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# On the mean values of an entire function and its derivatives represented by Dirichlet series, II

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**1.** Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \tag{1.1}$$

where $0 < \lambda_n < \lambda_{n+1} \to \infty$, $s = \sigma + it$, represent an everywhere absolutely convergent Dirichlet series. If

$$M(\sigma, f) = \operatorname*{l.u.b.}_{-\infty < t < \infty} |f(\sigma + it)|,$$

then $\log M(\sigma, f)$ is an increasing convex function of $\sigma$, and

$$\varrho = \limsup_{\sigma\to\infty} \frac{\log\log M(\sigma, f)}{\sigma} \tag{1.2}$$

is called the *Ritt-order* of $f(s)$.

We define the mean values of $f(s)$ as

$$W(\sigma) \equiv W(\sigma, f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt, \tag{1.3}$$

$$w_\delta(\sigma) \equiv w_\delta(\sigma, f) = \frac{1}{e^{\delta\sigma}} \int_{-\infty}^{\sigma} W(x) e^{\delta x} dx \tag{1.4}$$

$$= \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_{-\infty}^{\sigma} \int_{-T}^{T} |f(x+it)|^2 e^{\delta x} dx\, dt, \quad 0 < \delta < \infty.$$

One of the present authors has obtained in [2] a few properties of the mean values $v_\delta(\sigma, f)$ of $f(s)$, where $v_\delta(\sigma, f)$ are defined as

$$v_\delta(\sigma, f) = \frac{1}{e^{\delta\sigma}} \int_{0}^{\sigma} W(x) e^{\delta x} dx = w_\delta(\sigma, f) - A, \quad 0 < \delta < \infty, \tag{1.5}$$

where $A$ is a real constant depending on $\delta$ and $f$. It easily follows from (1.4) and (1.5) that for large $\sigma$ the behaviour of $w_\delta(\sigma, f)$ is the same as that of $v_\delta(\sigma, f)$, and all the results that have been derived for $v_\delta(\sigma, f)$ in [2] can be obtained for $w_\delta(\sigma, f)$. Thus we shall have ([2], p. 309)

$$(1.6) \qquad w_\delta(\sigma, f') - w_\delta(\sigma, f)\left(\frac{\log w_\delta(\sigma, f)}{2\sigma}\right)^2 \geqslant 0\,, \qquad \sigma > \sigma_0\,,$$

where $w_\delta(\sigma, f')$ is the mean value of $f'(s)$, the derivative of $f(s)$, i.e.,

$$\begin{aligned} w_\delta(\sigma, f') &= \frac{1}{e^{\delta\sigma}} \int_{-\infty}^{\sigma} W(x, f') e^{\delta x} dx \\ &= \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_{-\infty}^{\sigma} \int_{-T}^{T} |f'(x+it)|^2 e^{\delta x} dx\, dt\,, \qquad 0 < \delta < \infty\,. \end{aligned}$$

Rahman ([4], p. 1114) has proved the following lemma:

LEMMA A. *If in* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *all the coefficients* $\{a_n\}$, $n = 1, 2, \ldots$, *are non-negative, then for large values of* $\sigma$,

$$M(\sigma, f') \geqslant M(\sigma, f)\frac{\log M(\sigma, f)}{\sigma}\,.$$

In the present paper, using Lemma A, we establish a refinement of inequality (1.6) and derive some more results for $w_\delta(\sigma, f')$ and $w_\delta(\sigma, f)$.

**2.** We establish a refinement of (1.6) in the following theorem:

THEOREM 1. *If* $w_\delta(\sigma, f')$ *is the mean value of* $f'(s)$, *the first derivative of* $f(s)$, *then*

$$(2.1) \quad w_\delta(\sigma, f') - w_\delta(\sigma)\left(\frac{\log w_\delta(\sigma)}{2\sigma}\right)^2 \geqslant \frac{1}{(2\sigma)^2} w_\delta(\sigma) \log w_\delta(\sigma) \log\left(\frac{\log w_\delta(\sigma)}{2\sigma}\right)$$

*for* $\sigma > \sigma_0$.

Proof. For all $\sigma < \infty$, we have ([2], p. 308)

$$W(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}\,.$$

Therefore, by (1.4),

$$(2.2) \qquad w_\delta(\sigma) = \frac{1}{e^{\delta\sigma}} \int_{-\infty}^{\sigma} \left(\sum_{n=1}^{\infty} |a_n|^2 e^{2x\lambda_n}\right) e^{\delta x} dx\,.$$

The series under the integral sign is a uniformly convergent series of continuous functions for $\sigma < \infty$ and hence

$$(2.3) \qquad w_\delta(\sigma) = \frac{1}{e^{\delta\sigma}} \sum_{n=1}^{\infty} \int_{-\infty}^{\sigma} |a_n|^2 e^{2x\lambda_n} e^{\delta x}\, dx = \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n} .$$

Similarly, it can be shown that

$$(2.4) \qquad w_\delta(\sigma, f') = \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n} .$$

Now, consider the functions represented by the Dirichlet series

$$(2.5) \qquad \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(s\lambda_n) , \qquad \sum_{n=1}^{\infty} \frac{\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(s\lambda_n) .$$

For the first function we have, for every $\sigma < \infty$,

$$\max_{\operatorname{Re} s < 2\sigma} \left| \frac{|a_n|^2}{2\lambda_n + \delta} \exp(s\lambda_n) \right| \leqslant \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) ,$$

and $\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n)$ is convergent, its sum being $w_\delta(\sigma, f)$. Thus, the series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(s\lambda_n)$ represents an entire function $g(s)$, while the series $\sum_{n=1}^{\infty} \frac{\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(s\lambda_n)$ is $g'(s)$. The functions $g(s)$ and $g'(s)$ clearly satisfy the hypotheses of Lemma A, and so we have, for large values of $\sigma$,

$$(2.6) \qquad \sum_{n=1}^{\infty} \frac{\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \geqslant \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right) \frac{\log\left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right)}{2\sigma}$$

and

$$(2.7) \qquad \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \geqslant \left( \sum_{n=1}^{\infty} \frac{\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right) \frac{\log\left( \sum_{n=1}^{\infty} \frac{\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right)}{2\sigma} .$$

Thus, for sufficiently large $\sigma$, say $\sigma > \sigma_0$, we have from (2.4), (2.6) and (2.7),

$$w_\delta(\sigma, f') = \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}$$

$$\geqslant \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}\right) \left(\frac{\log\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}\right)}{2\sigma}\right)^2 +$$

$$+ \frac{1}{(2\sigma)^2} \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}\right) \log\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}\right) \times$$

$$\times \log\left(\frac{\log\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} e^{2\sigma\lambda_n}\right)}{2\sigma}\right)$$

$$= w_\delta(\sigma) \left(\frac{\log w\,(\sigma)}{2\sigma}\right)^2 + \frac{1}{(2\sigma)^2} w_\delta(\sigma) \log w_\delta(\sigma) \log\left(\frac{\log w_\delta(\sigma)}{2\sigma}\right),$$

which is (2.1).

Next we prove

THEOREM 2. *If $f(s)$ is of finite Ritt-order, then*

$$\log w_\delta(\sigma, f') \sim \log w_\delta(\sigma, f) \quad \textit{as } \sigma \to \infty\,. \tag{2.8}$$

Proof. It is known ([3], p. 140) that if $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ is of finite Ritt-order $\varrho$, then, for every $\varepsilon > 0$,

$$M(\sigma, f') \leqslant M(\sigma, f) \exp\{\sigma(\varrho + \varepsilon)\} \tag{2.9}$$

if $\sigma$ is sufficiently large.

Since

$$w_\delta(\sigma, f) = \frac{1}{e^{\delta\sigma}} \int_{-\infty}^{\sigma} W(x) e^{\delta x} dx \leqslant \frac{1}{e^{\delta\sigma}} \{M(\sigma, f)\}^2 \int_{-\infty}^{\sigma} e^{\delta x} dx = \frac{1}{\delta} \{M(\sigma, f)\}^2\,, \tag{2.10}$$

the Ritt-order of the function represented by the series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2s\lambda_n)$ is at most $\varrho$. Hence, for every $\varepsilon > 0$ and sufficiently large $\sigma$, (2.9) gives

$$\sum_{n=1}^{\infty} \frac{2\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \leqslant \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n)\right) \exp\{\sigma(\varrho + \varepsilon)\}\,. \tag{2.11}$$

Since the Ritt-order of a function is the same as that of its derivative, the Ritt-order of the function represented by the series $\sum_{n=1}^{\infty} \frac{2\lambda_n |a_n|^2}{2\lambda_n + \delta} \times$ $\times \exp(2s\lambda_n)$ is not greater than $\varrho$. So, (2.9) gives

$$(2.12) \quad \sum_{n=1}^{\infty} \frac{4\lambda_n^2 |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \leqslant \left\{ \sum_{n=1}^{\infty} \frac{2\lambda_n |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right\} \exp\{\sigma(\varrho + \varepsilon)\} .$$

Inequalities (2.11) and (2.12) lead to

$$(2.13) \qquad \begin{aligned} w_\delta(\sigma, f') &= \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \\ &\leqslant \frac{1}{4} \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right) \exp\{2\sigma(\varrho + \varepsilon)\} \\ &= \frac{1}{4} w_\delta(\sigma, f) \exp\{2\sigma(\varrho + \varepsilon)\} . \end{aligned}$$

This fact, together with (1.6), implies that, for functions of finite Ritt-order,

$$\log w_\delta(\sigma, f') \sim \log w_\delta(\sigma, f) \qquad \text{as } \sigma \to \infty .$$

**3.** Azpeitia [1] has proved that if

$$(3.1) \qquad \lim_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log n} = \infty ,$$

then the Ritt-order $\varrho$ of the function $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ is given by

$$\varrho = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} .$$

Hence if $f(s)$, defined by (1.1), is of Ritt-order $\varrho$ $(0 \leqslant \varrho \leqslant \infty)$ and (3.1) holds, the Ritt-order of the function defined by the series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \times$ $\times \exp(2s\lambda_n)$ is also $\varrho$. So, if (3.1) is satisfied, then

$$(3.2) \quad \limsup_{\sigma\to\infty} \frac{\log\log w_\delta(\sigma, f)}{\sigma} = \limsup_{\sigma\to\infty} \frac{\log\log \left\{ \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n + \delta} \exp(2\sigma\lambda_n) \right\}}{\sigma} = \varrho .$$

Further, if

$$\liminf_{n\to\infty} \frac{\lambda_n}{\log n} = \frac{1}{D} > 0\,, \tag{3.3}$$

then ([4], p. 1117)

$$\liminf_{\sigma\to\infty} \frac{\log\log W(\sigma, f)}{\sigma} = \lambda\,, \tag{3.4}$$

where

$$\lambda = \liminf_{\sigma\to\infty} \frac{\log\log M(\sigma, f)}{\sigma}$$

is the lower order of $f(s)$.

If (3.3) holds, then

$$\begin{aligned} W(\sigma, f) &= \sum_{n=1}^{\infty} |a_n|^2 \exp(2\sigma\lambda_n) \\ &\leqslant \Big(\sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n+\delta} \exp\{2\lambda_n(\sigma+\tfrac{1}{2}D+1+\varepsilon)\}\Big)\times \\ &\quad \times\Big(\sum_{n=1}^{\infty} (2\lambda_n+\delta)\exp\{-2\lambda_n(\tfrac{1}{2}D+1+\varepsilon)\}\Big) \\ &< Kw_\delta(\sigma+\tfrac{1}{2}D+1+\varepsilon, f)\,. \end{aligned}$$

From this, (2.10) and (3.4) it follows that

$$\liminf_{\sigma\to\infty} \frac{\log\log w_\delta(\sigma, f)}{\sigma} = \lambda\,. \tag{3.5}$$

We are now in a position to prove

THEOREM 3. *If* (3.1) *is satisfied, then*

$$\limsup_{\sigma\to\infty} \frac{\log\{w_\delta(\sigma, f')/w_\delta(\sigma, f)\}}{\sigma} = 2\varrho \tag{3.6}$$

*and if* (3.3) *holds, then*

$$\liminf_{\sigma\to\infty} \frac{\log\{w_\delta(\sigma, f')/w_\delta(\sigma, f)\}}{\sigma} = 2\lambda\,. \tag{3.7}$$

Proof. From (1.6), (2.13) and (3.2), the first result (3.6) easily follows. Now, (1.6) and (3.5) give

$$\liminf_{\sigma\to\infty} \frac{\log\{w_\delta(\sigma, f')/w_\delta(\sigma, f)\}}{\sigma} \geqslant 2\lambda\,. \tag{3.8}$$

To obtain the reverse inequality, we proceed as follows. It is known ([3], p. 139) that the inequality

$$M(\sigma, f') \leqslant \frac{1}{\gamma} M(\sigma+\gamma, f), \qquad \gamma > 0,$$

holds for the entire function $f$ defined by (1.1). Applying this result successively to the entire functions

$$\sum_{n=1}^{\infty} \frac{2\lambda_n |a_n|^2}{2\lambda_n+\delta} \exp\{2\lambda_n(s-2\gamma)\}, \qquad \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n+\delta} \exp\{2\lambda_n(s-2\gamma)\},$$

we get

$$\begin{aligned}
(3.9) \qquad w_\delta(\sigma-2\gamma, f') &= \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^2}{2\lambda_n+\delta} \exp\{2\lambda_n(\sigma-2\gamma)\} \\
&\leqslant \frac{1}{4\gamma} \sum_{n=1}^{\infty} \frac{2\lambda_n |a_n|^2}{2\lambda_n+\delta} \exp\{2\lambda_n(\sigma-\gamma)\} \\
&\leqslant \frac{1}{4\gamma^2} \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n+\delta} \exp(2\sigma\lambda_n) \\
&= \frac{1}{4\gamma^2} w_\delta(\sigma, f), \qquad \gamma > 0.
\end{aligned}$$

Now, since ([2], p. 308) $\log w_\delta(\sigma, f)$ is an increasing convex function of $\sigma$, we have

$$(3.10) \qquad \log w_\delta(\sigma, f) = \log w_\delta(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \Phi(x)\, dx,$$

where $\sigma_0 < \sigma$ and $\Phi(x)$ is a non-decreasing function of $x$. So, if $\lambda < \infty$ and $\varepsilon$ is a fixed positive number, (3.5) and (3.10) give

$$\int_{\sigma}^{\sigma+2} \Phi(x)\, dx < \log w_\delta(\sigma+2, f) < \exp\{(\sigma+2)(\lambda+\varepsilon)\}$$

for a sequence of values of $\sigma$ tending to infinity; say, for $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n, \ldots \to \infty$. Since $\Phi(x)$ is non-decreasing, we get

$$2\Phi(\sigma_n) < \exp\{(\sigma_n+2)(\lambda+\varepsilon)\}, \qquad n = 1, 2, \ldots,$$

and since $\varepsilon$ is arbitrary, we can write

$$\Phi(\sigma_n) < \exp\{\sigma_n(\lambda+\varepsilon)\}$$

for sufficiently large $n$. (3.10) then gives

$$\log w_\delta(\sigma_n, f) < \log w_\delta(\sigma_n-2\gamma, f) + 2\gamma \exp\{\sigma_n(\lambda+\varepsilon)\},$$

and if we take $\gamma = \frac{1}{2}\exp\{-\sigma_n(\lambda+\varepsilon)\}$, we obtain

$$\log w_\delta(\sigma_n, f) < \log w_\delta(\sigma_n - 2\gamma, f) + 1$$

for sufficiently large $n$. Substituting this value of $\gamma$ and the corresponding estimate for $w_\delta(\sigma_n, f)$ in (3.9), we see that, $\varepsilon > 0$ being given, there exists a sequence of values of $\sigma$ such that

$$w_\delta(\sigma - 2\gamma, f') < w_\delta(\sigma - 2\gamma, f)\exp\{2\sigma(\lambda+\varepsilon)\}\,. \tag{3.11}$$

Since $\gamma < 1$, we can even write

$$w_\delta(\sigma - 2\gamma, f') \leqslant w_\delta(\sigma - 2\gamma, f)\exp\{2(\sigma - 2\gamma)(\lambda+\varepsilon)\}$$

instead of (3.11). It follows that

$$\liminf_{\sigma\to\infty} \frac{\log\{w_\delta(\sigma, f')/w_\delta(\sigma, f)\}}{\sigma} \leqslant 2\lambda\,. \tag{3.12}$$

Thus, if $\lambda < \infty$, the inequality in (3.8) can be replaced by equality. If $\lambda = \infty$, then from (3.8) we have

$$\liminf_{\sigma\to\infty} \frac{\log\{w_\delta(\sigma, f')/w_\delta(\sigma, f)\}}{\sigma} = \infty\,.$$

**References**

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