Decomposable inverse limits with a single bonding map on $[0,1]$ below the identity

by

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1. Introduction. It has been known for some time that the collection of all limits of inverse sequences of mappings from the interval $[0,1]$ onto $[0,1]$ is the collection C of all non-degenerate chainable continua (compact, connected, metric spaces) [4]. The study of the collection S of all limits of inverse sequences with a single bonding map on $[0,1]$ is more recent. Henderson [5] showed that the pseudo-arc is such a limit, while Mahavier [6] showed that not every chainable continuum is.

We let B denote the collection of all limits of inverse sequences with a single bonding map f on $[0,1]$ such that if $0<x<1$, $f(x)<x$, and note that Henderson's paper also shows that the pseudo-arc is an element of B.

In this paper, we characterize the decomposable elements of B (Theorem 1), and show that B is a proper subcollection of S, since the $\sin\frac{1}{x}$ — continuum is not an element of B (by Theorem 3), but is the inverse limit with single bonding map f, where $f(0)=0, f(\frac{1}{2})=1, f(1)=\frac{1}{2}$, and f is linear on $[0,\frac{1}{2}]$ and on $[\frac{1}{2},1]$.

2. Preliminaries and main theorem. For a discussion of inverse limits, the reader is referred to [2], and for chainable continua, to [1]. A δ-regular ε-chain is a chain such that each link of it is of diameter less than ε, and the distance between any two non-intersecting links of it is greater than δ. A regular chain is a chain which is, for some $\delta>0$ and some $\varepsilon>0$, a δ-regular ε-chain. For more on this, see [3].

If f is a continuous function from $[0,1]$ onto $[0,1]$, then $\lim f$ denotes the limit of the inverse sequence with f as the only bonding map. The distance between two points $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ of $\lim f$ is $\sum\limits_{i=1} |x_i-y_i|\cdot 2^{-i}$.

DEFINITION. The continuum M is said to have *property* S with respect to the points A and B of M if and only if there exists a reversibly continuous transformation θ from M onto M such that $\theta(A)=A$, $\theta(B)=B$, and if $\varepsilon > 0$ and $\delta > 0$ then there exists a positive integer m such that if the distance from B to the point P of M is greater than δ, then the distance from A to $\theta^m(P)$ is less than ε.

THEOREM 1. *If M is a decomposable continuum then in order that there exist a continuous function f from $[0,1]$ onto $[0,1]$ such that if $0 < x < 1$ then $f(x) < x$, and such that M is topologically equivalent to* $\lim f$, *it is both necessary and sufficient that* (1) *M be chainable,* (2) *M be irreducible between some two of its points, A and B, and* (3) *M have property S with respect to A and B.*

3. The conditions are necessary. Suppose f is a continuous function from $[0,1]$ onto $[0,1]$ such that if $0 < x < 1$, then $f(x) < x$. Then $\lim f$ is chainable and irreducible from the point $A(0,0,\ldots)$ to the point $B(1,1,\ldots)$. Let θ denote the reversibly continuous transformation from $\lim f$ onto $\lim f$ such that if $P(p_1, p_2, \ldots)$ is a point of $\lim f$, then

$$\theta(P) = \big(f(p_1), f(p_2), \ldots\big) = \big(f(p_1), p_1, p_2, \ldots\big).$$

LEMMA 1. *If $\delta > 0$, and n is a positive integer, there is a positive number δ' such that if $P(p_1, p_2, \ldots)$ is a point of* $\lim f$ *at a distance from B greater than δ, then $1 - p_n > \delta'$.*

THEOREM 2. *The continuum* $\lim f$ *has property S with respect to the points $A(0,0,\ldots)$ and $B(1,1,\ldots)$.*

Proof. Suppose $\varepsilon > 0$ and $\delta > 0$. There exist (1) a positive integer n such that $(\frac{1}{2})^n < \frac{1}{2}\varepsilon$, (2) a number $\delta' > 0$ such that if $P(p_1, p_2, \ldots)$ is a point of $\lim f$ at a distance from B greater than δ then $1 - p_n > \delta'$ (by lemma 1), (3) a number z such that $0 < z < 1$ and if $z < x \leqslant 1$, then $f(z) > 1 - \delta'$, (4) a number k such that $0 < k < 1$ and if $\frac{1}{2}\varepsilon \leqslant x \leqslant z$, then $f(x) < kx$, and (5) a positive integer m such that $k^m < \dfrac{\varepsilon}{2}$.

Now, if $P(p_1, p_2, \ldots)$ is a point of $\lim f$ at a distance from B greater than δ, then $1 - p_n > \delta'$, and $p_n < z$. Thus

$$\tfrac{1}{2}\varepsilon > f^m(p_n) > f^m(p_{n-1}) > \ldots > f^m(p_1).$$

The distance from A to $\theta^m(P) = \big(f^m(p_1), f^m(p_2), \ldots\big)$ is easily shown to be less than $f^m(p_n) + (\frac{1}{2})^n < \varepsilon$.

THEOREM 3. *If* $\lim f$ *is decomposable, and irreducible from the point P to the point Q, then P is one of the points A and B, and Q is the other.*

Proof. The continuum $\lim f$ is irreducible either from A to P, or from B to P. If $\lim f$ is irreducible from A to P then, since $\lim f$ is decomposable, there is an open set R that contains A such that $\lim f$ is not irreducible from A to any point of R. If P is distinct from B then by

theorem 2 there exists a positive integer m such that $\theta^m(P)$ lies in R, and $\lim f$ is irreducible from A to a point of R, which is impossible. So P is B, and $\lim f$ is irreducible from Q to B. By a similar argument, Q can now be shown to be A.

Similarly, if $\lim f$ is irreducible from B to P, then P is A and Q is B.

4. The conditions are sufficient. Suppose M is a decomposable chainable continuum, with metric d, irreducible between two of its points, A and B, such that M has property S with respect to A and B. Let θ denote a transformation satisfying the requirements of property S with respect to A and B.

LEMMA 2. *If $\varepsilon > 0$, there is a regular ε-chain for M from A to B.*

DEFINITIONS. If N is a subcontinuum of M that separates A from B, then the two mutually separated connected point sets whose sum is $M - N$, and which contain A and B respectively, will be denoted, respectively, by $C_A(N)$ and $C_B(N)$. There exist (1) a subcontinuum N_1 of M that separates A from B such that for some number $\gamma > 0$, $d\big(C_A(N_1), C_B(N_1)\big) > \gamma$ and (2) a positive integer m such that the continuum $\theta^{-m}\big(N_1 \cup C_B(N_1)\big)$ does not intersect N_1. Let $\lambda = \theta^{-m}$ and $N_{n+1} = \lambda^n(N_1)$, for each n. Let L_1 denote the continuum $\big(N_1 \cup C_B(N_1)\big) \cap \cap \lambda\big(N_1 \cup C_A(N_1)\big)$, and for each n, let L_{n+1} denote $\lambda^n(L_1)$. For each n, let J_n denote $L_n - (N_n \cup N_{n+1})$ and let δ_n denote a positive number which is less than half of each of the following numbers: (1) $(\frac{1}{2})^n$, (2) the diameter of N_n, (3) the distance from N_n to $M - (J_{n-1} \cup N_n \cup J_n)$ if $n > 1$, and (4) $d\big(C_A(N_n), C_B(N_n)\big)$. Finally, for each n, let H_n denote $L_{2n} \cup L_{2n+1}$, K_n denote N_{2n}, and ω denote λ^{-2}.

THEOREM 4. *There exist sequences $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers, and $C_1, C_2, \ldots$ and $D_1, D_2, \ldots$ of chains such that if n is a positive integer, then*

(1) $\varepsilon_n < (\frac{1}{2})^n$ and C_n is a regular ε_n-chain that properly covers H_n,

(2) D_n and D_{n+1} are non-degenerate subchains of C_n such that

(a) the closure of any link of D_n does not intersect the closure of any link of D_{n+1}, and

(b) the first (last) link of D_n (D_{n+1}) is the first (last) link of C_n, and D_n (D_{n+1}) is the collection of all links of C_n that intersect K_n (K_{n+1}),

(3) every link of D_{n+1} lies in $H_n \cup H_{n+1}$,

(4) every link of $C_n - (D_n \cup D_{n+1})$ lies in $H_n - (K_n \cup K_{n+1})$, and

(5) if $1 \leqslant i < n$, then $\omega^i(C_n)$ is both an ε_n-chain and a strong refinement of C_{n-i}, such that no two adjacent links of it intersect two non-adjacent links of C_{n-j}, if $1 \leqslant j < n$.

Proof. If C is a chain for M from A to B, the first link of C is the one that contains A. There exist (1) a sequence $E_1, E_2, \ldots$ of regular chains for M from A to B and (2) decreasing sequences $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ of positive numbers such that for each n, (1) if $0 \leqslant i < n$, then $\lambda^{-i}(E_n)$ is a β_n-regular α_n-chain, (2) α_n is less than both δ_n and δ_{n+1}, and (3) if $n > 1$, then α_n is less than half of the Lebesgue number of the covering E_{n-1} of M, and $\alpha_n < \frac{1}{2}\beta_{n-1}$. For each n, let (1) E'_n denote the collection of all links of E_n that intersect L_n, (2) F_{n+1} denote the (non-degenerate) collection of all links of E_n that intersect N_{n+1} and (3) d_{n+1} denote the first link of E'_{n+1} that does not lie in some link of E'_n. If $n > 1$, then d_n lies in J_n, and intersects only the last link of E'_{n-1}, which must be a link of F_n.

For each $n > 1$, let F'_n denote the chain of which l' is a link if and only if for some link l of F_n, l' is the union of $l \cap J_n$ and all the links of E'_n that precede d_n and lie in l. Let E''_1 denote $(E'_1 - F_2) \cup F'_2$ and if $n > 1$, let E''_n denote the collection to which l belongs if and only if either (1) l is a link of $F'_n \cup F'_{n+1}$, or (2) l is a link of $E'_n - F_{n+1}$ that does not precede d_n.

For each n, let $C_n = E''_{2n} \cup E''_{2n+1}$, $D_n = F'_{2n}$, and $\varepsilon_n = \alpha_{2n-1}$. The sequences $\varepsilon_1, \varepsilon_2, \ldots, C_1, C_2, \ldots$, and $D_1, D_2, \ldots$ have the properties required by this theorem.

Definitions. With the aid of theorem 4, we now construct a function f such that $\lim f$ is topologically equivalent to M. For each positive integer i, and positive integer j such that the chain $C_1 \cup C_2 \cup \ldots \cup C_i$ has j links, let l_j denote the jth link of it. For each i, let n_i (n'_i) denote the positive integer such that l_{n_i} ($l_{n'_i}$) is the first (last) link of D_i. Then $n_1 = 1$, and the sequence $n_1, n'_1, n_2, n'_2, \ldots$ is increasing. Let C' denote $C_1 \cup C_2 \cup \ldots$.

Let $a_1, a_2, \ldots$ denote an increasing sequence of numbers that converges to 1 such that $a_1 = 0$. If l is a link of C', then $a(l)$ denotes a_i, where i is the positive integer such that $l = l_i$. Let f denote the function from $[0, 1]$ onto $[0, 1]$ such that (1) $f(0) = f(a_{n_2-1}) = 0$ and $f(1) = 1$, (2) if i is a positive integer and $i \geqslant n_2$, then (a) if $\omega(l_i)$ intersects only one element, l_j of C', then $f(a_i) = a_j$, and (b) if $\omega(l_i)$ intersects two elements, l_j and l_{j+1}, of C', then $f(a_i) = \frac{1}{2}(a_j + a_{j+1})$ ($\omega(l_i)$ cannot intersect three elements of C'), and (3) f is linear on each of the intervals $[0, a_{n_2-1}]$ and $[a_i, a_{i+1}]$, for each $i \geqslant n_2 - 1$. Note that f is continuous and if $0 < x < 1$, $f(x) < x$.

Finally, for each i, let s_i denote (1) the sect $[0, a_2)$ if $i = 1$, and (2) the segment (a_{i-1}, a_{i+1}) if $i > 1$. If l is a link of C', let $s(l) = s_i$, where i is the positive integer such that $l = l_i$.

The following theorems establish that $\lim f$ is topologically equivalent to M.

Lemma 3. *If i, j, and n are positive integers ($n_2 - 1 \leqslant i < j$) and there is a number y such that $a_j < y < a_{j+1}$ and $a_i \leqslant f^n(y) \leqslant a_{i+1}$, then f^n is linear on $[a_j, a_{j+1}]$, the interval $f^n[a_j, a_{j+1}]$ is a subset of the interval $[a_i, a_{i+1}]$, and*

$$|f^n(a_{j+1}) - f^n(a_j)| \leqslant (\tfrac{1}{2})^n (a_{i+1} - a_i) .$$

Lemma 4. *If i, j, and n are positive integers ($n_2 \leqslant i < j$), and $\omega^n(l_j)$ is a subset of l_i, then $f^n(\bar{s}_j)$ is a subset of s_i ($\bar{s}_j$ denotes the closure of s_j).*

Definition. Let H denote the set $B \cup H_2 \cup H_3 \cup \ldots$, and C denote the collection $C_2 \cup C_3 \cup \ldots$.

Lemma 5. *If $\varepsilon > 0$, $n \geqslant 0$, and $i > 0$, and P is a point of H that lies in $\omega^n(l_i)$, then there is a positive integer E such that if $e > E$, and j is a positive integer such that $\omega^{n+e}(l_j)$ contains P, then $f^e(\bar{s}_j)$ is a subset of s_i and the length of the interval $f^e(\bar{s}_j)$ is less than ε.*

Proof. It is easy to show that $i \geqslant n_2$ and that there is a positive integer E such that if $e > E$, then (1) if $\omega^{n+e}(l_j)$ contains P, then $\omega^{n+e}(l_j)$ lies in $\omega^n(l_i)$, from which $\omega^e(l_j)$ lies in l_i and, by lemma 4, $f^e(\bar{s}_j)$ lies in s_i; and (2) the number $\frac{1}{2}\varepsilon$ is greater than each of the numbers $(\frac{1}{2})^e(a_i - a_{i-1})$ and $(\frac{1}{2})^e(a_{i+1} - a_i)$. Hence, with the aid of lemma 3, each of the numbers $|f^e(a_{j+1}) - f^e(a_j)|$ and $|f^e(a_j) - f^e(a_{j-1})|$ is less than $\frac{1}{2}\varepsilon$. Since f^e is linear on both $[a_{j-1}, a_j]$ and $[a_j, a_{j+1}]$, the length of the interval $f^e(\bar{s}_j)$ is less than ε.

Definitions. Let V denote the collection of all sequences $v = v_1, v_2, \ldots$ such that for each positive integer n, (1) there is a link l (to be denoted by $L_n(v)$) of C such that $v_n = \omega^{n-1}(l)$, and (2) $\bar{v}_{n+1}$ lies in v_n. If v is a sequence of V, let P_v denote the point of H common to all the elements of v. The sequence v will be said to *determine* P_v.

Theorem 5. *Suppose v is a sequence in V and for each n, L_n denotes $L_n(v)$. Then if n is a positive integer, each term of the sequence $s(L_n), f(s(L_{n+1})), f^2(s(L_{n+2})), \ldots$ contains the closure of the next, and there is only one number common to all the elements of this sequence.*

This theorem follows easily from lemmas 4 and 5.

Definition. If n is a positive integer and v is a sequence in V and for each i, L_i denotes $L_i(v)$, then let $x_n(v)$ denote the number common to all the elements of the sequence $s(L_n)$, $f[s(L_{n+1})]$, $f^2[s(L_{n+2})], \ldots$

Theorem 6. *If v is a sequence in V and n is a positive integer, then $x_n(v) = f[x_{n+1}(v)]$.*

Theorem 7. *If the sequences v and v' of V both determine the point P of H, then $x_n(v) = x_n(v')$, for each n.*

Theorems 5, 6, and 7 justify the following:

Definition. Let T_1 denote the transformation from H into $\lim f$ such that if P is a point of H, then (a) if $P = B$, $T_1(P) = (1, 1, \ldots)$ and

(b) if $P \neq B$, then $T_1(P) = (x_1, x_2, \ldots)$, where for each n, $x_n = x_n(v)$, for any sequence v of V that determines P.

LEMMA 6. *If l is an element of C and P is a point of H in $\omega^{n-1}(l)$, for some positive integer n, and $T_1(P) = (x_1, x_2, \ldots)$, then x_n belongs to $s(l)$.*

THEOREM 8. *T_1 is reversibly continuous.*

Proof. With the aid of lemma 6, it is easy to show that T_1 is reversible. Since H is compact, we need show only that T_1 is continuous.

Suppose P is a point of H, $T_1(P) = (x_1, x_2, \ldots)$, and R is an open set in $\lim f$ that contains $T_1(P)$.

Suppose $P \neq B$, v is a sequence of V that determines P, L_i denotes $L_i(v)$ for each i, $\varepsilon > 0$ and n is a positive integer such that if $Q(q_1, q_2, \ldots)$ is a point of $\lim f$ and $|q_n - x_n| < \varepsilon$, then Q is in R. By lemma 5, there is a positive integer e such that, since P is in $v_{n+e} = \omega^{(n-1)+e}(L_{n+e})$, the length of the interval $f^e[\bar{s}(L_{n+e})]$ is less than ε. Also by lemma 6, if $Q(q_1, q_2, \ldots)$ is the image under T_1 of any point of the set $H \cap v_{n+e}$, (which is open with respect to H), then q_{n+e} lies in $s(L_{n+e})$, from which q_n lies in $f^e[s(L_{n+e})]$, as does x_n. So $|q_n - x_n| < \varepsilon$, and Q lies in R.

Suppose $P = B$, ε is a positive number such that if $Q(q_1, q_2, \ldots)$ is a point of $\lim f$ and $1 - q_1 < \varepsilon$, then Q is in R, and n is a positive integer such that $1 - a_n < \varepsilon$. Let D denote the set

$$H \cap (B \cup l_{n+1} \cup l_{n+2} \cup \ldots),$$

which is open with respect to H. If $Q(q_1, q_2, \ldots)$ is the image under T_1 of any point of D, then either $Q = B$ or Q is in l_i for some $i > n$, so that by lemma 6 q_1 is in $s(l_i)$. In any case, $1 - q_1 < \varepsilon$ and Q lies in R.

DEFINITION. For each n, let T_{n+1} denote the transformation from $\omega^n(H)$ into $\lim f$ such that if P belongs to $\omega^n(H)$ and $(x_1, x_2, \ldots) = T_n[\omega^{-1}(P)]$, then $T_{n+1}(P) = \big(f(x_1), f(x_2), \ldots\big)$.

THEOREM 9. *If the point P belongs to $\omega^{n-1}(H)$ for some positive integer n, then $T_n(P) = T_{n+1}(P)$.*

DEFINITION. Let T denote the transformation from M into $\lim f$ such that if P belongs to M, then (1) if $P = A$, $T(P) = (0, 0, \ldots)$, and (2) if $P \neq A$, then $T(P) = T_{n+1}(P)$, for every positive integer n such that $\omega^n(H)$ contains P.

LEMMA 7. *There is a positive integer n such that if P is a point of $M - \omega^n(H)$ and $T(P) = (x_1, x_2, \ldots)$, then $x_1 = 0$.*

THEOREM 10. *T is reversibly continuous.*

Proof. Clearly T is reversible, and since M is compact, it suffices to show that T is continuous. So suppose P is a point of M and R is a region in $\lim f$ that contains $T(P)$.

If $P = A$, then with the aid of lemma 7 it is not difficult to show that there exists a positive integer n such that if D denotes the open set $M - \omega^n(H)$, then $T(D)$ lies in R.

If $P \neq A$, there is a positive integer n such that P lies in an open subset Q of $\omega^n(H)$. $T(P) = T_{n+1}(P)$, and there is an open subset D of Q containing P such that $T_{n+1}(D)$ lies in R. But $T_{n+1}(D) = T(D)$.

THEOREM 11. $T(M) = \lim f$.

Proof. Since $f(0) = 0$ and $f(1) = 1$, $\lim f$ is irreducible from the point $(0, 0, \ldots)$ to the point $(1, 1, \ldots)$. But $T(M)$ is a subcontinuum of $\lim f$, and $T(A) = (0, 0, \ldots)$ and $T(B) = (1, 1, \ldots)$. Hence $T(M) = \lim f$.

Theorems 10 and 11 show that M and $\lim f$ are topologically equivalent.

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