ANNALES
POLONICI MATHEMATICI
XXII (1969)

On squares of differentiable functions

by M. KUCZMA (Katowice)

Squares are meant here in the sense of the superposition, i.e. $f^2(x) = f(f(x))$. Let D_k^r denote the family of all the mappings

$$f\colon R^k \to R^k$$

that are of class C^r in R^k, $1 \leqslant r \leqslant +\infty$, and have a positive Jacobian in R^k. Further let Q_k^r be the set of all squares of the functions from D_k^r:

$$Q_k^r = \{f\colon f = \varphi^2,\ \varphi \in D_k^r\}.$$

In [2] Z. Moszner has asked:

1° Whether $Q_k^r = D_k^r$,

or, more generally,

2° Whether every function $f \in D_k^r$ can be represented as a superposition of a finite number of functions from Q_k^r?

In the preceding paper [1] we have proved that the answer to 1° is negative, even in the one-dimensional case ($k = 1$). In the present paper we answer 2° in the positive in the case $k = 1$. For $k > 1$ question 2° remains still open.

Thus the purpose of the present paper is to prove the following

THEOREM. *Every function $f \in D_1^r$, $1 \leqslant r \leqslant +\infty$, can be represented as a superposition of at most four functions from the class Q_1^r.*

The proof of this theorem will be based on several lemmas.

LEMMA 1. *For any system of real numbers $\varepsilon > 0$, $d > 0$, $c_0 > 0$, $c_1 > 0$, $c_2, c_3, \ldots$, such that*

(1) $$c_0 < dc_1,$$

there exists a function $g(x)$ of class C^∞ on $\langle 0, d\rangle$ such that

(2) $$g^{(i)}(0) = 0, \quad g^{(i)}(d) = c_i, \quad i = 0, 1, 2, \ldots,$$

and

(3) $$0 \leqslant g'(x) \leqslant c_1 + \varepsilon \quad \textit{for } x \in \langle 0, d\rangle.$$

Proof. We shall only outline the proof. By a B-function we understand any function of the form

$$B(x) = \begin{cases} P(x)\exp[(x-a)^{-2}(x-b)^{-2}] & \text{for } x \in (a, b)\,, \\ 0 & \text{for } x \in (-\infty, a\rangle \cup \langle b, +\infty)\,, \end{cases}$$

where $a < b$ are constants and $P(x)$ is an arbitrary function which is positive and of class C^∞ on $\langle a, b\rangle$. As is well known, every B-function is of class C^∞ in $(-\infty, +\infty)$.

By Whitney's theorem [3] there exists a function $g_1(x)$ of class C^∞ on $\langle \frac{1}{2}d, d\rangle$ such that

$$g_1^{(i)}(\tfrac{1}{2}d) = 0\,, \quad g_1^{(i)}(d) = c_{i+2}\,, \quad i = 0, 1, 2, \ldots$$

We define $g_1(x)$ on $\langle 0, \frac{1}{2}d\rangle$ as a B-function with the support $\langle 0, \frac{1}{2}d\rangle$. Thus g_1 is of class C^∞ on $\langle 0, d\rangle$, positive on $(0, \frac{1}{2}d)$, and fulfils the conditions

$$g_1^{(i)}(0) = 0\,, \quad g_1^{(i)}(d) = c_{i+2}\,, \quad i = 0, 1, 2, \ldots$$

Let $M = \sup\limits_{\langle 0,d\rangle} |g_1(x)|$ and let $\delta > 0$ be chosen so that

$$0 < d-\delta < \min(\varepsilon/2M, c_1/2M)\,. \tag{4}$$

By adding, if necessary, to g_1 a B-function with the support $\langle 0, d\rangle$, we obtain a function g_2, of class C^∞ on $\langle 0, d\rangle$, positive on $(0, \delta)$, and such that

$$g_2^{(i)}(0) = 0\,, \quad g_2^{(i)}(d) = c_{i+2}\,, \quad i = 0, 1, 2, \ldots\,, \tag{5}$$

$$|g_2(x)| \leqslant 2M \quad \text{for } x \in \langle \delta, d\rangle\,. \tag{6}$$

Adding or subtracting from g_2 a finite number of B-functions with supports contained in $(0, \delta)$ we arrive at a new function, $g_3(x)$, of class C^∞ on $\langle 0, d\rangle$, positive on $(0, \delta)$ and fulfilling the following conditions:

$$g_3^{(i)}(0) = 0\,, \quad g_3^{(i)}(d) = c_{i+2}\,, \quad i = 0, 1, 2, \ldots\,, \tag{7}$$

$$\int_0^\delta g_3(t)\,dt = c_1 - \int_\delta^d g_2(t)\,dt\,. \tag{8}$$

(It follows from (4) and (6) that the expression on the right-hand side of (8) is positive.) Since

$$g_2(x) = g_3(x) \quad \text{for } x \in \langle \delta, d\rangle\,, \tag{9}$$

relation (8) implies that

$$\int_0^d g_3(t)\,dt = c_1\,. \tag{10}$$

Now we put

$$g_4(x) = \int_0^x g_3(t)\,dt\,. \tag{11}$$

By (7) and (10) the function $g_4(x)$ fulfils the conditions

$$g_4^{(i)}(0) = 0\,, \quad g_4^{(i)}(d) = c_{i+1}\,, \quad i = 0, 1, 2, \ldots \tag{12}$$

Moreover, since $g_3(t) > 0$ in $(0, \delta)$, we have by (11), (8), (4) and (6) for $x \in \langle 0, \delta\rangle$

$$0 \leqslant g_4(x) \leqslant g_4(\delta) = c_1 - \int_\delta^d g_2(t)\,dt \leqslant c_1 + \int_\delta^d |g_2(t)|\,dt < c_1 + \varepsilon\,.$$

On the other hand, for $x \in \langle \delta, d\rangle$ we have by (11), (9) and (8)

$$g_4(x) = g_4(\delta) + \int_\delta^x g_2(t)\,dt = c_1 - \int_x^d g_2(t)\,dt\,,$$

whence by (4) and (6)

$$|g_4(x) - c_1| \leqslant \int_x^d |g_2(t)|\,dt < \min(\varepsilon, c_1)\,.$$

Consequently

$$0 \leqslant g_4(x) < c_1 + \varepsilon \quad \text{for } x \in \langle 0, d\rangle\,. \tag{13}$$

Adding and subtracting a suitable combination of B-functions with supports contained in $(0, d)$ we may make $g_4(x)$ to satisfy additionally the condition

$$\int_0^d g_4(t)\,dt = c_0 \tag{14}$$

(cf. (1)). In virtue of (12), (14) and (13) the function

$$g(x) = \int_0^x g_4(t)\,dt$$

fulfils conditions (2) and (3) and evidently is of class C^∞ on $\langle 0, d\rangle$.

Lemma 2. *Let $F(x)$ be a function of class C^r on $(-\infty, +\infty)$, $1 \leqslant r \leqslant +\infty$, such that*

$$\limsup_{x\to-\infty} F(x) < +\infty \tag{15}$$

and

$$F'(x) > -1 \quad \textit{for } x \in (-\infty, +\infty)\,. \tag{16}$$

Then there exists a function $H(x)$ of class C^r on $(-\infty, +\infty)$ fulfilling the following conditions:

(17) $$H(x) > 0 \quad \textit{for } x \in (-\infty, +\infty)\,,$$

(18) $$H(x) > F(x) \quad \textit{for } x \in (-\infty, +\infty)\,,$$

(19) $$0 \leqslant H'(x) < F'(x)+1 \quad \textit{for } x \in (-\infty, +\infty)\,.$$

Proof. At first we shall define an auxiliary function $G(x)$. By (15) there exists a positive constant M such that

(20) $$F(x) \leqslant M \quad \text{for } x \in (-\infty, 0)\,.$$

We put $x_0 = 0$,

(21) $$G(x) = M+2 \quad \text{for } x \in (-\infty, 0\rangle = (-\infty, x_0\rangle\,,$$

and

(22) $$G(x) = \begin{cases} G(x_{2n}) & \text{for } x \in (x_{2n}, x_{2n+1}\rangle\,, \\ F(x)+1+k_n(x-x_{2n+1}) & \text{for } x \in (x_{2n+1}, x_{2n+2}\rangle\,, \\ & \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where

(23) $$x_{2n+1} = \inf\{x > x_{2n}\colon F(x) > G(x_{2n})-1\}\,, \quad n = 0, 1, 2, \ldots\,,$$

(24) $$x_{2n+2} = x_{2n+1}+2\,, \quad n = 0, 1, 2, \ldots\,,$$

and

(25) $$k_n = \tfrac{1}{2}(m_n+1)\,,$$

where

(26) $$m_n = \max\Big(\tfrac{1}{2}\,, \sup_{\langle x_{2n+1}, x_{2n+2}\rangle}\big(-F'(x)\big)\Big)\,.$$

If the set on the right-hand side of (23) is empty, then the sequence $\{x_n\}$ is finite, with last term x_{2n}, and $G(x) = G(x_{2n})$ in $(x_{2n}, +\infty)$. If x_{2n+1} exists, then

(27) $$F(x_{2n+1}) = G(x_{2n})-1 = G(x_{2n+1})-1\,.$$

On the other hand, we have for x_{2n}

(28) $$F(x_{2n}) \leqslant G(x_{2n})-2\,, \quad n = 0, 1, 2, \ldots$$

In fact, for $n = 0$ (28) holds in view of (20) and (21), and for $n > 0$ we have by (22), (24) and (25)

$$G(x_{2n})-F(x_{2n}) = 1+k_{n-1}(x_{2n}-x_{2n-1}) = 1+2k_{n-1} = 2+m_{n-1} > 2\,.$$

By (23) and (24) the sequence x_n increases to infinity and thus formulae (21) and (22) define the function $G(x)$ in the whole $(-\infty, +\infty)$.

$G(x)$ is continuous in $(-\infty, +\infty)$ (the continuity at x_{2n+1} results from (27)) and of class C^r in the set

$$(-\infty, x_1) \cup \bigcup_{n=1}^{\infty} (x_n, x_{n+1}) . \tag{29}$$

In intervals $(-\infty, x_1)$ and (x_{2n}, x_{2n+1}), $n = 1, 2, \ldots$, $G(x)$ is constant and thus in view of (16)

$$0 = G'(x) < F'(x) + 1 .$$

In intervals (x_{2n+1}, x_{2n+2}), $n = 0, 1, 2, \ldots$, we have

$$G'(x) = F'(x) + k_n . \tag{30}$$

By (26) and (16) we have $-F'(x) \leqslant m_n < 1$, whence according to (25)

$$-F'(x) < k_n < 1 ,$$

and by (30)

$$0 \leqslant G'(x) < F'(x) + 1 . \tag{31}$$

Thus relation (31) holds in the whole set (29). This shows that $G(x)$ is non-decreasing and hence positive in $(-\infty, +\infty)$, since M has been assumed positive. It follows from the definition of G that

$$G(x) \geqslant F(x) + 1 > F(x) \quad \text{in } (-\infty, +\infty) .$$

As we see, the function G has all the properties required except that it is not of class C^r in $(-\infty, +\infty)$. Now we shall modify the definition of G in a neighbourhood of each x_n, $n = 1, 2, \ldots$

Let us fix an x_{2n+1}. Let us put

$$\varepsilon = \tfrac{1}{3}(1 - k_n) \tag{32}$$

and let U be an open interval containing x_{2n+1} such that

$$|F'(x) - F'(x_{2n+1})| < \varepsilon \quad \text{for } x \in U \tag{33}$$

and

$$F(x) < G(x_{2n+1}) \quad \text{for } x \in U . \tag{34}$$

Next we fix $u_{2n+1}, v_{2n+1} \in U$ such that

$$x_{2n} < u_{2n+1} < x_{2n+1} < v_{2n+1} < x_{2n+2} , \tag{35}$$

$$F'(v_{2n+1}) > 0 , \tag{36}$$

$$G(v_{2n+1}) - G(x_{2n+1}) < (v_{2n+1} - u_{2n+1}) k_n < (v_{2n+1} - u_{2n+1}) G'(v_{2n+1}) . \tag{37}$$

In particular, condition (36) can be realized in virtue of (23) and (27).

By Lemma 1 there exists a function $h_{2n+1}(x)$ of class C^∞ in $\langle u_{2n+1}, v_{2n+1}\rangle$ and fulfilling the conditions

$$(38)\qquad h_{2n+1}^{(i)}(u_{2n+1}) = G^{(i)}(u_{2n+1}), \qquad h_{2n+1}^{(i)}(v_{2n+1}) = G^{(i)}(v_{2n+1}), \qquad i = 0, 1, 2, \ldots, r,$$

$$(39)\qquad 0 \leqslant h'_{2n+1}(x) \leqslant G'(v_{2n+1}) + \varepsilon \quad \text{for } x \in \langle u_{2n+1}, v_{2n+1}\rangle .$$

Relation (39) guarantees that $h_{2n+1}(x)$ is non-decreasing on $\langle u_{2n+1}, v_{2n+1}\rangle$, and hence by (38) ($i = 0$) it is positive, and by (34)

$$F(x) < G(x_{2n+1}) = G(u_{2n+1}) = h_{2n+1}(u_{2n+1}) \leqslant h_{2n+1}(x) \quad \text{for } x \in \langle u_{2n+1}, v_{2n+1}\rangle .$$

Moreover, we have in virtue of (39), (30), (33) and (32) for $x \in \langle u_{2n+1}, v_{2n+1}\rangle$

$$h'_{2n+1}(x) \leqslant G'(v_{2n+1}) + \varepsilon = F'(v_{2n+1}) + k_n + \varepsilon$$
$$< F'(x_{2n+1}) + k_n + 2\varepsilon < F'(x) + k_n + 3\varepsilon = F'(x) + 1 .$$

A similar construction leads us to a function $h_{2n}(x)$ which is positive and of class C^∞ in an interval $\langle u_{2n}, v_{2n}\rangle$ such that $x_{2n-1} < u_{2n} < x_{2n} < v_{2n} < x_{2n+1}$, and fulfils the conditions:

$$h_{2n}^{(i)}(u_{2n}) = G^{(i)}(u_{2n}), \qquad h_{2n}^{(i)}(v_{2n}) = G^{(i)}(v_{2n}), \qquad i = 0, 1, \ldots, r,$$

$$F(x) < h_{2n}(x) \quad \text{for } x \in \langle u_{2n}, v_{2n}\rangle ,$$

$$0 \leqslant h'_{2n}(x) < F'(x) + 1 \quad \text{for } x \in \langle u_{2n}, v_{2n}\rangle ,$$

$n = 1, 2, \ldots$

Putting

$$H(x) = \begin{cases} h_n(x) & \text{for } x \in \langle u_n, v_n\rangle , \quad n = 1, 2, \ldots, \\ G(x) & \text{for } x \in (-\infty, +\infty)\setminus \bigcup_{n=1}^{\infty} \langle u_n, v_n\rangle , \end{cases}$$

we obtain the required function $H(x)$ fulfilling all the conditions of the lemma.

Lemma 3. *If $f \in D_1^r$ and $f(x) \neq x$ in $(-\infty, +\infty)$, then $f \in Q_1^r$.*

This has been proved in [1].

For an arbitrary function f on $(-\infty, +\infty)$ we put

$$(40)\qquad \begin{cases} L^+ = \limsup\limits_{x\to+\infty} \big(f(x) - x\big), & L_+ = \liminf\limits_{x\to+\infty} \big(f(x) - x\big), \\ L^- = \limsup\limits_{x\to-\infty} \big(f(x) - x\big), & L_- = \liminf\limits_{x\to-\infty} \big(f(x) - x\big). \end{cases}$$

LEMMA 4. *If $f \in D_1^r$ and at least one of limits* (40) *is finite, then f is a superposition of two functions from the class Q_1^r.*

Proof. Let us suppose that $L^- < \infty$. We shall distinguish two cases.

I. $\lim\limits_{x\to+\infty} f(x) = +\infty$.

The function $F(x) = f(x) - x$ fulfils the hypotheses of Lemma 2 and consequently there exists a function $H(x)$ of class C^r in $(-\infty, +\infty)$ fulfilling conditions (17) through (19). We put

$$f_1(x) = f(x) - H(x) . \tag{41}$$

The function f_1 is of class C^r in $(-\infty, +\infty)$ and by (19)

$$f_1'(x) = f'(x) - H'(x) > f'(x) - \big(F'(x)+1\big) = 0 .$$

Consequently $f_1 \in D_1^r$. Further we have by (18)

$$f_1(x) < f(x) - F(x) = x \quad \text{ for } x \in (-\infty, +\infty) , \tag{42}$$

which shows that $\lim\limits_{x\to-\infty} f_1(x) = -\infty$. Since f_1 is increasing, the limit $\lim\limits_{x\to+\infty} f_1(x)$ exists. If the sequence x_n occurring in the proof of Lemma 2 is finite, then $H(x) = \text{const}$ for large x and consequently

$$\lim_{x\to+\infty} f_1(x) = +\infty . \tag{43}$$

If x_n is infinite, then for $x = x_{2n}$, $n = 1, 2, \ldots$, we have

$$H(x_{2n}) \leqslant H(v_{2n}) = G(v_{2n}) = G(x_{2n}) = F(x_{2n}) + 1 + 2k_n$$
$$< F(x_{2n}) + 3 = f(x_{2n}) - x_{2n} + 3 ,$$

whence

$$f_1(x_{2n}) = f(x_{2n}) - H(x_{2n}) > x_{2n} - 3 ,$$

and (43) holds all the same. Consequently the function

$$f_2(x) = x + H\big(f_1^{-1}(x)\big) \tag{44}$$

is defined and of class C^r in $(-\infty, +\infty)$, moreover, since f_1^{-1} is increasing, $f_2'(x) \geqslant 1$ and thus $f_2 \in D_1^r$. According to (17)

$$f_2(x) > x \quad \text{ for } x \in (-\infty, +\infty) . \tag{45}$$

By (42), (45) and Lemma 3 we have $f_1 \in Q_1^r$ and $f_2 \in Q_1^r$.

By (44) and (41) $f_2(f_1(x)) = f_1(x) + H(x) = f(x)$.

II. $\lim_{x\to+\infty} f(x) < +\infty$.

Then $f(x) < x$ for large x and we can find a positive constant H such that $H > f(x) - x$ for $x \in (-\infty, +\infty)$.

Then the functions $f_1(x) = f(x) - H$, $f_2(x) = x + H$, both belong to D_1^r and fulfil (42) and (45), respectively. Thus they belong to Q_1^r and the lemma follows as in the preceding case.

If $L^- = \infty$, but another of limits (40) is finite, the proof is analogous.

Proof of the theorem. In view of Lemma 4 it is enough to consider the case where all the four limits (40) are infinite ([1]). We take a $z \in (-\infty, +\infty)$ such that $f(z) = z$ and arbitrarily close to z there exist $x > z$ such that $f(x) > x$. (The existence of such a z is guaranteed by the condition $L^+ = \infty$.) Next we fix u, v such that $u < z < v$ and $f'(v) > 1$, $f(v) > v$. By Lemma 1 there exists ([2]) a function $g(x)$ of class C^∞ in $\langle u, v\rangle$ such that

$$g^{(i)}(u) = 0 \quad \text{for } i = 0, 1, 2, \ldots,$$

$$g^{(i)}(v) = f^{(i)}(v) \quad \text{for } i = 2, \ldots, r,$$

$$g(v) = f(v) - v, \quad g'(v) = f'(v) - 1,$$

$$g'(x) \geqslant 0 \quad \text{for } x \in \langle u, v\rangle .$$

The function

$$f_1(x) = \begin{cases} x & \text{for } x \in (-\infty, u), \\ x + g(x) & \text{for } x \in \langle u, v\rangle, \\ f(x) & \text{for } x \in (v, +\infty), \end{cases}$$

evidently belongs to D_1^r. Moreover, $\lim_{x\to-\infty} f_1(x) = -\infty$, and, since $L^+ = +\infty$, $\lim_{x\to+\infty} f_1(x) = +\infty$. Consequently the function $f_2(x) = f(f_1^{-1}(x))$ also belongs to D_1^r, and $f_2(f_1(x)) = f(x)$. Now

$$\lim_{x\to-\infty} (f_1(x) - x) = \lim_{x\to+\infty} (f_2(x) - x) = 0 .$$

By Lemma 4 each of the functions f_1, f_2 can be represented as a superposition of two functions from the class Q_1^r, and consequently f is a composition of four functions from the class Q_1^r, which was to be proved.

([1]) A function with such a property is constructed in [1], example V.

([2]) The condition $f(v) - v < (f'(v) - 1)(v - u)$, corresponding to (1), need not be fulfilled here, but this may have influence only on the second inequality in (3), which is irrelevant in the present case.

References

[1] M. Kuczma, *Fractional iteration of differentiable functions,* this fascicule, pp. 217-227.
[2] Z. Moszner, *Problème* P.2, Aequationes Math. 1 (1968), p. 150.
[3] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc. 36 (1934), pp. 63-89.

Reçu par la Rédaction le 6. 6. 1968