)_1| + |(Tf)_2|$ such that

(2a) $P(|(Tf)_1| \neq 0) \leqslant CP(f^* \neq 0)$;

(2b) $\|(Tf)_2\|_1 \leqslant C\|f\|_1$.

3. The mapping T satisfies the following norm inequalities:

(3a) $\|Tf\|_2 \leqslant C\|f\|_2$;

(3b) $\|(Tf)\chi(|Tf| > \lambda)\|_1 \leqslant C\|\sum_{i=1}^{n} |\varphi_i|\|_1$, where $f = (f_1, \ldots, f_n)$, $f_k = \sum_{i=1}^{k} \varphi_i$, $k \geqslant 1$, is bounded below by $-\lambda$, $\lambda > 0$. The symbol $\chi(|Tf| > \lambda)$ denotes the indicator function of the set in parentheses.

The modifications in the definition from [7] are in 2b and 3b. In probability applications the mappings are usually strictly local ($f^*(x) = 0$ implies $(Tf)(x) = 0$). To handle singular integrals, where "smearing" is present, we have added 2b. Condition 3b here is a weakening of the corresponding statement in [7] in that we demand that the inequalities hold only on functions that are bounded below.

The decomposition theorem allows us to prove the following

PROPOSITION 1. *A Class $\mathscr{B}$ mapping T is of weak type* (1, 1) *on the class of simple martingales.*

We remark that, usually, we may extend T to all L^1-bounded martingales as a weak type (1, 1) mapping. This possibility is realized when, for example, the map is linear.

The proof of Proposition 1, given in [7], requires some modification because of the addition of 2b and 3b. Condition 2 is to be applied to the martingale a, which is supported on a small set. We have

$$|Ta| \leqslant |(Ta)_1| + |(Ta)_2|,$$

so that it suffices to show

$$\boldsymbol{P}\big(|(Ta)_i| > \lambda\big) \leqslant C\|f\|_1/\lambda, \qquad i = 1, 2.$$

First,

$$\boldsymbol{P}\big(|(Ta)_1| > \lambda\big) \leqslant C\boldsymbol{P}(a^* \neq 0) \leqslant C\|f\|_1/\lambda$$

and, then

$$\boldsymbol{P}\big(|(Ta)_2| > \lambda\big) \leqslant C\|(Ta)_2\|_1/\lambda \leqslant C\|a\|_1/\lambda \leqslant C\|f\|_1/\lambda.$$

Condition 3b is to be applied to the martingale b of the decomposition. As we have remarked, b is uniformly bounded below by $-\lambda$ if $\lambda > \|f\|_1$. In this case,

$$\lambda\boldsymbol{P}(|Tb| > \lambda) \leqslant \|T(b)\chi(|Tb| > \lambda)\|_1 \leqslant C\Big\|\sum_{k=1}^{\infty}|\beta_k|\Big\|_1 \leqslant C\|f\|_1.$$

The treatment of the martingale $d = (d_1, d_2, \ldots)$ is the same as in [7].

Singular integral operators in periodic case, are Class $\mathscr{B}$ mappings if we let the probability space be the unit cube C^n in R^n and $\mathscr{F}_n$, $n \geqslant 1$ be the σ-fields generated by the usual partitioning. In this case, $(C^n, \mathscr{F}_n, n \geqslant 1, dx)$ is an (L^∞)-regular probability space. Integrable functions $f(x)$ are uniquely associated with martingales $\boldsymbol{E}(f\|\mathscr{F}_n)$, $n \geqslant 1$, and if $f = (f_1, \ldots, f_n)$ is a simple martingale on this space, f_n is a simple function. Given a kernel $k(x)$, define

$$T(f)(x) = \int_{C^n} k(x-y)f_n(y)\,dy.$$

Following Calderón, ([3], p. 436), we check that T is linear, L^2-bounded, and satisfies condition 2 for a class $\mathscr{B}$ mapping. We now may apply the Calderón-Zygmund two-fold decomposition to conclude that T is of weak type (1, 1) on simple martingales, and ultimately on all L^1-bounded martingales, without mention of the condition 3b. That is, in return for restricting the probability space, we have avoided the invocation of condition 3b. However, this is an illusion. The following theorem states, in effect, that condition 3b follows automatically in this case so that singular integrals on C^n may be considered as special cases of Class $\mathscr{B}$ mappings.

THEOREM. *Let $(\Omega, \mathscr{F}_n, n \geqslant 1, \boldsymbol{P})$ be an (L^∞)-regular probability space. Let T be a quasi-linear, local, L^2-bounded mapping from simple martingales to random variables. Then*

$$\|(Tf)\chi(|Tf| > \lambda)\|_1 \leqslant C\Big\|\sum_{k=1}^{\infty}|\varphi_i|\Big\|_1,$$

where f is a simple martingale bounded below by $-\lambda$, $\lambda > \|f\|$. That is, T satisfies condition 3b *and so is of Class $\mathscr{B}$.*

Proof. In what follows, to avoid subscripts we let f stand for the simple martingale $f = (f_1, f_2, \ldots, f_n)$, $f_k = \sum_{i=1}^{k}\varphi_i$, as well as the final term, i.e., $f = f_n$. Since T is of weak type (1, 1), we may adapt Marcinkiewicz's computation as follows: Let $f_{t(a)}$ be the martingale f stopped when it crosses $a\lambda$, $a \geqslant 1$. Then

$$\boldsymbol{P}(|Tf| > a\lambda) \leqslant \boldsymbol{P}(|T(f-f_{t(a)}| > a\lambda/2) + \boldsymbol{P}(|Tf_{t(a)}| > a\lambda/2)$$

$$\leqslant \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}|f-f_{t(a)}|\,d\boldsymbol{P} + \frac{C}{(a\lambda)^2}\int_{\{f_{t(a)} > a\lambda\}}|f_{t(a)}|^2\,d\boldsymbol{P} + \frac{C}{(a\lambda)^2}\int_{\{f_{t(a)} \leqslant a\lambda\}}|f_{t(a)}|^2\,d\boldsymbol{P}.$$

Since $0 \leqslant f^- \leqslant \lambda \leqslant a\lambda \leqslant f_{t(a)}$ on the range of integration, we may write

$$\frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}|f-f_{t(a)}|\,d\boldsymbol{P} \leqslant \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}|f| + f_{t(a)}\,d\boldsymbol{P}$$

$$\leqslant \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}f + 2f^- + f_{t(a)}\,d\boldsymbol{P} \leqslant \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}f_{t(a)}\,d\boldsymbol{P}$$

$$= \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}f\,d\boldsymbol{P} \leqslant C\boldsymbol{P}(f^* > a\lambda),$$

where the last inequality is obtained by using Theorem 2. Also

$$\frac{C}{(a\lambda)^2}\int_{\{f_{t(a)} > a\lambda\}}|f_{t(a)}|^2\,d\boldsymbol{P} \leqslant \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}f_{t(a)}\,d\boldsymbol{P}$$

$$= \frac{C}{a\lambda}\int_{\{f_{t(a)} > a\lambda\}}f\,d\boldsymbol{P} \leqslant C\boldsymbol{P}(f^* > a\lambda),$$

and

$$\frac{C}{(a\lambda)^2}\int_{\{f_{t(a)} \leqslant a\lambda\}}|f_{t(a)}|^2\,d\boldsymbol{P} = \frac{C}{(a\lambda)^2}\int_{\{f \leqslant a\lambda\}}|f|^2\,d\boldsymbol{P}.$$

If we recombine all of these inequalities and integrate with respect to a, we find

$$\int_1^\infty \boldsymbol{P}(|Tf| > a\lambda)\,da \leqslant C\int_1^\infty \Big(\boldsymbol{P}(f^* > a\lambda) + C\frac{1}{(a\lambda)^2}\int_{\{f\leqslant a\lambda\}} |f|^2 d\boldsymbol{P}\Big)\,da.$$

The left-hand side of the above inequality may be written

$$\int_1^\infty \boldsymbol{P}(|Tf| > a\lambda)\,da \geqslant \lambda^{-1}\|(Tf)\chi(|Tf| > \lambda)\|_1 - C\lambda^{-1}\|f\|_1.$$

The right-hand side consists of two terms:

$$\int_1^\infty \boldsymbol{P}(f^* > a\lambda)\,da \leqslant \lambda^{-1}\|f^*\|_1; \qquad \int_1^\infty 1/(a\lambda)^2\Big(\int_{\{f\leqslant a\lambda\}} |f|^2 d\boldsymbol{P}\Big)\,da \leqslant \lambda^{-1}\|f\|_1.$$

This leads to the final inequality

$$\|(Tf)\chi(|Tf| > \lambda)\|_1 \leqslant C(\|f^*\|_1 + \|f\|_1) \leqslant C\Big\|\sum_{k=1}^\infty |\varphi_i|\Big\|_1.$$

This completes the proof.

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References

[1] D. L. Burkholder, *Successive conditional expectation of an integrable function*, Ann. Math. Statist. 33 (1962), p. 887-893.

[2] — *Martingale transforms*, ibidem 37 (1966), p. 1494-1504.

[3] A. P. Calderón, *Singular integrals*, Bull. Amer. Math. Soc. 72 (1966), p. 427-465.

[4] — and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), p. 85-139.

[5] Y. S. Chow, *Martingales in a σ-finite measure space indexed by directed sets*, Trans. Amer. Math. Soc. 97 (1960), p. 254-285.

[6] J. L. Doob, *Stochastic processes*, New York 1953.

[7] R. F. Gundy, *A decomposition for L^1-bounded martingales*, Ann. Math. Statist. 39 (1968), p. 134-138.

[8] — *The martingale version of a theorem of Marcinkiewicz and Zygmund*, ibidem 38 (1967), p. 725-734.

[9] E. M. Stein, *Note on the class* $L\log L$, Studia Math. (to appear).

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