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On the maximum of the functional $|a_3 - a a_2^2|$ in the classes of quasi-starlike functions

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Consider the class $\mathfrak{G}^M$ of functions

(1) $$g(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots, \quad a_1 = 1/M, \text{ for } |z| < 1$$

determined by the equation

(2) $$F(g(z)) = (1/M)F(z) \quad \text{for } |z| < 1,$$

where

(3) $$F(\zeta) = \zeta + B_2\zeta^2 + B_3\zeta^3 + \dots \quad \text{for } |\zeta| < 1$$

is a starlike function, and $M > 1$. Let $\mathfrak{G}_m^M$ denote the class of functions $g(z) \in \mathfrak{G}^M$ which satisfy the equation

(4) $$G(g(z)) = (1/M)G(z) \quad \text{for } |z| < 1,$$

where

(5) $$G(\zeta) = \zeta / \prod_{k=1}^{m} (1 - \sigma_k \zeta)^{\beta_k} \quad \text{for } |\zeta| < 1$$

and

(6) $$\sigma_k = e^{i\varphi_k}, \quad \varphi_k \text{ real}, \quad k = 1, 2, \dots, m,$$
$$\sigma_j \neq \sigma_{j'} \quad \text{for } j \neq j';\ j, j' = 1, 2, \dots, m,$$

(7) $$\beta_k, \quad \beta_k \text{ positive}, \quad k = 1, 2, \dots, m, \quad \beta_1 + \beta_2 + \dots + \beta_m = 2,$$

are arbitrary parameters. Functions of the class $\mathfrak{G}^M$ are said to be *quasi-starlike* [1]. Next, let $\tilde{S}_M^*$ denote the class of functions

(8) $$\mathcal{G}(z) = z + A_2 z^2 + A_3 z^3 + \dots \quad \text{for } |z| < 1,$$

of the form $\mathcal{G}(z) = Mg(z)$, where $g(z) \in \mathfrak{G}^M$. Clearly,

(9) $$|\mathcal{G}(z)| \leqslant M \text{ for } |z| < 1, \quad \text{and} \quad A_n \doteq M a_n \text{ for } n = 2, 3, \dots$$

Functions (8) are called *normalized bounded quasi-starlike*. Finally, considering the class S^* of starlike functions (3) we easily see that, by the relation

$$(10)\qquad F(z) = \mathcal{G}(z) + (B_2/M)[\mathcal{G}(z)]^2 + (B_3/M^2)[\mathcal{G}(z)]^3 + \ldots,$$

which is itself a consequence of (2) and (3), each of them is the limit of some functions of form (8) as $M \to \infty$.

In this paper we evaluate the maximum of the functional $|a_3 - aa_2^2|$ in the classes $\mathfrak{G}^M$; a being an arbitrary real parameter. This result will imply the maximal value of the same functional in the classes $\widetilde{S}_M^*$ and S^*. Since if a function $g(z)$ of form (1) belongs to $\mathfrak{G}^M$, then so does $e^{i\theta}g(e^{-i\theta}z)$, θ being an arbitrary real number, the maximum of the functional under consideration equals to the maximum of

$$(11)\qquad \operatorname{re}(a_3 - aa_2^2) = x_3 - a(x_2^2 - y_2^2) = H(x_2, x_3, y_2),$$

as it may easily be verified. Here we have set

$$(12)\qquad a_n = x_n + iy_n, \quad n = 2, 3, \ldots$$

Consequently, we may confine ourselves to considering the latter functional. The above remarks imply that the maximum of (11) is non-negative. Moreover, for the extremal functions, when functional (11) attains its maximum, we have

$$(12')\qquad \operatorname{re}(a_3 - aa_2^2) = a_3 - aa_2^2.$$

Let us introduce for arbitrary functions (5) and (1), the latter being determined by (4), the following notations:

$$(13)\qquad G(\zeta) = \zeta + b_2\zeta^2 + b_3\zeta^3 + \ldots \quad \text{for } |\zeta| < 1,$$

$$(14)\qquad [g(z)]^p = a_p^{(p)}z^p + a_{p+1}^{(p)}z^{p+1} + \ldots \quad \text{for } |z| < 1,\ p = 1, 2, \ldots$$

Now, suppose that (1) is an extremal function of the class $\mathfrak{G}^M$, for which functional (11) attains its maximum. Theorem 2 in [1] implies that function (1) belongs either to $\mathfrak{G}_2^M$ or to $\mathfrak{G}_1^M$. We will consider, separately, both cases.

1. The case where the extremal function belongs to $\mathfrak{G}_2^M$. We assume that the extremal function (1) satisfies (4) and apply notations (13) and (14). Let us notice first that for the coefficients of (1) and (13) we have

$$(15)\qquad b_1a_k^{(1)} + b_2a_k^{(2)} + \ldots + b_ka_k^{(k)} - (1/M)b_k = 0, \quad k = 1, 2;$$
$$a_1 = 1/M,\ b_1 = 1.$$

Next, it is well known that, by (11), the function $\tilde{\mathfrak{R}}(\zeta)$, defined by formula (72) in [1], in our case may be written in the form

$$\tilde{\mathfrak{R}}(\zeta) = \sum_{p=1}^{2} \frac{1}{p}\left(\frac{D_p}{\zeta^p} + \bar{D}_p \zeta^p\right) + \lambda \,, \tag{16}$$

where λ is a real constant and

$$D_p = \sum_{l=1}^{3-p} C_l \sum_{k=l}^{3-p} [a_{k+p}^{(l+p)} - a_k^{(l)}] H_{k+p} \,, \qquad p = 1, 2 \,, \tag{17}$$

$$H_n = H'_{x_n} - iH'_{y_n} \,, \qquad n = 2, 3 \,, \tag{18}$$

$$C_1 = \begin{vmatrix} 1 & 0 \\ b_2 & 1 \end{vmatrix} = 1 \,, \qquad C_2 = \begin{vmatrix} 1 & 1 \\ 2b_2 & b_2 \end{vmatrix} = -b_2 \,. \tag{19}$$

On the other hand, by (5), in our case function (13) may be taken in the form

$$G(\zeta) = \zeta/(1-\sigma_1\zeta)^{\beta_1}(1-\sigma_2\zeta)^{\beta_2} \,, \qquad \beta_1 + \beta_2 = 2 \,, \tag{19'}$$

and we have

$$b_2 = \beta_1\sigma_1 + \beta_2\sigma_2 \,. \tag{20}$$

From (1) and (14) we get

$$a_2^{(2)} = a_1^2 \,, \qquad a_3^{(2)} = 2a_1a_2 \,, \qquad a_3^{(3)} = a_1^3 \,. \tag{21}$$

Further, by (15) with $k = 2$ we see that, since $1/M = a_1$,

$$a_2 = -b_2(a_1^2 - a_1) \,. \tag{22}$$

Finally, from (18), (11) and (12) we obtain

$$H_2 = -2\alpha a_2 \,, \qquad H_3 = 1 \,. \tag{23}$$

Now, by (19)-(23), we get subsequently from (17) the relations

$$D_1 = (\beta_1\sigma_1 + \beta_2\sigma_2)(a_1^2 - a_1)[1 - 3a_1 + 2\alpha(a_1^2 - a_1)] \,, \qquad D_2 = a_1^3 - a_1 \,. \tag{24}$$

Applying (24) to (16) we have

$$\begin{aligned} \tilde{\mathfrak{R}}(\zeta) = {} & (\beta_1\sigma_1 + \beta_2\sigma_2)(a_1^2 - a_1)[1 - 3a_1 + 2\alpha(a_1^2 - a_1)]\zeta^{-1} + \\ & + \tfrac{1}{2}(a_1^3 - a_1)\zeta^{-2} + (\beta_1\bar{\sigma}_1 + \beta_2\bar{\sigma}_2)(a_1^2 - a_1)[1 - 3a_1 + 2\alpha(a_1^2 - a_1)]\zeta + \\ & + \tfrac{1}{2}(a_1^3 - a_1)\zeta^2 + \lambda \,. \end{aligned} \tag{25}$$

Besides, it is well known [1] that the function $\mathfrak{R}(\zeta)$ has two distinct double zeros σ_1 and σ_2 in the case under consideration. Therefore (25) and the Viète formulae yield

$$\sigma_1^2\sigma_2^2 = 1 , \quad \sigma_1+\sigma_2 = \frac{3a_1-1-2\,aa_1(a_1-1)}{a_1+1}\,(\beta_1\sigma_1+\beta_2\sigma_2) .$$

Consequently, by the second of conditions (19′), we obtain the following system of relations:

$$\sigma_1^2\sigma_2^2 = 1 , \tag{26}$$

$$\sigma_1+\sigma_2 = \varkappa(\beta_1\sigma_1+\beta_2\sigma_2) , \tag{27}$$

$$\beta_1+\beta_2 = 2 , \tag{28}$$

where, for brevity, we have set

$$\varkappa = [3a_1-1-2aa_1(a_1-1)]/(a_1+1) . \tag{28'}$$

We shall find those solutions σ_k, β_k $(k = 1, 2)$ of system (26)-(28), for which the corresponding function (1) gives the maximum of (11) and, in addition, it belongs to $\mathfrak{S}_2^M$. The other solutions are to be excluded. We distinguish three particular cases.

1.1. The particular case where $\varkappa \neq 0$ and $\varkappa \neq 1$. From (27) and (28) we have immediately

$$\begin{aligned} \beta_1\sigma_1+\beta_2\sigma_2 &= (1/\varkappa)(\sigma_1+\sigma_2) , \\ \beta_1\bar{\sigma}_1+\beta_2\bar{\sigma}_2 &= (1/\varkappa)(\bar{\sigma}_1+\bar{\sigma}_2) , \\ \beta_1+\beta_2 &= 2 , \end{aligned} \tag{29}$$

whence

$$\begin{vmatrix} \sigma_1 & \sigma_2 & (1/\varkappa)(\sigma_1+\sigma_2) \\ \bar{\sigma}_1 & \bar{\sigma}_2 & (1/\varkappa)(\bar{\sigma}_1+\bar{\sigma}_2) \\ 1 & 1 & 2 \end{vmatrix} = 0 . \tag{30}$$

Consequently, since $\bar{\sigma}_1 = 1/\sigma_1$ and $\bar{\sigma}_2 = 1/\sigma_2$, we easily get

$$\sigma_1^2-\sigma_2^2 = 0 . \tag{31}$$

The system of equations (26) and (31) has the following solutions:

(i) $\sigma_1 = 1$, $\sigma_2 = 1$;	(v) $\sigma_1 = i$, $\sigma_2 = -i$;
(ii) $\sigma_1 = -1$, $\sigma_2 = -1$;	(vi) $\sigma_1 = -i$, $\sigma_2 = i$;
(iii) $\sigma_1 = i$, $\sigma_2 = i$;	(vii) $\sigma_1 = 1$, $\sigma_2 = -1$;
(iv) $\sigma_1 = -i$, $\sigma_2 = -i$;	(viii) $\sigma_1 = -1$, $\sigma_2 = 1$.

Let us notice now that σ_1 and σ_2 are assumed to be distinct since the extremal function $g(z)$ belongs to $\mathfrak{G}_2^M$. Hence solutions (i)-(iv) are to be rejected. Next, we see that (v), and (vi) as well, implies $\beta_1 = \beta_2$ as a consequence of (29) (the first equation). Therefore, by the third of equations (29), we get $\beta_1 = \beta_2 = 1$. Consequently, by (4) and (19′), the function $g(z)$ that corresponds to solution (v), $\beta_1 = \beta_2 = 1$ of system (26)-(28) is determined by the equation

$$\frac{g(z)}{1+[g(z)]^2} = (1/M)\frac{z}{1+z^2}\,. \tag{32}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = -(1/M)(1-1/M^2) < 0\,. \tag{32'}$$

By (12′) it follows from (32′) that the function $g(z)$ determined by (32) is not an extremal function for which functional (11) attains its maximum, because this maximum is non-negative as it has been remarked at the beginning of our paper. Analogously, we conclude that in case of (vii) as well as (viii) we also have $\beta_1 = \beta_2 = 1$. Consequently, by (4) and (19′), the function $g(z)$ that corresponds to solution (vii), $\beta_1 = \beta_2 = 1$ of system (26)-(28) is determined by the equation

$$\frac{g(z)}{1-[g(z)]^2} = (1/M)\frac{z}{1-z^2}\,. \tag{33}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = (1/M)(1-1/M^2)\,, \tag{34}$$

where, by (12′), the expression on the left-hand side of (34) is equal to its real part.

Considering (M, a) as coordinates of points in the plane, we may formulate the results of Section 1.1 as follows: if a point (M, a) is situated in the half-plane $M > 1$ with two curves excluded, given by the equations

$$\varkappa = 0\,, \quad \text{i.e. } a = M(3-M)/2(1-M)\,, \quad \text{where } 1 < M < \infty\,, \tag{35}$$

and

$$\varkappa = 1\,, \quad \text{i.e. } a = M\,, \quad \text{where } 1 < M < \infty\,, \tag{36}$$

respectively, then the extremal function $g(z) \in \mathfrak{G}_2^M$, for which functional (11) attains its maximum, is determined by equation (33), and the expression $a_3 - aa_2^2$ that corresponds to this function is given by formula (34).

1.2. The particular case where $\varkappa = 0$. If $\varkappa = 0$, then, as it can easily be seen, system (26)-(28) has infinitely many solutions of the form:

(ix) $\sigma_1 = i$, $\sigma_2 = -i$; $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1 + \beta_2 = 2$;

(x) $\sigma_1 = -i$, $\sigma_2 = i$; $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1 + \beta_2 = 2$;

(xi) $\sigma_1 = 1$, $\sigma_2 = -1$; $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1 + \beta_2 = 2$;

(xii) $\sigma_1 = -1$, $\sigma_2 = 1$; $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1 + \beta_2 = 2$.

We notice first that, by (4) and (19′), the function $g(z)$ that corresponds to (ix) ((x) gives the same) is determined by the equation

$$\frac{g(z)}{[1-ig(z)]^{\beta_1}[1+ig(z)]^{\beta_2}} = (1/M)\frac{z}{(1-iz)^{\beta_1}(1+iz)^{\beta_2}}. \tag{37}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = -(1/M)(1-1/M^2) < 0 . \tag{37′}$$

Similarly as in case of the function $g(z)$ determined by (32) it follows from (37′) and (12′) that the function $g(z)$ determined by (37) is not the desired extremal function. Next, we see that, by (4) and (19′), the function $g(z)$ that corresponds to (xi) ((xii) gives the same) is determined by the equation

$$\frac{g(z)}{[1-g(z)]^{\beta_1}[1+g(z)]^{\beta_2}} = (1/M)\frac{z}{(1-z)^{\beta_1}(1+z)^{\beta_2}}. \tag{38}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = (1/M)(1-1/M^2) , \tag{39}$$

where, by (12′), the expression on the left-hand side of (39) is equal to its real part.

The conclusion of Section 1.2 is that if a point (M, a) is situated on curve (35), then the extremal function $g(z) \in \mathfrak{S}_2^M$, for which functional (11) attains its maximum, is determined by equation (38), and the expression $a_3 - aa_2^2$ that corresponds to this function is given by formula (39). We recall that there are infinitely many functions satisfying (39).

1.3. The particular case where $\varkappa = 1$. If $\varkappa = 1$ we consider β_1 and β_2 as unknowns in system (27)-(28), and conclude that the principal determinant of this system equals $\sigma_1 - \sigma_2 \neq 0$ (the extremal function belongs to $\mathfrak{S}_2^M$). By a direct calculation, $\beta_1 = \beta_2 = 1$. Hence it easily follows that system (26)-(28) has infinitely many solutions of the form:

(xiii) $\sigma_1 = e^{i\varphi}$, $\sigma_2 = e^{-i\varphi}$, $\beta_1 = \beta_2 = 1$; $\varphi \neq k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ arbitrary and real;

(xiv) $\sigma_1 = e^{i\varphi}$, $\sigma_2 = -e^{-i\varphi}$, $\beta_1 = \beta_2 = 1$; $\varphi \neq (2k+1)\frac{1}{2}\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ arbitrary and real;

k being an arbitrary integer. Now we notice that, by (4) and (19′), the function $g(z)$ that corresponds to (xiii) is determined by the equation

$$\frac{g(z)}{[1-e^{i\varphi}g(z)][1-e^{-i\varphi}g(z)]} = (1/M)\frac{z}{(1-e^{i\varphi}z)(1-e^{-i\varphi}z)}\,. \tag{40}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = -(1/M)(1-1/M^2) < 0\,. \tag{40′}$$

Similarly as in case of the function $g(z)$ determined by (32) it follows from (40′) and (12′) that the function $g(z)$ determined by (40) is not the desired extremal function. Next, we see that, by (4) and (19′), the function $g(z)$ that corresponds to (xiv) is determined by the equation

$$\frac{g(z)}{[1-e^{i\varphi}g(z)][1+e^{-i\varphi}g(z)]} = (1/M)\frac{z}{(1-e^{i\varphi}z)(1+e^{-i\varphi}z)}\,. \tag{41}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = (1/M)(1-1/M^2)\,, \tag{42}$$

where, by (12′), the expression on the left-hand side of (42) is equal to its real part.

The conclusion of Section 1.3 is that if a point (M, a) is situated on curve (36), then the extremal function $g(z) \in \mathfrak{S}_2^M$, for which functional (11) attains its maximum, is determined by equation (41), and the expression $a_3 - aa_2^2$ that corresponds to this function is given by formula (42). We recall that there are infinitely many functions satisfying (42).

2. The case where the extremal function belongs to $\mathfrak{S}_1^M$. In the case under consideration the function $G(\zeta)$ of form (5) is defined by

$$G(\zeta) = \zeta/(1-\sigma\zeta)^2\,, \quad \text{where } |\sigma| = 1\,.$$

Consequently, by (4), the extremal function $g(z) \in \mathfrak{S}_1^M$, for which functional (11) attains its maximum, is determined by the equation

$$\frac{g(z)}{[1-\sigma g(z)]^2} = (1/M)\frac{z}{(1-\sigma z)^2}\,. \tag{43}$$

Hence, after a direct calculation,

$$a_3 - aa_2^2 = \left[\frac{1}{M}\left(1-\frac{1}{M}\right)\left(3-\frac{5}{M}\right) - \frac{4a}{M^2}\left(1-\frac{1}{M}\right)^2\right]\sigma^2\,. \tag{44}$$

As remark at the beginning of the paper, the expression $a_3 - aa_2^2$ assigned to an extremal function is non-negative. Therefore we may replace the expression on the right-hand side of (44) by its modulus. Hence, by $|\sigma| = 1$, we have

$$a_3 - aa_2^2 = \left|\frac{1}{M}\left(1-\frac{1}{M}\right)\left[\left(3-\frac{5}{M}\right) - \frac{4a}{M}\left(1-\frac{1}{M}\right)\right]\right|\,. \tag{45}$$

Consequently,

$$(45') \quad a_3 - a a_2^2 = \frac{1}{M}\left(1-\frac{1}{M}\right)\left(\frac{4a}{M^2} - \frac{4a+5}{M} + 3\right) \quad \text{for } a \leqslant \frac{M(3M-5)}{4(M-1)}$$

and

$$(45'') \quad a_3 - a a_2^2 = \frac{1}{M}\left(1-\frac{1}{M}\right)\left(-\frac{4a}{M^2} + \frac{4a+5}{M} - 3\right) \quad \text{for } a > \frac{M(3M-5)}{4(M-1)}.$$

The conclusion of Section 2 is that, given $M > 1$ and a real, the extremal function $g(z) \in \mathfrak{G}_1^M$, for which functional (11) attains its maximum, is determined by equation (43), and the expression $a_3 - a a_2^2$ that corresponds to this function is given by formula (45).

The considerations of Section 1 and 2 yielded formulae (34) and (45) which gave the maximal value of the expression $a_3 - a a_2^2$ in $\mathfrak{G}_2^M$ and $\mathfrak{G}_1^M$, respectively. By (12') this value is the maximum of functional (11) in the classes under consideration.

Furthermore, we see that for $M > 1$ the inequality

$$\left|\frac{1}{M}\left(1-\frac{1}{M}\right)\left[\left(3-\frac{5}{M}\right) - \frac{4a}{M}\left(1-\frac{1}{M}\right)\right]\right| \geqslant \frac{1}{M}\left(1-\frac{1}{M^2}\right)$$

holds if and only if $-\infty < a \leqslant M(M-3)/2(M-1)$ or $M \leqslant a < \infty$. We remark that $a \geqslant M$ implies $a > M(3M-5)/4(M-1)$ whilst $a \leqslant M(M-3)/$ $/2(M-1)$ implies $a < M(3M-5)/4(M-1)$. Finally, we show that for $M > 1$ the inequality

$$\left|\frac{1}{M}\left(1-\frac{1}{M}\right)\left[\left(3-\frac{5}{M}\right) - \frac{4a}{M}\left(1-\frac{1}{M}\right)\right]\right| \leqslant \frac{1}{M}\left(1-\frac{1}{M^2}\right)$$

holds if and only if $M(M-3)/2(M-1) \leqslant a \leqslant M$.

Summing up the above considerations we obtain the following

THEOREM. *For any function of form* (1) *which belongs to* $\mathfrak{G}^M$ *we have the following* (*sharp*) *estimates*:

$$(46) \quad |a_3 - a a_2^2|$$

$$\leqslant \begin{cases} \dfrac{1}{M}\left(1-\dfrac{1}{M}\right)\left(-\dfrac{4a}{M^2} + \dfrac{4a+5}{M} - 3\right) & \textit{for } \mathfrak{D}_1 = \begin{cases} 1 < M < \infty, \\ M \leqslant a < \infty; \end{cases} \\[2ex] \dfrac{1}{M}\left(1-\dfrac{1}{M^2}\right) & \textit{for } \mathfrak{D}_2 = \begin{cases} 1 < M < \infty, \\ \dfrac{M(M-3)}{2(M-1)} \leqslant a \leqslant M; \end{cases} \\[2ex] \dfrac{1}{M}\left(1-\dfrac{1}{M}\right)\left(\dfrac{4a}{M^2} - \dfrac{4a+5}{M} + 3\right) & \textit{for } \mathfrak{D}_3 = \begin{cases} 1 < M < \infty, \\ -\infty < a \leqslant \dfrac{M(M-3)}{2(M-1)}. \end{cases} \end{cases}$$

Equality holds for function (1) determined by (43) if $(M, a) \in \mathfrak{D}_1$ or $(M, a) \in \mathfrak{D}_3$ while it holds for function (1) determined by (33) if $(M, a) \in \mathfrak{D}_2$. In case where the point (M, a) lies on curve (35) or (36), there are infinitely many extremal functions for which functional (11) attains its maximum in the class $\mathfrak{G}^M$. These functions are determined by (38) in case of curve (35), and by (41) in case of curve (36).

In particular, from the third of inequalities (46) for $a = 0$ we obtain an estimate earlier derived by Dziubiński [1].

If we replace a by aM in (46), and multiply by M both sides of each inequality, we obtain the following

COROLLARY 1. *For any function of form* (8) *which belongs to* $\widetilde{S}^*_M$ *we have the following* (*sharp*) *estimates*:

$$(47)\qquad |A_3 - aA_2^2| \leqslant \begin{cases} \dfrac{4a-5}{M^2} - \dfrac{8(a-1)}{M} + 4a - 3 & \text{for } \widetilde{\mathfrak{D}}_1 = \begin{cases} 1 < M < \infty, \\ 1 \leqslant a < \infty; \end{cases} \\ 1 - \dfrac{1}{M^2} & \text{for } \widetilde{\mathfrak{D}}_2 = \begin{cases} 1 < M < \infty, \\ \dfrac{M-3}{2(M-1)} \leqslant a \leqslant 1; \end{cases} \\ \dfrac{5-4a}{M^2} + \dfrac{8(a-1)}{M} + 3 - 4a & \text{for } \widetilde{\mathfrak{D}}_3 = \begin{cases} 1 < M < \infty, \\ -\infty < a \leqslant \dfrac{M-3}{2(M-1)}. \end{cases} \end{cases}$$

By (43) equality holds for function (8) determined by

$$(48)\qquad \frac{\mathcal{G}(z)}{[1 - \sigma\mathcal{G}(z)/M]^2} = \frac{z}{(1-\sigma z)^2}, \quad \text{where } |\sigma| = 1,$$

if $(M, a) \in \widetilde{\mathfrak{D}}_1$ or $(M, a) \in \widetilde{\mathfrak{D}}_3$. By (33) equality holds for function (8) determined by

$$(49)\qquad \frac{\mathcal{G}(z)}{1 - [\mathcal{G}(z)]^2/M} = \frac{z}{1-z^2}$$

if $(M, a) \in \widetilde{\mathfrak{D}}_2$. In case where the point (M, a) lies on the curve

$$(50)\qquad a = (3-M)/2(1-M), \quad \text{where } 1 < M < \infty,$$

or

$$(51)\qquad a = 1, \quad \text{where } 1 < M < \infty,$$

there are infinitely many extremal functions for which the functional $|A_3 - A_2^2|$ attains its maximum in the class $\widetilde{S}^*_M$. By (38) and (41) these functions are determined by the equation

$$(52)\qquad \frac{\mathcal{G}(z)}{[1 - \mathcal{G}(z)/M]^{\beta_1}[1 + \mathcal{G}(z)/M]^{\beta_2}} = \frac{z}{(1-z)^{\beta_1}(1+z)^{\beta_2}}.$$

where $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1+\beta_2 = 2$, in case of curve (50), and by the equation

$$\frac{\mathcal{G}(z)}{[1-e^{i\varphi}\mathcal{G}(z)/M][1+e^{-i\varphi}\mathcal{G}(z)/M]} = \frac{z}{(1-e^{i\varphi}z)(1+e^{-i\varphi}z)}, \tag{53}$$

where $\varphi \neq (2k+1)\frac{1}{2}\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ is an arbitrary real number, in case of curve (51).

As remarked at the beginning of the paper, every starlike function (3) is the limit of some functions of form (8) as $M \to \infty$. Hence, as it is easily seen, letting $M \to \infty$ in (47) we obtain the following

COROLLARY 2. *For any function of form* (3) *which belongs to* S^* *we have the following* (*sharp*) *estimates*:

$$|B_3 - aB_2^2| \leqslant \begin{cases} 4a-3 & \text{for } 1 \leqslant a < \infty, \\ 1 & \text{for } \frac{1}{2} \leqslant a \leqslant 1, \\ 3-4a & \text{for } -\infty < a \leqslant \frac{1}{2}. \end{cases} \tag{54}$$

By (48) equality holds for function (3) of the form

$$F(z) = z/(1-\sigma z)^2, \quad \text{where } |\sigma| = 1, \tag{55}$$

if $1 \leqslant a < \infty$ or $-\infty < a \leqslant \frac{1}{2}$. By (49) equality holds for function (3) of the form

$$F(z) = z/(1-z^2) \tag{56}$$

if $\frac{1}{2} \leqslant a < 1$. In case where $a = 1$ or $a = \frac{1}{2}$ it may be verified that there are infinitely many extremal functions for which the functional $|B_3 - aB_2^2|$ attains its maximum in the class S^*. By (52) and (53) these functions are given by the formula

$$F(z) = z/(1-z)^{\beta_1}(1+z)^{\beta_2}, \tag{57}$$

where $\beta_1, \beta_2 > 0$, arbitrary and such that $\beta_1+\beta_2 = 2$, in case of $a = 1$, and by the formula

$$F(z) = z/(1-e^{i\varphi}z)(1+e^{-i\varphi}z), \tag{58}$$

where φ is an arbitrary real number, in case of $a = \frac{1}{2}$.

The functional $|A_3 - aA_2^2|$ within the class S_M has been investigated by Jakubowski [4] in the case $0 \leqslant a < 1$, by Zawadzki [5] in the case $a = 1$, and for arbitrary real a by Jakubowski [3]. We remark that the first of the estimates in (47) is the same as the corresponding result for the class S_M established by Jakubowski [3]. The above functional has been investigated within the full class S of univalent functions for $0 \leqslant a \leqslant 1$ by Golusin ([2], pp. 197-199), while for $a < 0$ and $a \geqslant 1$ by

Jakubowski [3]. Finally, we remark that the first of the estimates in (54) is the same as the corresponding result for the class S established by Jakubowski [3].

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