Accordingly in this case φ_U is a homeomorphism of $(S(U), \tau_l(U))$ onto $S(U'), \tau_l(U')) = G(U)$ and by (a) a natural equivalence between F and G has been established.

The second case is dual.

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Cup product, duality and periodicity for generalized group cohomology*

by

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Introduction. A finite permutation representation (G, X) of a group G consists of a finite non-empty set X, with G acting on the left, such that $(\varrho\sigma)x = \varrho(\sigma x)$ for all $\varrho, \sigma \in G$ and all $x \in X$ and such that $1x = x$ for all $x \in X$, where 1 denotes the identity element of G.

Out of a given arbitrary finite permutation representation (G, X), one can form the "standard complex" $C(X; G)$ (see [6], p. 135) which generalizes the standard complex for ordinary group cohomology. By means of this "standard complex", a „cohomology theory of finite permutation representations" was defined and investigated in [6], [7], [8] and [9].

Using recent developments in relative homological algebra, this "cohomology of permutation representations" was axiomatized and investigated in [4]. In this paper we continue this study.

In Chapter I we investigate the cup product in this relative homological algebra setting, thereby extending illuminating and giving new proofs for the results of [7].

In Chapter II we examine the results of [8] in this relative setting and generalize the results of [8] to arbitrary (i.e. not necessarily transitive) finite permutation representations.

In Chapter III we go on to investigate question of periodicity and to generalize the results on periodicity for ordinary group cohomology given in [2], Chapter XII, § 11.

If (G, X) is a finite permutation representation, then $f((G, X))$ will denote the finite collection $\mathfrak{H}$ of the subgroups of finite index in G which fix the points of X. Clearly $f((G, X)) = \mathfrak{H}$ is closed under conjugation by elements of G. Moreover, if we are given a finite collection $\mathfrak{H}$ of subgroups of G of finite index which is closed under conjugation, then there exists a finite permutation representation (G, X) such that $f((G, X)) = \mathfrak{H}$.

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If R is any ring with an identity, then ${}_{R[G]}\mathfrak{C}$ will denote the category of left $R[G]$-modules; however, for simplicity, if $R = Z$, the ring of rational integers, then ${}_{Z[G]}\mathfrak{C}$ will be denoted by ${}_G\mathfrak{C}$. Also, Ab will denote the category of Abelian groups. If K is a subgroup of G and if $A \in {}_{R[G]}\mathfrak{C}$, then $A^K = \{a \in A \mid ka = a \text{ for all } k \in K\}$ and $T_K\colon {}_{R[G]}\mathfrak{C} \to {}_{R[K]}\mathfrak{C}$ will denote the "forgetful" functor.

If now K is assumed to be of finite index in G and if $\{x_1, \ldots, x_n\}$ is a representative choice for the left cosets of K in G and if $A \in {}_{R[G]}\mathfrak{C}$, then the trace map $S_{G|K}\colon A^K \to A^G$ is defined by: $S_{G|K}(a) = \sum_{i=1}^{n} x_i a$ for all $a \in A^K$ and is clearly independent of the coset representative choice. If $A \in {}_{R[G]}\mathfrak{C}$ is such that there is a $u \in \big(\mathrm{Hom}_R(A, A)\big)^K = \mathrm{Hom}_{R[K]}(A, A)$ such that $S_{G|K}(u) = 1_A$, then A is said to be $G|K$-*special*.

If now $\mathfrak{H}$ denotes an arbitrary collection of subgroups of G, then $\overline{\mathfrak{H}}$ will denote the set of all finite intersections of elements of $\mathfrak{H}$. $\mathfrak{F}_R(\mathfrak{H})$ will denote the class of sequences in ${}_{R[G]}\mathfrak{C}$ which are exact when for each $H \in \mathfrak{H}$ the functor $\mathrm{Hom}_H\big(R, T_H(\ast)\big)$ is applied (note that if $A \in {}_{R[K]}\mathfrak{C}$, then $\mathrm{Hom}_{R[K]}(R, A) \cong A^K$ for any group K) and $Q_R(\mathfrak{F})$ will denote the short exact sequences in $\mathfrak{Z}_R(\mathfrak{H})$; thus $Q_R(\mathfrak{H})$ forms a proper class in the sense of [5], Chapter XII § 4 (see [4], Chapter I, § 1). Again, if $\mathfrak{H}$ denotes an arbitrary set of subgroups of G, then $\mathfrak{E}_0^R(\mathfrak{H})$ will denote the sequences of ${}_{R[G]}\mathfrak{C}$ which upon application of the functor T_H are split exact in ${}_{R[H]}\mathfrak{C}$ in the sense of [3], Chapter I, § 1 for every $H \in \mathfrak{H}$; $\mathfrak{E}^0_R(\mathfrak{H})$ is similarly defined using cosplit coexactness and $P_R(\mathfrak{H})$ will denote the short exact sequences of $\mathfrak{E}_0^R(\mathfrak{H})$ and hence of $\mathfrak{E}^0_R(\mathfrak{H})$. Moreover, $\mathfrak{E}_0^R(\mathfrak{H})$ is a projective class of sequences in the sense of [3], Chapter I, § 2 (see [4], Chapter I, § 1), and, similarly, $\mathfrak{E}^0_R(\mathfrak{H})$ is an injective class of sequences. Also, $P_R(\mathfrak{H})$ is a proper class of short exact sequences, and $P_R(\mathfrak{H}) \subset Q_R(\overline{\mathfrak{H}})$ for any such $\mathfrak{H}$.

If now $\mathfrak{H}$ is assumed to be an arbitrary set of subgroups of finite index in G, then $U_{\mathfrak{H}}$ will denote the functor $U_{\mathfrak{H}}\colon {}_{R[G]}\mathfrak{C} \to Ab$ defined by: $U_{\mathfrak{H}}(A) = A^G / \sum_{H \in \mathfrak{H}} S_{G|H}(A^H)$ and if P_R denotes any proper class of short exact sequences in ${}_{R[G]}\mathfrak{C}$, then by a $(P_R, \mathfrak{H})$-comology theory (see [4], Definition 2.1.1) is meant a set $\{F_R^n \mid n \in Z\}$ of covariant additive functors from ${}_{R[G]}\mathfrak{C}$ into Ab, such that F_R^0 is naturally equivalent with $U_{\mathfrak{H}}$ and, such that for each $n \in Z$, (F_R^n, F_R^{n+1}) is a P_R-connected pair of functors which is both left P_R-couniversal and right P_R-universal in the sense of [5], Chapter XII, § 7.

In all of this, if $R = Z$, then the symbol R will be dropped.

Finally, it should be noted that all of the results of [6], [7] and [4] are valid if the ring Z is replaced throughout by any commutative ring with an identity.

I. Cup product.

Although all of the definitions and results of this Chapter are given for ${}_G\mathfrak{C}$, it should be noted that they are also valid if the ring Z is replaced by any commutative ring with an identity.

§ 1. The Definition. For any $A, B \in {}_G\mathfrak{C}$, one turns $A \otimes_Z B$ into a left G-module by setting $g(a \otimes_Z b) = ga \otimes_Z gb$ for all $g \in G$, $a \in A$ and $b \in B$. (For simplicity of notation, when tensoring with respect to Z, $\otimes_Z$, the subscript Z will be omitted provided that no confusion can arise.)

LEMMA 1.1.1. *If $\mathfrak{H}$ is a collection of subgroups of finite index in G and if $A, B \in {}_G\mathfrak{C}$, then the map $\psi\colon A^G \otimes B^G \to (A \otimes B)^G$ in Ab given by $\psi(a \otimes b) = a \otimes b$ for all $a \in A^G$ and $b \in B^G$ induces a map $\Phi\colon U_{\mathfrak{H}}(A) \otimes U_{\mathfrak{H}}(B) \to U_{\mathfrak{H}}(A \otimes B)$ in Ab.*

Proof. Let $H, K \in \mathfrak{H}$ and let $G = \bigcup_{i=1}^{n} x_i H = \bigcup_{j=1}^{m} y_j K$ be corresponding left coset decompositions of G. Then if $a \in A^H$ and $b \in B^G$, $S_{G|H}(a) \otimes b = \big(\sum_{i=1}^{n} x_i a\big) \otimes b = \sum_{i=1}^{n} x_i(a \otimes b) \in S_{G|H}\big((A \otimes B)^H\big)$. Similarly, if $a \in A^G$ and $b \in B^K$, then $a \otimes S_{G|H}(b) \in S_{G|H}\big((A \otimes B)^K\big)$.

Let P be a proper class of short exact sequences in ${}_G\mathfrak{C}$; let $\mathfrak{H}$ be a collection of subgroups of finite index in G; and assume that $\{F^n \mid n \in Z\}$ is a $(P, \mathfrak{H})$-cohomology theory. Observe that a covariant additive functor $J^n\colon {}_G\mathfrak{C} \times {}_G\mathfrak{C} \to Ab$ for any $n \in Z$ is defined by setting $J^n(A, B) = F^n(A \otimes B)$ for all $A, B \in {}_G\mathfrak{C}$. Similarly, for any $p, q \in Z$, a covariant additive functor $M^{p,q}\colon {}_G\mathfrak{C} \times {}_G\mathfrak{C} \to Ab$ is defined by setting $M^{p,q}(A, B) = F^p(A) \otimes F^q(B)$ for all $A, B \in {}_G\mathfrak{C}$.

DEFINITION 1.1.1. A cup product for a $(P, \mathfrak{H})$ cohomology theory is a collection $\{f_{p,q} \mid p, q \in Z\}$ of natural transformations $f_{p,q}\colon M^{p,q} \to J^{p+q}$ for all $p, q \in Z$ such that:

1) if $p = q = 0$, then $f_{0,0}$ corresponds to Φ of Lemma 1.1.1 via the natural equivalence $F^0 \cong U$.

2) If $E\colon 0 \to A' \to A \to A'' \to 0$ is in P and if $B \in {}_G\mathfrak{C}$ is such that $E \otimes B\colon 0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ is also in P, then the following square commutes for all $p, q \in Z$:

$$\begin{array}{ccc}
M^{p,q}(A, B) = F^p(A'') \otimes F^q(B) & \xrightarrow{f_{p,q}(A'', B)} & F^{p+q}(A'' \otimes B) = J^{p+q}(A'', B) \\
\Big\downarrow{\scriptstyle E_* \otimes 1} & & \Big\downarrow{\scriptstyle (E \otimes B)_*} \\
M^{p+1,q}(A', B) = F^{p+1}(A') \otimes F^q(B) & \xrightarrow{f_{p+1,q}(A', B)} & F^{p+q+1}(A' \otimes B) = J^{p+q+1}(A', B)
\end{array}$$

where the subscript star denotes the P-connecting morphism of $\{F^n \mid n \in Z\}$ for the appropriate short exact sequence of P.

3) If E: $0 \to B' \to B \to B'' \to 0$ is in P and if $A \in {}_G\mathfrak{C}$ is such that $A \otimes E$: $0 \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$ is also in P, then for any $p, q \in Z$, the following square commutes up to $(-1)^p$:

$$\begin{array}{ccc} M^{p,q}(A, B'') = F^p(A) \otimes F^q(B'') & \xrightarrow{f_{p,q}(A,B'')} & F^{p+q}(A \otimes B'') = J^{p+q}(A, B'') \\ \Big\downarrow{\scriptstyle 1 \otimes E_*} & & \Big\downarrow{\scriptstyle (A \otimes E)_*} \\ M^{p,q+1}(A, B') = F^p(A) \otimes F^{q+1}(B') & \xrightarrow{f_{p,q+1}(A,B')} & F^{p+q+1}(A \otimes B') = J^{p+q+1}(A, B'). \end{array}$$

§ 2. Existence of a cup product.

LEMMA 1.2.1. *If H is a subgroup of finite index in G and if $A \in {}_G\mathfrak{C}$ is $G|H$-special, then for any $B \in {}_G\mathfrak{C}$, $A \otimes B$ and $B \otimes A$ are also $G|H$-special.*

Proof. If $u \in \operatorname{Hom}_H(A, A)$ is such that $S_{G|H}(u) = 1_A$, then $u \otimes 1_B \in \operatorname{Hom}_H(A \otimes B, A \otimes B)$ and $S_{G|H}(u \otimes 1_B) = S_{G|H}(u) \otimes 1_B = 1_{A \otimes B}$. Finally $A \otimes B$ and $B \otimes A$ are isomorphic in ${}_G\mathfrak{C}$.

We now restrict considerations of cohomology theories to those discussed in [4]. Hence, we may assume that we are given a finite permutation representation (G, X) of G such that $f\big((G, X)\big) = \mathfrak{H}$ is the (finite) collection of subgroups (of finite index) in G which fix the points of X and let $\{F^n |\ n \in Z\}$ denote a $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory which exists by [4], Theorem 2.2.2. Moreover, if the $Q(\bar{\mathfrak{H}})$-connecting morphisms are restricted to $P(\mathfrak{H})$, then $\{F^n |\ n \in Z\}$ becomes a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology as follows from the results of [4]. Chapter II, § 2.

In [6], [7], [8], [9] and [4], the more cumbersome notation $\{H^n(X, G, \divideontimes) |\ n \in Z\}$ was used for these cohomology theories; where no confusion can arise, we shall use the simpler notation $\{F^n |\ n \in Z\}$ for such cohomology theories.

A sequence $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ in ${}_G\mathfrak{C}$ is defined in [7]. Definition 6.1, to be (G, X) exact if both the sequence and the induced sequence of cochain complexes $\operatorname{Hom}_G\big(C.(X, G), A'\big) \to \operatorname{Hom}_G\big(C, (X, G), A\big) \to \operatorname{Hom}_G\big(C.(X, G), A''\big)$ is exact where $C.(X, G)$ denotes the "standard complex" of the permutation representation (G, X) as given in [6], p. 135.

LEMMA 1.2.2. *An exact sequence $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ in ${}_G\mathfrak{C}$ is (G, X) exact if and only if it lies in $\mathfrak{F}(\bar{\mathfrak{H}})$.*

Proof. If $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ is in $\mathfrak{F}(\bar{\mathfrak{H}})$, then, since the objects in the "standard complex" are $\mathfrak{F}(\bar{\mathfrak{H}})$-projective ([4], Chapter I, § 1 and Lemma 2.2.1), it follows that $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ is (G, X) exact. Conversely, assume that $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ is (G, X) exact. Now, for any $H \in \bar{\mathfrak{H}}$, there is a positive integer ϱ such that X has transitive constituents $T_1, \ldots, T_u$ with H, the subgroup of G fixing a point of T_1. But, $\operatorname{Hom}_G(Z[X^\varrho], A') \to \operatorname{Hom}_G(Z[X^\varrho], A) \to \operatorname{Hom}_G(Z[X^\varrho], A'')$ is exact

and $Z[X^\varrho] = \bigoplus_{i=1}^{u} Z[T_i]$ and $Z[T_1] \cong Z[G] \otimes_H Z$ in ${}_G\mathfrak{C}$. Hence, $\operatorname{Hom}_G(Z[G] \otimes_H Z, A') \to \operatorname{Hom}_G(Z[G] \otimes_H Z, A) \to \operatorname{Hom}_G(Z[G] \otimes_H Z, A'')$ is exact and thus $\operatorname{Hom}_H\big(Z, T_H(A')\big) \to \operatorname{Hom}_H\big(Z, T_H(A)\big) \to \operatorname{Hom}_H\big(Z, T_H(A'')\big)$ is also exact. This implies that $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ lies in $\mathfrak{F}(\bar{\mathfrak{H}})$.

COROLLARY 1.2.1. *A short exact sequence of ${}_G\mathfrak{C}$ is (G, X) exact if and only if it lies in $Q(\bar{\mathfrak{H}})$.*

Hence, [7], Propositions 5.1 and 6.1, demonstrate:

THEOREM 1.2.1. *Every $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory $\{F^n |\ n \in Z\}$ has a cup product.*

Moreover, if the $Q(\bar{\mathfrak{H}})$-connecting morphisms of the $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory $\{F^n |\ n \in Z\}$ are restricted to $P(\mathfrak{H})$, then $\{F^n |\ n \in Z\}$ becomes a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory and the cup product for the $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory $\{F^n |\ n \in Z\}$ satisfies the definition for the cup product of the $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory $\{F^n |\ n \in Z\}$. Hence:

THEOREM 1.2.2. *Every $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory $\{F^n |\ n \in Z\}$ has a cup product.*

However, a different proof of Theorems 1.2.1 and 1.2.2 can be given, using only "universal properties", which we feel is of some interest and hence, we proceed to do this. (We assume that $\{F^n |\ n \in Z\}$ denotes both a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$ and a $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory where the $P(\mathfrak{H})$-connecting morphisms are obtained by restricting the $Q(\bar{\mathfrak{H}})$-connecting morphisms to $P(\mathfrak{H})$.)

LEMMA 1.2.3. *For any $B \in {}_G\mathfrak{C}$, $\{F^n(\divideontimes \otimes B) |\ n \in Z\}$ is an exact $P(\mathfrak{H})$-connected sequence of functors such that $\big(F^n(\divideontimes \otimes B), F^{n+1}(\divideontimes \otimes B)\big)$ is left $P(\mathfrak{H})$-couniversal and right $P(\mathfrak{H})$-universal for any $n \in Z$. The same holds for $\{F^n(B \otimes \divideontimes) |\ n \in Z\}$.*

Proof ([7], Prop. 7.1; Lemma 1.2.1; and [5], Chapter XII, Theorems 7.2 and 7.6; and [4], Chapter I, § 3).

LEMMA 1.2.4. *For any $B \in {}_G\mathfrak{C}$ and for any fixed $q \in Z$, $\{F^p(\divideontimes) \otimes \otimes F^q(B) |\ p \in Z\}$ is a $P(\mathfrak{H})$-connected sequence of functors with $P(\mathfrak{H})$-connecting morphism $E_* \otimes 1$ for E: $0 \to A' \to A \to A'' \to 0$ in $P(\mathfrak{H})$ such that $\big(F^p(\divideontimes) \otimes F^q(B), F^{p+1}(\divideontimes) \otimes F^q(B)\big)$ is right $P(\mathfrak{H})$-universal for all $p \in Z$.*

Proof. Apply ([5], Chapter XII, Theorem 7.6) noting that $\otimes F^q(B)$ is right exact in Ab.

Similarly we have:

LEMMA 1.2.5. *For any $A \in {}_G\mathfrak{C}$ and for any fixed $p \in Z$, $\{F^p(A) \otimes \otimes F^q(\divideontimes) |\ q \in Z\}$ is a $P(\mathfrak{H})$-connected sequence of functors with connecting morphism for E': $0 \to B' \to B \to B'' \to 0$ in $P(\mathfrak{H})$ given by: $(1 \otimes E'_*)(\sigma \otimes \sigma')$*

$= \sigma \otimes E'_* \sigma'$ *for all* $\sigma \in F^p(A)$ *and* $\sigma' \in F^q(B'')$. *Moreover,* $\big(F^p(A) \otimes F^q(\ast), F^p(A) \otimes F^{q+1}(\ast)\big)$ *is right* $P(\mathfrak{H})$-*universal for all* $q \in Z$.

We now proceed to derive Theorem 1.2.2, using only the "universal properties".

Let $A \in {}_G\mathfrak{C}$ and let $f_{0,0}(A, \ast)$: $F^0(A) \otimes F^0(\ast) \to F^0(A \otimes \ast)$ denote the unique natural transformation corresponding to Φ of Lemma 1.1.1. By universality, there exist unique natural transformations $f_{0,q}(A, \ast)$: $F^0(A) \otimes F^q(\ast) \to F^q(A \otimes \ast)$ for all $q \in Z$ such that $\{f_{0,q}(A, \ast) \mid q \in Z\}$ is a morphism of $P(\mathfrak{H})$-connected sequences of functors. Thus, if E': $0 \to B' \to B \to B'' \to 0$ is in $P(\mathfrak{H})$, then

$$\begin{tikzcd}
F^0(A) \otimes F^q(B'') \arrow[r, "{f_{0,q}(A,B'')}"] \arrow[d, "1 \otimes E'_*"'] & F^q(A \otimes B'') \arrow[d, "(A \otimes E')_*"] \\
F^0(A) \otimes F^{q+1}(B') \arrow[r, "{f_{0,q+1}(A,B')}"] & F^{q+1}(A \otimes B')
\end{tikzcd}$$

commutes.

We claim that the natural transformations $f_{0,q}(A, \ast)$ are functorial in A. For, let $A \xrightarrow{\gamma} A'$ be a map in ${}_G\mathfrak{C}$ and consider $\{F^q(\gamma \otimes 1) \circ f_{0,q}(A, \ast) \mid q \in Z\}$ and $\{f_{0,q}(A', \ast) \circ (F^0(\gamma) \otimes 1) \mid q \in Z\}$. It is easy to see that both sets are morphisms of the $P(\mathfrak{H})$-connected sequence of functors $\{F^0(A) \otimes \otimes F^q(\ast) \mid q \in Z\}$ which for $q = 0$ correspond to the map of $U_{\mathfrak{H}}(A) \otimes \otimes U_{\mathfrak{H}}(\ast) \to U_{\mathfrak{H}}(A' \otimes \ast)$ induced by $\Phi \circ \big(U_{\mathfrak{H}}(\gamma) \otimes 1\big) = U_{\mathfrak{H}}(\gamma \otimes 1) \circ \Phi$. Thus, by uniqueness, $F^q(\gamma \otimes 1) \circ f_{0,q}(A, \ast) = f_{0,q}(A', \ast) \circ \big(F^0(\gamma) \otimes 1\big)$ for all $q \in Z$ and hence $f_{0,q}(A, \ast)$ is functorial in A. Now, fix $q \in Z$ and $B \in {}_G\mathfrak{C}$, then $f_{0,q}(\ast, B)$: $F^0(\ast) \otimes F^q(B) \to F^q(\ast \otimes B)$ is a natural transformation of functors and thus extends uniquely to a morphism of $P(\mathfrak{H})$-connected sequences of functors $\{f_{p,q}(\ast, B) \mid p \in Z\}$: $\{F^p(\ast) \otimes F^q(B) \mid p, q \in Z\} \to \{F^{p+q}(\ast \otimes B) \mid p \in Z\}$. Thus, $\{f_{p,q} \mid p, q \in Z\}$ satisfies 2) of Definition 1.1.1 and is functorial in the first variable. For the second variable, suppose that $B \xrightarrow{\omega} B'$ is a map in ${}_G\mathfrak{C}$ and consider the natural transformations $f_{p,q}(\ast, B') \circ \big(1 \otimes F^q(\omega)\big)$ and $F^{p+q}(1 \otimes \omega) \circ f_{p,q}(\ast, B)$ of the functors $F^p(\ast) \otimes F^p(B) \to F^{p+q}(\ast \otimes B')$ for any $p \in Z$. It is easy to see that $\big\{f_{p,q}(\ast, B') \circ \big(1 \otimes F^q(\omega)\big) \mid p \in Z\big\}$ and $\{F^{p+q}(1 \otimes \omega) \circ f_{p,q}(\ast, B') \mid p \in Z\}$ are actually morphisms of $P(\mathfrak{H})$-connected sequences of functors. But, $f_{0,q}(\ast, B') \circ \big(1 \otimes F^q(\omega)\big) = F^q(1 \otimes \otimes \omega) \circ f_{0,q}(\ast, B')$, since $f_{0,q}$ is functorial in the second variable. Hence, $f_{p,q}(\ast, B') \circ \underline{\big(1 \otimes F^q(\omega)\big)} = F^{p+q}(1 \otimes \omega) \circ f_{p,q}(\ast, B')$ for all $p \in Z$. Thus, it remains to verify 3) of Definition 1.1.1. Let E': $0 \to B' \to B \to B'' \to 0$ be in $P(\mathfrak{H})$ and fix $q \in Z$; for each $A \in {}_G\mathfrak{C}$ and each $p \in Z$ set $\lambda^p(A) = f_{p,q+1}(A, B') \circ (1 \otimes E'_*)$ and $\psi^p(A) = (-1)^p (A \otimes E')_* \circ f_{p,q}(A, B'')$. Then, for each $p \in Z$, it is easy to see that λ^p and ψ^p are natural transformations of functors: $F^p(\ast) \otimes F^q(B'') \to F^{p+q+1}(\ast \otimes B')$. Using the generalization

of [2]. Chapter III, Prop. 4.1, to proper classes of short exact sequences, it follows that $\{\lambda^p \mid p \in Z\}$ and $\{\psi^p \mid p \in Z\}$ are actually morphisms of $P(\mathfrak{H})$-connected sequences of functors. Finally, for $p = 0$ and for any $A \in {}_G\mathfrak{C}$, $\lambda^0(A) = f_{0,q+1}(A, B') \circ (1 \otimes E'_*) = (A \otimes E'_*) \circ f_{0,q}(A, C') = \psi^0(A)$ as has been previously noted and hence, $\lambda^p = \psi^p$ for all $p \in Z$ — completing this proof of Theorem 1.2.2.

Corollary 1.2.2. *Each* $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-*cohomology theory has a unique cup product.*

Next, we give another proof of Theorem 1.2.1 using only the "universal properties."

Lemma 1.2.6. *If* E: $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ *is in* $Q(\bar{\mathfrak{H}})$, *then there exists a commutative diagram*

$$\begin{tikzcd}
\bar{E}\colon 0 \arrow[r] & A \arrow[r, "\bar{u}"] \arrow[d, "1"] & \bar{B} \arrow[r, "\bar{v}"] \arrow[d, "\pi"] & \bar{C} \arrow[r] \arrow[d, "\alpha"] & 0 \\
E\colon 0 \arrow[r] & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r] & 0
\end{tikzcd} \tag{1}$$

such that $\bar{E} \in P(\mathfrak{H})$ *and such that* $F^n(\alpha)$ *is an isomorphism for all* $n \in Z$.

Proof. Since $\mathfrak{E}^0(\mathfrak{H})$ is an injective class, there is a sequence $0 \to A \xrightarrow{u_1} D$ in $\mathfrak{E}^0(\mathfrak{H})$ such that D is an $\mathfrak{E}^0(\mathfrak{H})$-injective. Let $\bar{u}$: $A \to B \oplus D$ be defined by: $\bar{u}(a) = \big(u(a), u_1(a)\big)$; then $\bar{u}$ is clearly monic. Setting $\bar{v} =$ coker $\bar{u}$ and letting π be the projection of $B \oplus D$ onto the first component we obtain the commutative diagram (1) with α unique and $\bar{E} \in P(\mathfrak{H})$. Applying $\big(F^n(\ast), F^{n+1}(\ast)\big)$ to the diagram and observing that $F^n(B \oplus D) = F^n(B) \oplus \oplus F^n(D)$ and $F^n(D) = 0$ render $F^n(\pi)$ an isomorphism, the desired conclusion is a consequence of the five lemma.

Now, the $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory $\{F^n \mid n \in Z\}$ after restricting the connecting morphisms to $P(\mathfrak{H})$ becomes a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory and hence, has a unique cup product $\{f_{p,q} \mid p, q \in Z\}$ as a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory. We show that $\{f_{p,q} \mid p, q \in Z\}$ is also a cup product for $\{F^n \mid n \in Z\}$ as a $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory. Thus, it remains to verify 2) and 3) of Definition 1.1.1 for the proper class $Q(\bar{\mathfrak{H}})$.

Suppose that E: $0 \to A \to B \to C \to 0$ is in $Q(\bar{\mathfrak{H}})$ and $D \in {}_G\mathfrak{C}$ is such that $E \otimes D$: $0 \to A \otimes D \to B \otimes D \to C \otimes D \to 0$ is also in $Q(\bar{\mathfrak{H}})$. Applying Lemma 1.2.6 to E and tensoring diagram (1) of this Lemma with D, we get the commutative diagram:

$$\begin{tikzcd}
\bar{E} \otimes D\colon 0 \arrow[r] & A \otimes D \arrow[r] \arrow[d, "1"] & \bar{B} \otimes D \arrow[r] \arrow[d, "\pi \otimes 1"] & \bar{C} \otimes D \arrow[r] \arrow[d, "\alpha \otimes 1"] & 0 \\
E \otimes D\colon 0 \arrow[r] & A \otimes D \arrow[r] & B \otimes D \arrow[r] & C \otimes D \arrow[r] & 0
\end{tikzcd}$$

where $\bar{E}\otimes D$ is in $P(\mathfrak{H})$ ([7], Prop. 7.1) and $E\otimes D$ is in $Q(\bar{\mathfrak{H}})$. This gives rise to the cube:

$$\begin{tikzcd}
& F^p(C)\otimes F^q(D) \arrow[rr, "f_{p,q}(C,D)"] \arrow[dd, "E_*\otimes 1" near start] & & F^{p+q}(C\otimes D) \arrow[dd, "(E\otimes D)_*"] \\
F^p(\bar{C})\otimes F^q(D) \arrow[ur, "F^p(a)\otimes 1"] \arrow[rr, "f_{p,q}(\bar{C},D)" near start] \arrow[dd, "\bar{E}_*\otimes 1"'] & & F^{p+q}(\bar{C}\otimes\bar{D}) \arrow[ur, "F^{p+q}(a\otimes 1)"] \arrow[dd, "(\bar{E}\otimes D)_*" near start] & \\
& F^{p+1}(A)\otimes F^q(D) \arrow[rr, "f_{p+1,q}(A,D)"' near start] & & F^{p+q+1}(A\otimes D) \\
F^{p+1}(A)\otimes F^q(D) \arrow[ur, "1"] \arrow[rr, "f_{p+1,q}(A,D)"'] & & F^{p+q+1}(A\otimes D) \arrow[ur, "1"'] &
\end{tikzcd}$$

where all faces except the rear face are known to commute. But $F^p(a)\otimes 1$ is an isomorphism and 2) of Definition 1.1.1 follows. A similar proof yields 3) of Definition 1.1.1, concluding the proof of Theorem 1.2.1.

COROLLARY 1.2.3. *Each $(Q(\bar{\mathfrak{H}}),\bar{\mathfrak{H}})$-cohomology theory has a unique cup product.*

Proof. A cup product for a $(Q(\bar{\mathfrak{H}}),\bar{\mathfrak{H}})$-cohomology theory $\{\bar{F}^n|\ n\in Z\}$ is also a cup product for $\{F^n|\ n\in Z\}$ as a $(P(\mathfrak{H}),\mathfrak{H})$-cohomology theory and hence, is unique.

§ 3. Applications of uniqueness.

As usual, (G,X) denotes a finite permutation representation of the group G and $\mathfrak{H}=f((G,X))$. Let $U_1(\divideontimes),\ldots,U_n(\divideontimes),V(\divideontimes)$ each represent an exact $P(\mathfrak{H})$-connected sequence of covariant additive functors from ${}_G\mathfrak{C}$ into Ab. A map $F\colon U_1\otimes\otimes U_2\otimes\ldots\otimes U_n\to V$ is a family of morphisms $F\colon U_1^{i_1}(A_1)\otimes\ldots\otimes U_n^{i_n}(A_n)\to V^{i_1+\ldots+i_n}(A_1\otimes\ldots\otimes A_n)$ defined for all $A_1,\ldots,A_n\in{}_G\mathfrak{C}$ and all $i_1,\ldots,i_n\in Z$ which is natural relative to each of the variables $A_1,\ldots,A_n$ and which commutes with the connecting morphisms as follows: if $0\to A_j'\to A_j\to A_j''\to 0$ is in $P(\mathfrak{H})$, then the diagram:

$$\begin{tikzcd}
U_1(A_1)\otimes\ldots\otimes U_j(A_j'')\otimes\ldots\otimes U_n(A_n) \arrow[r, "F"] \arrow[d] & V(A_1\otimes\ldots\otimes A_j''\otimes\ldots\otimes A_n) \arrow[d] \\
U_1(A_1)\otimes\ldots\otimes U_j(A_j')\otimes\ldots\otimes U_n(A_n) \arrow[r, "F"] & V(A_1\otimes\ldots\otimes A_j'\otimes\ldots\otimes A_n)
\end{tikzcd}$$

(where we have omitted $i_1,\ldots,i_n$) "skew commutes" for all choices of $(i_1,\ldots,i_n)\in Z^n$.

Using the fact that $P(\mathfrak{H})$ has enough projectives and injectives (which are the same class of G-modules), one can use the same method of proof as in [2], Chapter XII, Theorem 5.1, or as in [7], Prop. 8.1, to demonstrate:

THEOREM 1.3.1. *Assume in the above that all of the functors $U_1,U_2,\ldots,U_n,V$ vanish on the $\mathfrak{E}_0(\mathfrak{H})$-projectives (and hence on the $\mathfrak{E}^0(\mathfrak{H})$-injectives); then if two maps, $F,G\colon U_1\otimes\ldots\otimes U_n\to V$ coincide in any one dimension $(i_1,\ldots,i_n)\in Z^n$, then $F=G$.*

This theorem has Corollaries 1.2.2 and 1.2.3 as immediate consequences. However, this theorem has several other consequences, including those paralleling the results of [2], Chapter XII, § 5 and of [7], Sections 9–15.

For any $A,B\in Ab$, we always have the isomorphism $\lambda(A,B)\colon A\otimes\otimes B\to B\otimes A$ given by: $\lambda(A,B)(a\otimes b)=b\otimes a$ for all $a\in A$ and $b\in B$; when $A,B\in{}_G\mathfrak{C}$, then $\lambda(A,B)$ is also a G-isomorphism. Assuming that $\{f_{p,q}|\ p,q\in Z\}$ is the cup product for the $(P(\mathfrak{H}),\mathfrak{H})$-cohomology theory $\{F^n|\ n\in Z\}$, then for any $A,B\in{}_G\mathfrak{C}$ and for any $p,q\in Z$ we have a morphism in Ab:

$$(-1)^{pq}F^{p+q}\big(\lambda(B,A)\big)\circ f_{q,p}(B,A)\circ\lambda\big(F^p(A),F^q(B)\big)\colon F^p(A)\otimes F^q(B)\to \to F^{p+q}(A\otimes B)\,.$$

PROPOSITION 1.3.1. *The above defines a $(P(\mathfrak{H}),\mathfrak{H})$-cup product for $\{F^n|\ n\in Z\}$ and hence for any $A,B\in{}_G\mathfrak{C}$ and any $p,q\in Z$ we have:*

$$(-1)^{pq}F^{p+q}\big(\lambda(B,A)\big)\circ f_{q,p}(B,A)\circ\lambda\big(F^p(A),F^q(B)\big)=f_{p,q}(A,B)\,.$$

Proof. Straightforward verification.

This is another proof of [7], Prop. 11.1.

If $A\in{}_G\mathfrak{C}$, $p\in Z$ and $x\in F^p(A)$ are fixed, then, for each $q\in Z$, a natural transformation of functors $f_q^x\colon F^q(\divideontimes)\to F^{p+q}(A\otimes\divideontimes)$ is defined by: if $B\in{}_G\mathfrak{C}$ and $y\in F^q(B)$, then $f_q^x(y)=f_{p,q}(x\otimes y)$. Obviously, a natural transformation of functors $\bar{f}_q^x\colon F^q(\divideontimes)\to F^{p+q}(\divideontimes\otimes A)$ is also defined by: if $B\in{}_G\mathfrak{C}$ and $y\in F^q(B)$, then $\bar{f}_y^x(y)=f_{q,p}(y\otimes x)$.

For fixed $A\in{}_G\mathfrak{C}$ and fixed $a\in A^G$, a natural transformation ψ_a of functors is given by: if $B\in{}_G\mathfrak{C}$, then $\psi_a\colon B\to A\otimes B$ is defined by $\psi_a(b)=a\otimes b$ for all $b\in B$. Obviously, a natural transformation of functors $\bar{\psi}$ is given by: if $B\in{}_G\mathfrak{C}$, then $\bar{\psi}_a\colon B\to B\otimes A$ is defined by $\bar{\psi}_a(b)=b\otimes a$ for all $b\in B$.

PROPOSITION 1.3.2. *If $A\in{}_G\mathfrak{C}$, if $a\in A^G$ and if $x\in F^0(A)$ denotes the element corresponding to $a+\sum_{H\in\mathfrak{H}}S_{G|H}(A^H)$ of $U_{\mathfrak{H}}(A)$, then:*

$$F^q(\psi_a)=f_q^x\quad \textit{for all } q\in Z$$

and

$$F^q(\bar{\psi}_a)=\bar{f}_q^x\quad \textit{for all } q\in Z\,.$$

Proof. Show that $\{f_q^x|\ q\in Z\}$ and $\{F^q(\psi_a)|\ q\in Z\}$ are "maps" of the exact $P(\mathfrak{H})$-connected sequences of covariant functors $\{F^q(\divideontimes)|\ q\in Z\}\to\{F^q(A\otimes\divideontimes)|\ q\in Z\}$ in the sense of Theorem 1.3.1 which agree for $q=0$, for the last part, observe that $\overline{\psi}_a=\lambda(A,\divideontimes)\circ\overline{\psi}_a$ and $\overline{f}_q^x=F^q\big(\lambda(A,\divideontimes)\big)\circ f_q^x$.

This is another proof of [7], Prop. 9.1.

Proposition 1.3.3. *For any* $A, B, C\in {}_G\mathfrak{C}$ *and any* $p, q, r\in Z$,

$$f_{p+q,r}(A\otimes B, C)\circ\big(f_{p,q}(A, B)\otimes 1\big)=f_{p,q+r}(A, B\otimes C)\circ\big(1\otimes f_{q,r}(B, C)\big).$$

Proof. Apply Theorem 1.3.1, viewing the above as defining maps of $F^p(A)\otimes F^q(B)\otimes F^r(C)\to F^{p+q+r}(A\otimes B\otimes C)$ which agree for $p=q=r=0$.

For the rest of this section, it is necessary to switch notation and so we shall now let $\{H^n(X; G, \divideontimes)|\ n\in Z\}$ denote a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory.

If K is a subgroup of G and if $x\in G$, then (K, X) and (xKx^{-1}, X) are finite permutation representations with $f\big((K, X)\big)=\mathfrak{L}=\{K\cap H|\ H\in\mathfrak{H}\}$ and $f\big((xKx^{-1}, X)\big)=\mathfrak{M}=\{(xKx^{-1})\cap H|\ H\in\mathfrak{H}\}$. Let $\{H^n(X; K, \divideontimes)|\ n\in Z\}$ denote a $\big(P(\mathfrak{L}), \mathfrak{L}\big)$ cohomology theory with cup product $\{f_{p,q}^K|\ p, q\in Z\}$, let $\{H^n(X; xKx^{-1}, \divideontimes)|\ n\in Z\}$ denote a $\big(P(\mathfrak{L}), \mathfrak{L}\big)$-cohomology theory with cup product $\{f_{p,q}^{xKx^{-1}}|\ p, q\in Z\}$ and let T_K: ${}_G\mathfrak{C}\to{}_K\mathfrak{C}$ and $T_{xKx^{-1}}$: ${}_G\mathfrak{C}\to{}_{xKx^{-1}}\mathfrak{C}$ denote the usual "forgetful" functors.

Proposition 1.3.4. *For any* $A, B\in{}_G\mathfrak{C}$ *and any* $p, q\in Z$ *we have*:

$$c_x\circ f_{p,q}^K\big(T_K(A), T_K(B)\big)=f_{p,q}^{xKx^{-1}}\big(T_{xKx^{-1}}(A), T_{xKx^{-1}}(B)\big)\circ(c_x\otimes c_x)$$

where c_x *is as defined in* [4], *Chapter* II, § 4.

Proof. Observe that for $p=q=0$, we have equality and apply Theorem 1.3.1.

Suppose now that K is an arbitrary subgroup of G and that L is an arbitrary subgroup of K; let $f\big((K, X)\big)=\mathfrak{L}=\{H\cap K|\ H\in\mathfrak{H}\}$ and let $f\big((L, X)\big)=\mathfrak{M}=\{H\cap L|\ H\in\mathfrak{H}\}$. Then we have $\{\mathrm{Res}^n(L, K)|\ n\in Z\}$: $\big\{H^n\big(X; K, T_K(\divideontimes)\big)\big|\ n\in Z\big\}\to\big\{H^n\big(X; L, T_L(\divideontimes)\big)\big|\ n\in Z\big\}$, a morphism of exact $P(\mathfrak{H})$-connected sequences of functors, defined as in [4]. Chapter II, § 4. Assume that $\{f_{p,q}^K|\ p, q\in Z\}$ is the cup product for $\{H^n(X; K, \divideontimes)|\ n\in Z\}$ and that $\{f_{p,q}^L|\ p, q\in Z\}$ is the cup product for $\{H^n(X; L, \divideontimes)|\ n\in Z\}$.

Proposition 1.3.5. *For any* $A, B\in{}_G\mathfrak{C}$ *and any* $p, q\in Z$ *we have*:

$$\mathrm{Res}^{p+q}(L, K)\circ f_{p,q}^K\big(T_K(A), T_K(B)\big)$$
$$=f_{p,q}^L\big(T_L(A), T_L(B)\big)\circ\big((\mathrm{Res}^p(L, K)\otimes\mathrm{Res}^q(L, K)\big).$$

Proof. Similar to that of Prop. 1.3.4.

Now, further assume that L is of finite index in K; then we have $\{\mathrm{Cor}^n(K, L)|\ n\in Z\}$: $\big\{H^n\big(X; L, T_L(\divideontimes)\big)\big|\ n\in Z\big\}\to\big\{H^n\big(X; K, T_K(\divideontimes)\big)\ n\in Z\big\}$, a morphism of exact $P(\mathfrak{H})$-connected sequences of functors, defined as in [4], Chapter II, § 4.

Proposition 1.3.6. *For any* $A, B\in{}_G\mathfrak{C}$ *and any* $p, q\in Z$, *we have*:

$$\mathrm{Cor}^{p+q}(K, L)\circ f_{p,q}^L\big(T_L(A), \mathrm{Res}^q(L, K)\big(T_K(B)\big)\big)$$
$$=f_{p,q}^K\big(\mathrm{Cor}^p(K, L)\big(T_L(A)\big), T_K(B)\big)$$

and

$$\mathrm{Cor}^{p+q}(K, L)\circ f_{p,q}^L\big(T_L(A), \mathrm{Res}^q(L, K)\big(T_K(B)\big)\big)$$
$$=f_{p,q}^K\big(T_K(A), \mathrm{Cor}^q(K, L)\big(T_L(B)\big)\big).$$

Proof. Similar to that of Prop. 1.3.4.

In order to complete the derivation of the basic results of [7] by means of our axiomatic relative homological algebra setting, it remains to deal with sections 14 and 15 of [7] in which generalizations of the restriction and corestriction mappings are introduced and utilized. Once these generalizations are illuminated, all of the basic results of sections 14 and 15 of [7] will be consequences of our previous results.

Suppose that (G, X) and (G', X') are finite permutation representations of two groups where $f\big((G, X)\big)=\mathfrak{H}$ and $f\big((G', X')\big)=\mathfrak{H}'$ and suppose that φ: $G'\to G$ is a group epimorphism such that:

1) $\mathrm{Ker}(\varphi)\subseteq H'$ for all $H'\in\mathfrak{H}'$,
2) for every $H'\in\mathfrak{H}'$, $\varphi(H')\in\mathfrak{H}$, and
3) for every $H\in\mathfrak{H}$ there exists an $H'\in\mathfrak{H}'$ such that $\varphi(H')=H$.

(As an example, suppose that $X'=X$ and that (φ, l_x): $(G', X)\to\to(G, X)$ is a morphism of permutation representations — as defined in [6], p. 134, Definition 1 — with φ onto.) Letting φ induce by "pull back" the functor T_φ: ${}_G\mathfrak{C}\to{}_{G'}\mathfrak{C}$:

Lemma 1.3.1. $\big\{H^n\big(X', G', T_\varphi(\divideontimes)\big)\big|\ n\in Z\big\}$ *is a* $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-*cohomology theory.*

Proof. It is straight forward to see that $U_{\mathfrak{H}'}\circ T_\varphi=U_{\mathfrak{H}}$ and that if $E\in P(\mathfrak{H})$, then $T_\varphi(E)\in P(\mathfrak{H}')$. Moreover, if $H\in\mathfrak{H}$ and $B\in{}_H\mathfrak{C}$, then it is easy to see that if $H'\in\mathfrak{H}'$ is such that $\varphi(H')=H$, then $T_\varphi(Z[G]\otimes\otimes_H B)\cong Z[G']\otimes_{H'}\overline{B}$ where $\overline{B}$ is the left H'-module obtained from B by "pull back" along φ.

It is now apparent that in the above, if G is a subgroup of L where (L, X) is a permutation representation of L, then $\{\mathrm{Res}^n(G, L)|\ n\in Z\}$: $\{H^n(X; L, \divideontimes)|\ n\in Z\}\to\{H^n(X; G, T_G(\divideontimes)|\ n\in Z\}$ induces the morphism $\{i_q^*|\ q\in Z\}$: $\{H^n(X; L, \divideontimes)|\ n\in Z\}\to\big\{H^n\big(X', G', T_\varphi\circ T_G(\divideontimes)\big)\big|\ n\in Z\big\}$ of exact

$P(\mathfrak{H})$-connected sequences of functors. This is the restriction mapping of [7], Definition 14.1. An analogous result holds for the corestriction mapping of [7], Definition 15.1. Now, Proposition 14.1 and 15.2 of [7] easily follow from Propositions 1.3.5 and 1.3.6 respectively.

For any $A, B, C \in {}_G\mathfrak{C}$ and any G-homomorphism $\theta\colon A\otimes B\to C$, we obtain a "cup product relative to θ" defined by: $f^{\theta}_{p,q}(A,B) = H^{p+q}(X; G, \theta)\circ$ $\circ f_{p,q}(A,B)\colon H^p(X; G, A)\otimes H^q(X; G, B)\to H^{p+q}(X; G, C)$.

In particular, if $R\in {}_G\mathfrak{C}$ is also a ring such that $g(r_1, r_2) = (gr_1)(gr_2)$ for all $g\in G$ and all $r_1, r_2\in R$, then $\theta\colon R\otimes R\to R$ defined by: $\theta(r_1\otimes r_2) = r_1r_2$ for all $r_1, r_2\in R$ is a G-homomorphism of $R\otimes R\to R$ and the "cup product relative to θ" turns $H^*(R) = \{H^n(X; G, R)\mid n\in Z\}$ into a graded set of Abelian groups with a multiplication which is associative by Prop. 1.3.3 and is skew commutative by Prop. 1.3.1 if R is commutative. Thus, $H^*(R)$ is a graded ring. Moreover, if R has a unit element, then it is easy to see from Prop. 1.3.2 that (since $1\in R^G$) the element of $H^0(X; G, R)$ corresponding to the element $1+\sum_{H\in\mathfrak{H}} S_{G|H}(R^H)$ of $U_{\mathfrak{H}}(R)$ is a unit element of $H^*(R)$, thereby giving another proof of [7], Prop. 13.2.

Finally, [7], Prop. 14.2, is a consequence of the definition of restriction in dimension 0.

Thereby, all of the results of [7] have been subsumed by our axiomatic relative homological algebra setting.

II. Duality

§ 1. The functors $h_{p,q}$. For this chapter on duality, we shall always refer to a fixed finite permutation representation (G, X). Let $f\big((G,X)\big) = \mathfrak{H}$ and let $\{F^n\mid n\in Z\}$ denote a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory with cup product $\{f_{p,q}\mid p, q\in Z\}$. For ease of notation $\otimes$ will mean $\otimes_Z$, $\otimes_G$ will mean $\otimes_{Z[G]}$, $\mathrm{Hom}(A,B)$ will denote $\mathrm{Hom}_Z(A,B)$ and $\mathrm{Hom}_G(A,B)$ will denote $\mathrm{Hom}_{Z[G]}(A,B)$ throughout this chapter.

For any $A, B\in {}_G\mathfrak{C}$, the mapping $\theta\colon \mathrm{Hom}(A,B)\otimes A\to B$, defined by $\theta(f\otimes a) = f(a)$ for all $f\in\mathrm{Hom}(A,B)$ and $a\in A$, is a homomorphism of left G-modules and hence gives rise to a "cup product relative to θ", $\{f_{p,q}\mid p, q\in Z\}$, as previously defined. Then, as in [8], Section 2, for $A, B\in {}_G\mathfrak{C}$ and any $p, q\in Z$, we define: $h_{p,q}(A,B)\colon F^p\big(\mathrm{Hom}(A,B)\big)$ $\mathrm{Hom}\big(F^q(A), F^{p+q}(B)\big)$ by: if $\gamma\in F^p\big(\mathrm{Hom}(A,B)\big)$ and $a\in F^q(A)$, then $h_{p,q}(A,B)(\gamma)(a) = f^{\theta}_{p,q}(\gamma\otimes a)$. It is easy to see that, for any $p, q\in Z$, $h_{p,q}$ is a natural transformation of functors which are contravariant in the first variable and covariant in the second variable.

We then derive, as in [2], Chapter XII, Prop. 6.2:

PROPOSITION 2.1.1. *If $B\in {}_G\mathfrak{C}$ has the property that $h_{p,q}(A,B)$ is an isomorphism for all $A\in {}_G\mathfrak{C}$ for some fixed $p, q\in Z$, then $h_{u,v}(A,B)$ is an isomorphism for all $A\in {}_G\mathfrak{C}$ and all $u, v\in Z$ such that $u+v = p+q$.*

§ 2. Change of rings. Now, observe that all of the results of [6], [7], [4] and of this paper up to this point can be completely rewritten (with at most insignificant changes) by replacing Z throughout by any commutative ring with an identity. For our duality program we utilize two such rings besides Z.

For the rest of this chapter, we assume that G is a finite group.

Let d' denote the index of the permutation representation (see [6], p. 140 for the definition) and let $h' = \text{l.c.m.}\ \{|H\colon 1|/H\in\mathfrak{H}\}$; it is easily seen that $|G\colon 1| = h'd'$. Set $d = (h', d')$ and let $h' = hd$, $d' = md$ where $(h, m) = 1$. Set $R = Z[1/d]$ and $S = Z[1/h'] = R[1/h]$ and let ${}_{R[G]}\mathfrak{C}$ and ${}_{S[G]}\mathfrak{C}$ denote the categories of left $R[G]$ and $S[G]$-modules respectively. One can view ${}_{R[G]}\mathfrak{C}$ as the class of G-modules which are uniquely divisible by d and hence ${}_{R[G]}\mathfrak{C}$ forms a complete subcategory of ${}_G\mathfrak{C}$; a similar statement holds for ${}_{S[G]}\mathfrak{C}$.

Thus, we obtain cohomology theories with respect to $\mathfrak{H} = f\big((G,X)\big)$: $\{F^n_R\mid n\in Z\}$ for ${}_{R[G]}\mathfrak{C}$ and $\{F^n_S\mid n\in Z\}$ for ${}_{S[G]}\mathfrak{C}$.

LEMMA 2.2.1. *If $u\colon A\to B$ is a map in ${}_{S[G]}\mathfrak{C}$ which splits over S, then u splits over $S[H]$ for all $H\in\mathfrak{H}$.*

Proof. Let $\alpha\colon B\to A$ be an S-map such that $u\circ\alpha = 1_B$ and let $H\in\mathfrak{H}$ and define $\bar\alpha\colon B\to A$ by $\bar\alpha(b) = \dfrac{1}{|H\colon 1|}\sum_{a\in H} a^{-1}\alpha(ab)$ for all $b\in B$. Then, it is easy to see that $\bar\alpha$ is an $S[H]$-map such that $u\circ\bar\alpha = 1_B$. A similar proof holds if there is an S-map $\alpha\colon B\to A$ such that $\alpha\circ u = 1_A$.

Letting $\mathfrak{K}$ denote the set consisting of just the identity subgroup of G, we see that the class $\mathfrak{E}^S_0(\mathfrak{H})$ of all sequences in ${}_{S[G]}\mathfrak{C}$ which are $S[H]$-split exact for every $H\in\mathfrak{H}$ is just the class $\mathfrak{E}^S_0(\mathfrak{K})$ and similarly for $\mathfrak{E}^0_S(\mathfrak{H})$ and $\mathfrak{E}^0_S(\mathfrak{K})$ and thus $P_S(\mathfrak{H}) = P_S(\mathfrak{K})$.

THEOREM 2.2.1. *If $\mathfrak{K}$ is the set consiting of just the identity subgroup of G, then the $\big(P_S(\mathfrak{H}), \mathfrak{H}\big)$ and $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theories in ${}_{S[G]}\mathfrak{C}$ coincide. (Here $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$ is the S-split "ordinary" cohomology theory in ${}_{S[G]}\mathfrak{C}$).*

Proof. It suffices to show that if $A\in {}_{S[G]}\mathfrak{C}$, then $U_{\mathfrak{H}}(A) = U_{\mathfrak{K}}(A)$. Clearly, if $A\in {}_{S[G]}\mathfrak{C}$ and $H\in\mathfrak{H}$, then $N(A)\subseteq S_{G|H}(A^H)$ where $N = S_{G|1}$. If $a\in A^H$ and if $\{x_1, \ldots, x_n\}$ are representatives for the distinct left cosets of H in G, then $N(a) = \sum_{i=1}^{n}\sum_{h\in H} x_i ha = |H\colon 1|\,S_{G|H}(A)$. But, since $H\in\mathfrak{H}$,

$|H:1|$ has in inverse in S and hence $S_{G|H}(A^H) \subseteq N(A)$. Thus, $S_{G|H}(A^H) = N(A)$ for all $H \in \mathfrak{H}$ which implies that $U_{\mathfrak{H}}(A) = U_{\mathfrak{K}}(A)$.

§ 3. Interrelationships between the cohomology theories.

Let T_R: ${}_{R[G]}\mathfrak{C} \to {}_G\mathfrak{C}$ and let T_S: ${}_{S[G]}\mathfrak{C} \to {}_{R[G]}\mathfrak{C}$ denote the usual "forgetful" functors. If $E \in P_R(\mathfrak{H})$, then $T_R(E) \in P(\mathfrak{H})$ and T_R sends $G|H$-special modules of ${}_{R[G]}\mathfrak{C}$ into $G|H$-special modules of ${}_G\mathfrak{C}$ for all $H \in \mathfrak{H}$; moreover, $F^0 \circ T_R$ is naturally equivalent to the functor $U_{\mathfrak{H}}$. A similar result holds using the functors T_S: ${}_{S[G]}\mathfrak{C} \to {}_{R[G]}\mathfrak{C}$ and $T_R \circ T_S$: ${}_{S[G]}\mathfrak{C} \to {}_G\mathfrak{C}$. Hence:

THEOREM 2.3.1. *$\{(F^n \circ T_R)|\ n \in Z\}$ is a $(P_R(\mathfrak{H}), \mathfrak{H})$-cohomology theory for ${}_{R[G]}\mathfrak{C}$ and $\{(F^n_R \circ T_S)|\ n \in Z\}$ and $\{(F^n \circ T_R \circ T_S)|\ n \in Z\}$ are $(P_S(\mathfrak{H}), \mathfrak{H})$-cohomology theories for ${}_{S[G]}\mathfrak{C}$.*

On the other hand, consider the functors: V: ${}_G\mathfrak{C} \to {}_{R[G]}\mathfrak{C}$ defined by: $V(A) = R \otimes A \cong R[G] \otimes_G A$ for $A \in {}_G\mathfrak{C}$ and W: ${}_{R[G]}\mathfrak{C} \to {}_{S[G]}\mathfrak{C}$ defined by: $W(B) = S \otimes_R B \cong S[G] \otimes_{R[G]} B$ for $B \in {}_{R[G]}\mathfrak{C}$. For $A \in {}_H\mathfrak{C}$, $V(Z[G] \otimes_H A) \cong R[G] \otimes_G Z[G] \otimes_H A \cong R[G] \otimes_H A \cong R[G] \otimes_{R[H]} R[H] \otimes_H A$. Hence, V sends $\mathfrak{E}_0(\mathfrak{H})$-projectives (and $\mathfrak{E}^0(\mathfrak{H})$-injectives) into $\mathfrak{E}_0^R(\mathfrak{H})$-projectives (and $\mathfrak{E}^0_R(\mathfrak{H})$-injectives); also, V sends $P(\mathfrak{H})$ into $P_R(\mathfrak{H})$.

THEOREM 2.3.2. *$\{(F^n_R \circ V)|\ n \in Z\}$ is an exact $P(\mathfrak{H})$-connected sequence of covariant additive functors from ${}_G\mathfrak{C}$ into Ab such that, for each $n \in Z$, $(F^n_R \circ V, F^{n+1}_R \circ V)$ is both left $P(\mathfrak{H})$-couniversal and right $P(\mathfrak{H})$-universal in the sense of* [5], *Chapter* XII, § 7, *and the functor $F^0_R \circ V$ is naturally equivalent to the functor $U_{\mathfrak{H}} \circ V$. Moreover, $\{F^n_R \circ V|\ n \in Z\}$ with these properties is unique up to isomorphism of doubly infinite $P(\mathfrak{H})$-connected sequences of functors. A similar situation holds for the sequences of functors $\{(F^n_S \circ W \circ V)|\ n \in Z\}$ and $\{(F^n_S \circ W)|\ n \in Z\}$.*

Proof. Apply the results of [5], Chapter XII, § 7.

LEMMA 2.3.1. *If $S = R[1/h]$ and G are as above, then for any finitely generated left $R[G]$-module A and any $B \in {}_{R[G]}\mathfrak{C}$, the natural homomorphism Φ: $S \otimes_R \mathrm{Hom}_{R[G]}(A, B) \to \mathrm{Hom}_{R[G]}(A, S \otimes_R B)$, defined by: $\Phi(s \otimes_R f)(a) = s \otimes_R f(a)$ for all $s \in S$, $f \in \mathrm{Hom}_{R[G]}(A, B)$ and $a \in A$, is an isomorphism.*

Proof. Since R is noetherian and G is finite, $R[G]$ is left noetherian and hence A is a finitely presented left $R[G]$-module. Now ([1], Chapitre I, § 2, No. 9, Prop. 10) applies with left and right interchanged.

Now, returning to the situation before the lemma, if we view R as a trivial G-module, then for any $B \in {}_{R[G]}\mathfrak{C}$ and for any subgroup K of G, $S \otimes_R (B^K) \cong S \otimes_R \mathrm{Hom}_{R[K]}(R, B) \cong \mathrm{Hom}_{R[K]}(R, S \otimes_R B) \cong (S \otimes_R B)^K$ where the isomorphism λ: $S \otimes_R (B^K) \to (S \otimes_R B)^K$ is given by: $\lambda(s \otimes_R b) = s \otimes_R b$ for all $b \in B^K$ and $s \in S$.

LEMMA 2.3.2. *The functors $F^0_S \circ W$ and $U_{\mathfrak{H}}$ are naturally equivalent.*

Proof. $F^0_S \circ W$ is naturally equivalent to the functor $U_{\mathfrak{H}} \circ W$. For $B \in {}_{R[G]}\mathfrak{C}$,

$$U_{\mathfrak{H}}(W(B)) = (S \otimes_R B)^G / \sum_{H \in \mathfrak{H}} S_{G|H}(S \otimes_R B)^H$$

$$\cong S \otimes_R (B^G) / S \otimes_R \Big(\sum_{H \in \mathfrak{H}} S_{G|H}(B^H)\Big) \cong S \otimes_R U_{\mathfrak{H}}(B)\,.$$

But, $0 = d' U_{\mathfrak{H}}(B) = md\, U_{\mathfrak{H}}(B) = m\, U_{\mathfrak{H}}(B)$ since d is a unit on $U_{\mathfrak{H}}(B)$; thus, $(m, h) = 1$ implies that h is a unit on $U_{\mathfrak{H}}(B)$ and hence $S \otimes_R U_{\mathfrak{H}}(B) \cong U_{\mathfrak{H}}(B)$. Finally, note that all of the isomorphisms are functorial in B.

Applying Theorem 2.3.2, we obtain

THEOREM 2.3.3. *$\{(F^n_S \circ W)|\ n \in Z\}$ is a $(P_R(\mathfrak{H}), \mathfrak{H})$-cohomology theory for ${}_{R[G]}\mathfrak{C}$.*

§ 4. The Duality Theorems.

Let $\{f^S_{p,q}|\ p, q \in Z\}$ denote the cup product for the $(P_S(\mathfrak{H}), \mathfrak{H})$-cohomology theory $\{F^n_S|\ n \in Z\}$ and, for any $A, B \in {}_{R[G]}\mathfrak{C}$, let $t(A, B)$: $(S \otimes_R A) \otimes_S (S \otimes_R B) \to S \otimes_R (A \otimes_R B)$ denote the obvious $S[G]$-isomorphism determined by: $t(A, B)\big((s_1 \otimes_R a) \otimes_S (s_2 \otimes_R b)\big) = s_1 s_2 \otimes_R (a \otimes_R b)$ for all $s_1, s_2 \in S$ and all $a \in A$ and $b \in B$. Also, if A, B are R-modules (respectively left $R[G]$-modules), then $A \otimes B = A \otimes_Z B = A \otimes_R B$ as R-modules (respectively left $R[G]$-modules). The same holds true for the pairs of rings (Z, S) and (R, S) and these facts will be used in what follows without further notice.

THEOREM 2.4.1. *For any $p, q \in Z$ and any $A, B \in {}_{R[G]}\mathfrak{C}$ the morphism*

$$F^{p+q}_S(t(A, B)) \circ f^S_{p+q}(W(A), W(B)) \colon F^p_S(W(A)) \otimes_R F^q_S(W(B)) \to F^{p+q}_S(W(A \otimes_R B))$$

defines the cup product for the $(P_R(\mathfrak{H}), \mathfrak{H})$-cohomology theory $\{(F^n_S \circ W)|\ n \in Z\}$.

Proof. It is easy to check the cup product defining properties.

Noting that $\{(f_{p,q}|\ p, q \in Z\}$ i sthe cup product for the $(P(\mathfrak{H}), \mathfrak{H})$-cohomology theory $\{F^n|\ n \in Z\}$, the same method of proof yields:

THEOREM 2.4.2. *For any $p, q \in Z$ and any $A, B \in {}_{R[G]}\mathfrak{C}$ the morphism*

$$f_{p,q}(T_R(A), T_R(B)) \colon F^p(T_R(A)) \otimes_R F^q(T_R(A)) \to F^{p+q}(T_R(A \otimes_R B))$$

defines the cup product for the $(P_R(\mathfrak{H}), \mathfrak{H})$-cohomology theory $\{(F^n \circ T_R)|\ n \in Z\}$.

The following generalizes [8], Theorem 4.1, since we do not assume that (G, X) is transitive:

THEOREM 2.4.3. *If $B \in {}_{S[G]}\mathfrak{C}$ is a trivial G-module which is also divisible as an S (or Z)-module and if $A \in {}_{R[G]}\mathfrak{C}$, then the map*

$$h_{p-1,-1}\big(T_R(A), T_R(B)\big)\colon F^{p-1}\big(\mathrm{Hom}\big(T_R(A), T_R(B)\big)\big)$$
$$= F^{p-1}\big(\mathrm{Hom}_R\big(T_R(A), T_R(B)\big)\big) \to \mathrm{Hom}\big(F^{-1}\big(T_R(A)\big), F^{p-2}\big(T_R(B)\big)\big)$$
$$= \mathrm{Hom}_R\big(F^{-1}\big(T_R(A)\big), F^{p-2}\big(T_R(B)\big)\big),$$

defined by:

$$h_{p-1,-1}\big(T_R(A), T_R(B)\big)(\gamma)(a) = \big(F^{p-2}(\theta) \circ f_{p-1,-1}\big(T_R(A), T_R(B)\big)\big)(\gamma \otimes a)$$
$$= \big(F^{p-2}(\theta) \circ f_{p-1,-1}\big(T_R(A), T_R(B)\big)\big)(\gamma \otimes_R a)$$

for all

$$\gamma \in F^{p-1}\big(\mathrm{Hom}_R\big(T_R(A), T_R(B)\big)\big) = F^{p-1}\big(\mathrm{Hom}\big(T_R(A), T_R(B)\big)\big)$$

and all $a \in F^{-1}\big(T_R(A)\big)$ where θ: $\mathrm{Hom}_R(A, B) \otimes_R A = \mathrm{Hom}(A, B) \otimes A \to B$ is defined by: $\theta(f \otimes a) = \theta(f \otimes_R a) = f(a)$ for all $f \in \mathrm{Hom}_R(A, B)$ $= \mathrm{Hom}(A, B)$ and all $a \in A$, is an isomorphism for all $p \in Z$.

Proof. Using Theorem 2.4.2 and the analog of Prop. 2.1.1 in the category ${}_{R[G]}\mathfrak{C}$, it suffices to show that $h_{0,-1}\big(T_R(A), T_R(B)\big)$ is an isomorphism for B as above and for A arbitrary in ${}_{R[G]}\mathfrak{C}$. Thus, by the "uniqueness" of a $\big(P_R(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory and the "uniqueness" of the cup product, it suffices to prove that

$$h'_{0,-1}(A, B)\colon F^0_S\big(W\big(\mathrm{Hom}_R(A, B)\big)\big) \to \mathrm{Hom}_R\big(F_S^{-1}\big(W(A)\big), F_S^{-1}\big(W(B)\big)\big)$$

defined by:

$$h'_{0,-1}(A, B)(\gamma)(a)$$
$$= \Big(F_S^{-1}\big(W(\theta)\big)F_S^{-1}\big(t\big(\mathrm{Hom}_R(A, B), A\big)\big) \circ f^S_{0,-1}\big(\mathrm{Hom}_R(A, B), A\big)\Big)(\gamma \otimes_R a)$$

for all $\gamma \in F^0_S\big(W\big(\mathrm{Hom}_R(A, B)\big)\big)$ and all $a \in F_S^{-1}\big(W(A)\big)$, is an isomorphism. On the other hand, consider the morphism

$$h''_{0,-1}(S \otimes_R A, B)\colon F^0_S\big(\mathrm{Hom}_S(S \otimes_R A, B)\big) \to \mathrm{Hom}_S\big(F_S^{-1}(S \otimes_R A), F_S^{-1}(B)\big)$$

which is the analog in the category ${}_{S[G]}\mathfrak{C}$ of the mapping $h_{0,-1}$ and which utilizes the $S[G]$-morphism θ': $\mathrm{Hom}_S(S \otimes_R A, B) \otimes_S (S \otimes_R A) \to B$ defined by: $\theta'\big(f \otimes_S (s \otimes_R a)\big) = f(s \otimes_R a)$ for all $s \in S$, $a \in A$ and $f \in \mathrm{Hom}_S(S \otimes_R A, B)$. But, by Theorem 2.2.1, $\{F^n_S \mid n \in Z\}$ can be computed by using the ordinary standard complex with coefficients from S instead of Z. Hence, by analogs of [2], Chapter XII, Prop. 6.3 and Theorem 6.4, we find that $h''_{0,-1}(S \otimes \otimes_R A, B)$ is an isomorphism.

If ϱ denotes the natural $S[G]$-isomorphism ϱ: $S \otimes_R \mathrm{Hom}_R(A, B)$ $\mathrm{Hom}_S(S \otimes A, B)$ and if $\mathfrak{S}$ denotes the natural $S[G]$-isomorphism $\mathfrak{S}$: $S \otimes_R B \to B$, then $\mathfrak{S} \circ W(\theta) \circ t\big(\mathrm{Hom}_R(A, B), A\big) = \theta' \circ (\varrho \otimes_S 1)$ as $S[G]$-maps of $\big(S \otimes_R \mathrm{Hom}_R(A, B)\big) \otimes_S (S \otimes_R A) \to B$. From this, it is easy to see that $h'_{0,-1}(A, B) = F_S^{-1}(\mathfrak{S}^{-1}) \circ h''_{0,-1}(S \otimes_R A, B) \circ F^0_S(\varrho)$ and hence $h'_{0,-1}(A, B)$ is also an isomorphism, which concludes the proof.

If we assume that the $\big(P(\mathfrak{H}), \mathfrak{H}\big)$-cohomology theory $\{F^n \mid n \in Z\}$ is obtained by restricting to $P(\mathfrak{H})$ the $Q(\overline{\mathfrak{H}})$-connecting morphisms of the $\big(Q(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$-cohomology theory $\{F^n \mid n \in Z\}$, then the cup product for $\{F^n \mid n \in Z\}$ as both types of cohomology theories comprise the same set of functors $\{f_{p,q} \mid p, q \in Z\}$.

Note that the short exact sequences $0 \to R \to S \to R/S \to 0$ and $0 \to R$ $\to Q \to Q/R \to 0$, where Q denotes the rational numbers and where G acts trivially on all modules, lie in $Q(\overline{\mathfrak{H}})$. Then, using Theorem 2.4.3 (without assuming (G, X) transitive), we can generalize [8], Theorem 6.1, to:

THEOREM 2.4.4. *The homomorphism f^q: $F^{-q}(R) \to \mathrm{Hom}\big(F^q(R), F^0(R)\big)$, defined by: $f^q(a)(\beta) = f^0_{p,q}(a \otimes \beta)$ for all $a \in F^{-q}(R)$ and $\beta \in F^q(F)$ where θ is as defined at the end of Chapter* I, § 3, *is an isomorphism.*

Since Q is uniquely divisible by d', $F^n(Q) = 0$ for all $n \in Z$ and $0 \to R$ $\to Q \to Q/R \to 0 \in \mathfrak{F}(\overline{\mathfrak{H}})$. It follows that

$$F^{-1}(Q/R) \cong F^0(R)$$

and

$$F^{-1}(Q/R) \cong F_S^{-1}\big(S \otimes_R (Q/R)\big) = \mathrm{Ker}\big(N\colon \big(S \otimes_R (Q/R)\big) \to \big(S \otimes_R (Q/R)\big)\big)$$
$$= \big\{a \in \big(S \otimes_R (Q/R)\big) \big| \ |G\colon 1|\ a = 0\big\}$$

by Theorem 2.2.1. But $|G\colon 1| = d'h'$ and hence $F_S^{-1}\big(S \otimes_R (Q/R)\big) = \big\{a \in \big(S \otimes$ $\otimes_R (Q/R)\big) \big| \ ma = 0\big\} \cong S \otimes_R \{a \in Q/R \mid ma = 0\} \cong \{a \in Q/R \mid ma = 0\}$ since $(m, h) = 1$. However, $mF^q(R) = 0$ and thus, the result of [8], Remark 6.2 has also been generalized to arbitrary finite permutation representations:

THEOREM 2.4.5. $F^{-q}(R) \cong \mathrm{Hom}\big(F^q(R), Q/R\big)$ *for all $q \in Z$.*

III. Periodicity

§ 1. Shapiro's Lemma Generalized. For this chapter, we have as usual a finite permutation representation (G, X) of the group G and we assume that K is an arbitrary subgroup of finite index in G, that $f\big((G, X)\big) = \mathfrak{H}$ and that $f\big((K, X)\big) = \mathfrak{L} = \{K \cap H \mid H \in \mathfrak{H}\}$.

LEMMA 3.1.1. *If $\{F^n \mid n \in Z\}$ is a $\big(Q(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$ (resp. $\big(P(\mathfrak{H}), \mathfrak{H}\big)$)-cohomology theory, then $\big\{F^n\big(\mathrm{Hom}_K(Z[G], *)\big) \mid n \in Z\big\}$ is a $\big(Q(\overline{\mathfrak{L}}), \overline{\mathfrak{L}}\big)$ (resp. $\big(P(\mathfrak{L}), \mathfrak{L}\big)$)-cohomology theory for ${}_K\mathfrak{C}$.*

Proof. If $B \in {}_K\mathfrak{C}$, then

$$F^0\big(\mathrm{Hom}_K(Z[G], B)\big) \cong \big(\mathrm{Hom}_K(Z[G], B)\big)^G / \sum_{H \in \mathfrak{H}} S_{G|H}\big(\mathrm{Hom}_K(Z[G], B)^H\big).$$

Now $f \in \mathrm{Hom}_K(Z[G], B)$ is fixed by the action of G if and only if $f(g) = f(1)$ for all $g \in G$ and since f is a $Z[K]$-homomorphism, $\mathrm{Hom}_K(Z[G], B) \cong B^K$ under the map $f \to f(1)$. Also, $f \in \mathrm{Hom}_K(Z[G], B)$ is fixed by the action of $H \in \mathfrak{H}$ if and only if $f(gh) = f(g)$ for all $g \in G$ and $h \in H$. If $G = \bigcup_{i=1}^{n} Kx_iH$ is the (K, H) double coset decomposition where $\{x_1, \ldots, x_n\}$ is a representative choice from the distinct double cosets and if $\{k_{ij} \mid 1 \leqslant j \leqslant m(i)\}$ is a representative choice from the distinct left cosets of $K \cap (x_iHx_i^{-1})$ in K, then $\{k_{ij}x_i \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m(i)\}$ is a representative choice for the distinct left cosets of H in G. Now, it is clear that

$$\Big(\bigoplus_{i=1}^{n} B^{K \cap (x_iHx_i^{-1})}\Big) = \big(\mathrm{Hom}_K(Z[G], B)\big)^H$$

under the mapping which sends

$$(b_1, \ldots, b_n) \in \bigoplus_{i=1}^{n} B^{K \cap (x_iHx_i^{-1})}$$

into the unique $f \in \big(\mathrm{Hom}_K(Z[G], B)\big)^H$ determined by the property that $f(x_i) = b_i$ for all $1 \leqslant i \leqslant n$. Moreover, if $f \in \big(\mathrm{Hom}_K(Z[G]B)\big)^H$, then

$$\big(S_{G|H}(f)\big)(1) = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} f(k_{ij}x_i) = \sum_{i=1}^{n} S_{K|K \cap (x_iHx_i^{-1})}\big(f(x_i)\big).$$

By varying H in $\mathfrak{H}$, it is now apparent that

$$F^0\big(\mathrm{Hom}_K([G], B)\big) \cong B^K / \sum_{L \in \mathfrak{L}} S_{K|L}(B^L).$$

If $A \in {}_L\mathfrak{C}$ for some $L \in \mathfrak{L}$, then $\mathrm{Hom}_K(Z[G], \mathrm{Hom}_L(Z[K], A)) \cong \mathrm{Hom}_L(Z[G], A)$ and hence is $G|L$-special by [4], Theorem 1.3.1 (here L is of finite index in G). But, $L = K \cap H$ for some $H \in \mathfrak{H}$ and [4], Prop. 1.3.1 implies that $\mathrm{Hom}_L(Z[G], A)$ is $G|H$-special and thus $F^m\big(\mathrm{Hom}_K(Z[G], \ast)\big)$ vanishes on the $\mathfrak{E}_0(\mathfrak{L})$-projectives (and $\mathfrak{E}^0(\mathfrak{L})$-injectives) of ${}_K\mathfrak{C}$. If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is in $Q(\mathfrak{L})$ and if $H \in \bar{\mathfrak{H}}$, then (using the notation above: $G = \bigcup_{i=1}^{n} Kx_iH$, etc.) the sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} (A^{K \cap (x_iHx_i^{-1})}) \xrightarrow{\alpha^*} \bigoplus_{i=1}^{n} (B^{K \cap (x_iHx_i^{-1})}) \xrightarrow{\beta^*} \bigoplus_{i=1}^{n} (C^{K \cap (x_iHx_i^{-1})}) \longrightarrow 0$$

is exact where α^*, β^* are just α, β acting component-wise. Consequently, the analysis above shows that

$$0 \longrightarrow \big(\mathrm{Hom}_K(Z[G], A)\big)^H \xrightarrow{\alpha'} \big(\mathrm{Hom}_K(Z[G], B)\big)^H \xrightarrow{\beta'} \big(\mathrm{Hom}_K(Z[G], C)\big)^H \longrightarrow 0$$

is exact and hence

$$0 \to \mathrm{Hom}_K(Z[G], A) \to \mathrm{Hom}_K(Z[G], B) \to \mathrm{Hom}_K(Z[G], C) \to 0$$

is in $Q(\bar{\mathfrak{H}})$. For $H \in \mathfrak{H}$, let $G = \bigcup_{i=1}^{n} Kx_iH$ be as above and for $1 \leqslant i \leqslant n$ let $\{h_{ij} \mid 1 \leqslant j \leqslant m(i)\}$ be a representative choice for the distinct right cosets of $H \cap (x_iKx_i^{-1})$ in H, then $\{x_ih_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m(i)\}$ is a representative choice for the distinct right cosets of K in G. As a left K-module and a right H-module $Z[G] \cong \bigoplus_{i=1}^{n} M_i$ where $M_i = \bigoplus_{j=1}^{m(i)} Z[K]x_ih_{ij}$.

Now let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be in $P(\mathfrak{L})$; to finish the proof of this lemma, it suffices to prove that $0 \to \mathrm{Hom}_K(M_i, A) \xrightarrow{\alpha'} \mathrm{Hom}_K(M_i, B) \xrightarrow{\beta'} \mathrm{Hom}_K(M_i, C) \to 0$ is split exact as a sequence of left H-modules for any $1 \leqslant i \leqslant n$. Since $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is in $P(\mathfrak{L})$, there is a $K \cap (x_iHx_i^{-1})$-map $\varrho\colon B \to A$ such that $\varrho \circ \alpha = 1_A$. We define a map $\varrho^*\colon \mathrm{Hom}_K(M_i, B) \to \mathrm{Hom}_K(M_i, A)$ by: $(\varrho^* g)(kx_ih_{ij}) = k\varrho\big(g(x_ih_{ij})\big)$ for any $g \in \mathrm{Hom}_K(M_i, B)$ and any $k \in K$; clearly, $\varrho^* g \in \mathrm{Hom}_K(M_i, A)$ and $\varrho^* \circ \alpha' = 1$. We claim that ϱ^* is actually an H-map. For, let $h \in H$ and let $h_{ij}h = x_i^{-1}k'x_ih_{ij}$ where $k' \in K$, then if $g \in \mathrm{Hom}_K(M_i, B)$ we get $\varrho^*(hg)(kx_ih_{ij}) = k\varrho(hg)(x_ih_{ij}) = k\varrho\big(g(x_ih_{ij}h)\big) = k\varrho\big(g(k'x_ih_{ij'})\big) = kk'\varrho\big(g(x_ih_{ij'})\big)$ since $x_i^{-1}k'x_i \in H$ implies that $k' \in K \cap x_iHx_i^{-1}$ and $\big(h(\varrho^* \circ g)\big)(kx_ih_{ij}) = (\varrho^* \circ g)(kx_ih_{ij}h) = (\varrho^* \circ g)(kk'x_ih_{ij'}) = kk'\varrho\big(g(x_ih_{ij'})\big)$ which finishes the proof.

§ 2. Periodicaty in $Z[G]$. Assume the usual setup (G, X), $f\big((G, X)\big) = \mathfrak{H}$ and let $\{F^n \mid n \in Z\}$ denote a $\big(P(\mathfrak{H}), \mathfrak{H}\big)$ or a $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$-cohomology theory.

Lemma 3.2.1. *If for some integer q and some integer n the functors F^{n+q} and F^n are naturally equivalent, then the functors F^{n+q} and F^n are naturally equivalent for all $n \in Z$.*

Proof. The natural equivalence of F^n and F^{n+q} extends by universality to a natural equivalence for all $n \in Z$.

Definition 3.2.1. Any integer q, for which the functors F^m and F^{m+q} are naturally equivalent for some n and hence all $n \in Z$ is called a *period* of (G, X) or of $\big(P(\mathfrak{H}), \mathfrak{H}\big)$ or of $\big(Q(\bar{\mathfrak{H}}), \bar{\mathfrak{H}}\big)$.

Obviously, the periods of (G, X) form a subgroup of Z and (G, X) is said to be *periodic* if this subgroup of Z is not just the zero subgroup.

COROLLARY 3.2.1. *If K is a subgroup of finite index in G, then any period of (G, X) is also a period for (K, X).*

Proof. Apply Lemma 3.1.1.

COROLLARY 3.2.1. *If the finite group G has an Abelian subgroup K of type (p, p) such that (K, X) is fixed point free, then G is not periodic.*

Proof. Since (K, X) is fixed point free $f\big((K, X)\big) = \mathfrak{L}$ consists of just the identity subgroup and hence the $\big(P(\mathfrak{L}), \mathfrak{L}\big)$ and $\big(Q(\overline{\mathfrak{L}}), \overline{\mathfrak{L}}\big)$ cohomology theories are just the "ordinary" cohomology theory for K which is not periodic.

COROLLARY 3.2.3. *If G is a finite group and if h' is as defined in Chapter* II, § 1 *and if the prime p divides $|G\colon 1|$ but does not divide h' and if (G, X) is periodic, then the p-Sylow subgroup of G is cyclic or generalized quaternion.*

Proof. The result follows from a well known group theoretic argument, since G does not have an Abelian subgroup of type (p, p).

§ 3. Periodicity in $R[G]$ and $S[G]$. In this section, G will denote a finite group and (G, X) and $f\big((G, X)\big) = \mathfrak{H}$ are as usual. Also, $h' = hd$ and $d' = md$ where $(m, h) = 1$ are as defined in Chapter II, § 1 and $R = Z[1/d]$ and $S = Z[1/h'] = R[1/h]$. Let $\mathfrak{K}$ denote the set consisting of just the identity subgroup of G (hence $\mathfrak{K} = \overline{\mathfrak{K}}$) and let $I = I(G, X)$ $= \{p \in Z|\ p$ is a prime, $p|d', p\nmid h'\}$. If $I = \Phi$, then d' is a unit in S and hence any $\big(P_S(\mathfrak{H}), \mathfrak{H}\big)$ (which is the same as $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$) or $\big(Q_S(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$ or $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ or $\big(P_R(\mathfrak{H}), \mathfrak{H}\big)$ or $\big(Q_R(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$-cohomology theory (by Theorem 2.3.2) consists only of the zero functor. Thus, for the remainder of the section, we assume that $I \neq \Phi$ and we let $d' = \pi\delta$ where π is an I number (the primes dividing π are in I) and where δ is an I' number (the primes dividing δ are not in I). Thus, $\pi \neq 1$ and $|G\colon 1| = d'h'$ $= \pi(\delta h')$ where $\delta h'$ is an I' number; hence, if $\{F_S^n|\ n \in Z\}$ denotes a $\big(P_S(\mathfrak{H}), \mathfrak{H}\big)$ (or equivalently a $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$ or a $\big(Q_S(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$ or a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ cohomology theory), then $\pi F_S^n = 0$ for all $n \in Z$ since δ and $\delta h'$ are units in S. Here, it should be noted that if $\{F_S^n|\ n \in Z\}$ denotes a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ cohomology theory, since $Q_S(\mathfrak{K}) \supseteq Q_S(\overline{\mathfrak{H}}) \supseteq P_S(\mathfrak{K}) = P_S(\mathfrak{H})$, then $\{F_S^n|\ n \in Z\}$ can be turned into a $\big(Q_S(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$ (resp. $\big(P_S(\mathfrak{H}), \mathfrak{H}\big)$) (which is the same as a $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$)-cohomology theory by restricting the connecting morphisms to $Q_S(\overline{\mathfrak{H}})$ (resp. $P_S(\mathfrak{H}) = P_S(\mathfrak{K})$). Consequently, it will usually only be necessary to work with just a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory.

If S is viewed as a trivial G-module, then $F_S^0(S) \cong S/d'S \cong S/\pi S$ $\cong Z/\pi Z$ which is cyclic of order π and this holds for all of the cohomology theories mentioned above.

As has been noted before, the results of [6], [7], [4] and of Chapter I can be completely rewritten in the category ${}_{S[G]}\mathfrak{C}$ for the three cohomology theories mentioned above. The same can be said for the results of Chapter III, § 1 and § 2 provided that in Corollary 3.2.1 the prime p is assumed to be in I. Moreover, since $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$ and $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ form the "ordinary" cohomology theory in ${}_{S[G]}\mathfrak{C}$, the results of [2] Chapter XII, can be examined for S-analogs.

In particular, [2], Chapter XII, Theorem 10.1, holds for any $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory $\{F_S^n|\ n \in Z\}$ where it is to be noted that if a prime $p \notin I$, then $F_S^n(A)$ has no p primary component for all $n \in Z$ and all $A \in {}_{S[G]}\mathfrak{C}$ and that the "ordinary" cohomology theory over S for a p-Sylow subgroup of G consists only of the 0 functors.

Using [2], Chapter XII, § 7, it is easy to see that a cyclic group has a periodic $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory (of period 2) if the cohomology theory is not 0 and a similar result holds for generalized quaternion groups. Also, [2], Chapter XII, Propositions 6.1, 6.2, and 6.3 and Theorem 6.4 carry over directly to $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theories as has been noted in the proof of Theorem 2.4.3 and hence these results carry over to $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theories. Also, using the $Q_S(\mathfrak{K})$ sequence $0 \to S \to$ $\to Q \to Q/S \to 0$ where Q denotes the rational numbers and where G acts trivially on all modules and recalling that $F_S^0(S) \cong S/\pi S \cong Z/\pi Z$, the S-analog of [2], Chapter XII, Theorem 6.6 follows via a parallel proof for the $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory $\{F_S^n|\ n \in Z\}$.

DEFINITION 3.3.1. An element $\alpha \in F_S^q(S)$ (where S is a trivial G-module) will be called a *maximal generator* if α is a generator of $F_S^q(S)$ of order π.

With this definition, S-analogs of [2], Chapter XII, Propositions 11.1 and 11.2 follow for the $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory $\{F_S^n|\ n \in Z\}$. Then from the S-analogs of Lemma 3.2.1 and of Definition 3.2.1, it becomes obvious that q is a period of $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ if and only if $F_S^q(S)$ has a maximal generator. Also, it is clear that the $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ periods form a subgroup of Z. Also, the S-analog of Lemma 3.1.1 shows that if K is a subgroup of G and if q is a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ period for G, then q is a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$ period for K.

If the subgroup K is such that $|K\colon 1| = \pi_K k'$ where π_K is an I number and k' is an I' number and where $\pi_K \neq 1$ and if g is a maximal generator of $F_S^q(S)$, then the proof of [2], Chapter XII, Prop. 11.3 can be used to show that then $\mathrm{Res}^q(K, G)g$ is a maximal generator for $H_S^q\big(K, T_K(S)\big)$ where $\{H_S^n(K, \ast)|\ n \in Z\}$ denotes a $\big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theory for K. If, now, we assume that K is a p-Sylow subgroup of G for any $p \in I$, then the S-analog of [2], Chapter XII, Prop. 11.4 also follows directly. Finally, the S-analog of [2], Chapter XII, Theorem 11.6 gives:

THEOREM 3.3.1. *For the finite group G, with $I \neq \Phi$, the following statements are equivalent*:

a) *At least one of the following cohomology theories*:

$$\big(P_S(\mathfrak{H}), \mathfrak{H}\big), \ \big(Q_S(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big), \ \big(P_S(\mathfrak{K}), \mathfrak{K}\big), \ \big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$$

has period > 0.

b) *All of the cohomology theories*:

$$\big(P_S(\mathfrak{H}), \mathfrak{H}\big), \ \big(Q_S(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big), \ \big(P_S(\mathfrak{K}), \mathfrak{K}\big), \ \big(Q_S(\mathfrak{K}), \mathfrak{K}\big)$$

have a period > 0.

c) *Every Abelian subgroup of G whose order is an I-number is cyclic.*

d) *If $p \in I$, then every p-subgroup of G is either cyclic or is generalized quanternion.*

e) *If $p \in I$, then every p-Sylow subgroup of G is either cyclic or is a generalized quaternion group.*

Since $\big(P_S(\mathfrak{H}), \mathfrak{H}\big)$ and $\big(P_S(\mathfrak{K}), \mathfrak{K}\big)$-cohomology theories coincide, Theorems 2.3.1 and 2.3.3 can be applied to give:

THEOREM 3.3.2. *For the finite permutation representation (G, X) of the finite group G with $I \neq \Phi$, the 5 equivalent statements of Theorem* 3.3.1 *are also equivalent to each of the two equivalent statements*:

a) *The $\big(P_R(\mathfrak{H}), \mathfrak{H}\big)$-cohomology has a period* > 0.

b) *The $\big(Q_R(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$-cohomology has a period* > 0.

If $d = 1$, then $R = Z$ and these results apply to the $\big(P(\mathfrak{H}), \mathfrak{H}\big)$ and $\big(Q(\overline{\mathfrak{H}}), \overline{\mathfrak{H}}\big)$-cohomology theories. Thus, [2], Chapter XII, Theorem 11.6 has been generalized.

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