References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] — *Concerning the notion of the shape of compacta*, Proc. of the Symposium on Topology and its Applications, Herceg Novi 1968.

[3] — *Fundamental retracts and extensions of fundamental sequences*, Fund. Math. 64 (1968), pp.55–85.

[4] — *On movable compacta*, Fund. Math. 66 (1969), pp. 137–146.

[5] — *Some remarks concerning the shape of pointed compacta*. Fund. Math. this volume, pp. 221–240.

[6] J. Dugundji, *Modified Vietoris theorems for homotopy*, Fund. Math. 66 (1970), pp.223–235.

[7] H. Patkowska, *A homotopy extension theorem for fundamental sequences*, Fund. Math. 64 (1969), pp. 87-89.

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Regulated bases and completions of regular spaces

by

Solomon Leader (New Brunswick, N. J.)

0. Introduction. One can construct the completion of a metric space from any topological base of open balls knowing only the binary relation $S \Subset T$ on the base B defined to mean S is uniformly interior to T with the diameter of S at most half that of T. Motivated by such constructions involving a "regulator" $S \Subset T$ we introduce here the "abstract regulated base" and show that it has a representation as a base of regularly open subsets of a regular Hausdorff space with the regulator on the base somewhat like a semi-topogenous order [2]. $S \Subset T$ always implies $\bar{S} \subseteq T$, the weakest regulator.

The representation theorem yields a technique for "completing" a regular Hausdorff space relative to a base of regularly open subsets and a regulator on the base. Such completions include all Hausdorff compactifications and local compactifications as well as all metric completions, but not all completions of uniform spaces.

The concept of abstract regulated base comes under the program set forth by K. Menger [8]. The compingent algebra of H. de Vries [13] is a special kind of abstract regulated base. Our representation theorem subsumes that of de Vries and thereby that of M. H. Stone [11].

Our "end" is a generalization of the concept introduced to construct compactifications by H. Freudenthal [4] and P. S. Alexandroff [1]. (See [6] and Chapter 21 of [12].)

1. Abstract regulated bases. An *abstract regulated base* $(\boldsymbol{B}, \ll)$ consists of a non-empty set $\boldsymbol{B}$ and a binary relation $\ll$ on $\boldsymbol{B}$ subject under Definition 1 to the four axioms A_1-A_4 listed below.

Definition 1. Given a, b in $\boldsymbol{B}$ we say that a *meets* b if there exists c in $\boldsymbol{B}$ with both $c \ll a$ and $c \ll b$.

A_1. If $a \ll b$ and $b \ll c$ then $a \ll c$.

A_2. If $c \ll a$ is equivalent to $c \ll b$ for all c in $\boldsymbol{B}$ then $a = b$.

A_3. If $a \ll b$ and c meets every $z \ll b$ then $a \ll c$.

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A_4. Every member of $\boldsymbol{B}$ belongs to some subset $\boldsymbol{E}$ of $\boldsymbol{B}$ satisfying:

E(i). Given a, b in $\boldsymbol{E}$, there exists c in $\boldsymbol{E}$ with both $c \ll a$ and $c \ll b$.

E(ii). If $a \ll b$ and a meets every member of $\boldsymbol{E}$ then b belongs to $\boldsymbol{E}$.

A non-empty subset $\boldsymbol{E}$ of $\boldsymbol{B}$ satisfying E(i) and E(ii) will be called an *end*. The relation $\ll$ will be called a *regulator*.

LEMMA 1. *Given b in $\boldsymbol{B}$ there exists c in $\boldsymbol{B}$ with $c \ll b$.*

Proof. Apply A_4 to get an end $\boldsymbol{E}$ to which b belongs. Then apply E(i) with $a = b$ to get c.

DEFINITION 2. Let $a \leqslant b$ mean $z \ll b$ for all $z \ll a$.

LEMMA 2. (i) *$a \leqslant b$ under Definition 2 is a partial ordering on $\boldsymbol{B}$.*

(ii) *$a \ll b$ implies $a \leqslant b$.*

(iii) *$a \ll b \leqslant c$ implies $a \ll c$.*

(iv) *If c meets every $z \leqslant b$ then $b \leqslant c$.*

Proof. To prove (i) note that reflexivity and transitivity follow from Definition 2 while antisymmetry is A_2. A_1 and Definition 2 give (ii). (iii) follows from Definition 2. (iv) is a reformulation of A_3 in terms of Definition 2.

LEMMA 3. *Every end $\boldsymbol{E}$ in $(\boldsymbol{B}, \ll)$ is a filter in $(\boldsymbol{B}, \leqslant)$. That is,*

F(i). *Given a, b in $\boldsymbol{E}$ there exists c in $\boldsymbol{E}$ with both $c \leqslant a$ and $c \leqslant b$.*

F(ii). *If a belongs to $\boldsymbol{E}$ and $a \leqslant b$ then b belongs to $\boldsymbol{E}$.*

Proof. F(i) follows from E(i) and (ii) of Lemma 2. Given a in $\boldsymbol{E}$ and $a \leqslant b$, use E(*i*) to get c in $\boldsymbol{E}$ with $c \ll a$. Then $c \ll b$ by (iii) of Lemma 2. So by E(i) and Definition 1 every member of $\boldsymbol{E}$ meets c. Hence b belongs to $\boldsymbol{E}$ by E(ii). So F(ii) holds.

LEMMA 4. *The following are equivalent for a, b in $\boldsymbol{B}$:*

(i) *a meets b.* (See Definition 1.)

(ii) *There exists c in $\boldsymbol{B}$ such that both $c \leqslant a$ and $c \leqslant b$.*

(iii) *There exists an end $\boldsymbol{E}$ to which both a and b belong.*

Proof. That (i) implies (ii) follows from (ii) of Lemma 2. Given (ii) use A_4 to choose an end $\boldsymbol{E}$ to which c belongs. Then both a and b belong to $\boldsymbol{E}$ by F(ii) of Lemma 3. So (ii) implies (iii). That (iii) implies (i) follows directly from E(i) and Definition 1.

LEMMA 5. *For b, c in $\boldsymbol{B}$ the following are equivalent:*

(i) *$b \leqslant c$.*

(ii) *c belongs to every end to which b belongs.*

(iii) *Every z which meets b meets c.*

(iv) *c meets every $z \leqslant b$.*

Proof. That (i) implies (ii) is F(ii) of Lemma 3. Let (ii) hold and let z meet b. Then z and b belong to some end $\boldsymbol{E}$ by Lemma 4. Then c belongs to $\boldsymbol{E}$ by (ii). Hence z meets c by Lemma 4. So (ii) implies (iii). Let (iii) hold and let $z \leqslant b$. Then z meets b by Lemma 4 and (i) of Lemma 2. Hence z meets c by (iii). So (iii) implies (iv). That (iv) implies (i) is (iv) of Lemma 2.

LEMMA 6. *Let $\boldsymbol{D}$ and $\boldsymbol{E}$ be ends in $(\boldsymbol{B}, \ll)$. Then*

(i) *If every member of $\boldsymbol{D}$ meets every member of $\boldsymbol{E}$ then $\boldsymbol{D} = \boldsymbol{E}$.*

(ii) *If $\boldsymbol{D} \subseteq \boldsymbol{E}$ then $\boldsymbol{D} = \boldsymbol{E}$.*

Proof. Consider any b in $\boldsymbol{D}$. By E(i) choose a in $\boldsymbol{D}$ with $a \ll b$. a meets every member of $\boldsymbol{E}$ by the hypothesis of (i). Hence b belongs to $\boldsymbol{E}$ by E(ii). So $\boldsymbol{D} \subseteq \boldsymbol{E}$. Interchanging $\boldsymbol{D}$ and $\boldsymbol{E}$ in the proof we get (i). (ii) follows from (i) and Lemma 4.

2. Stone bases.

DEFINITION 3. An abstract regulated base $(\boldsymbol{B}, \ll)$ is a *Stone base* if $a \ll b$ is equivalent to $a \leqslant b$, that is, if $z \ll b$ for all $z \ll a$ implies $a \ll b$.

THEOREM 1. *A Stone base is a pair $(\boldsymbol{B}, \leqslant)$ where $\boldsymbol{B}$ is a non-empty set and $\leqslant$ is a partial ordering on $\boldsymbol{B}$ satisfying under the definition of "meets" provided by* (ii) *of Lemma* 4:

S_1. *If c meets every $z \leqslant b$ then $b \leqslant c$.*

S_2. *Every member of $\boldsymbol{B}$ belongs to some subset $\boldsymbol{E}$ of $\boldsymbol{B}$ satisfying:*

U(i). *Given a, b in $\boldsymbol{E}$, there exists c in $\boldsymbol{E}$ with both $c \leqslant a$ and $c \leqslant b$.*

U(ii). *If b meets every member of $\boldsymbol{E}$ then b belongs to $\boldsymbol{E}$.*

Proof. That a Stone base has these properties follows from (i) and (iv) of Lemma 2 and A_4. Conversely, given $(\boldsymbol{B}, \leqslant)$ with the above properties define $a \ll b$ to be $a \leqslant b$. Then A_1, A_2, and A_4 are trivial. S_1 implies A_3.

An end $\boldsymbol{E}$ in a Stone base (a non-empty subset $\boldsymbol{E}$ satisfying U(i) and U(ii)) will be called an *ultrafilter*. Every ultrafilter is a maximal filter, that is, a maximal subset of $\boldsymbol{B}$ satisfying U(i). However, unless $a \wedge b$ exists in $\boldsymbol{B}$ whenever a meets b, the Axiom of Choice may yield a maximal filter for which U(ii) fails. So a maximal filter in a Stone base need not be an ultrafilter.

3. Regulated spaces. A *regulated space* is a triple $(X, \mathfrak{B}, \Subset)$ with X a non-empty set and $\mathfrak{B}$ a collection of non-empty subsets of X such that $(\mathfrak{B}, \Subset)$ is a regulated base with A_2 and A_4 replaced by:

B_1. *For every point x in X the set $\mathfrak{B}_x$ of all members of $\mathfrak{B}$ which contain x is an end.*

B_2. *If $\mathfrak{B}_x = \mathfrak{B}_y$ then $x = y$.*

It is obvious that B_1 makes A_4 redundant. That B_1 also makes A_2 redundant can be shown as follows. Consider A not contained in B. Choosing x in $A-B$ we have A in $\mathcal{B}_x$ and B not in $\mathcal{B}_x$. By E(i) choose C in $\mathcal{B}_x$ with $C \Subset A$. Now if $C \Subset B$ then B would belong to every end $\mathcal{E}$ to which C belongs. But the latter is contradicted for $\mathcal{E} = \mathcal{B}_x$. So $C \not\Subset B$.

THEOREM 2. *Let $(X, \mathcal{B}, \Subset)$ be a regulated space. Then*

(i) $B \leqslant C$ *in* $\mathcal{B}$ (Definition 2) *if and only if* $B \subseteq C$.

(ii) A *meets* B *in* $\mathcal{B}$ (Definition 1) *if and only if* $A \cap B \neq \emptyset$.

(iii) $\mathcal{B}$ *is a base for a topology* $\mathcal{C}$ *in* X.

(iv) $A \Subset B$ *implies* $\bar{A} \subseteq B$ *under* $\mathcal{C}$.

(v) X *is a regular Hausdorff space under* $\mathcal{C}$.

(vi) *Under* $\mathcal{C}$ *every member of* $\mathcal{B}$ *is regularly open* (*i.e. is the interior of its closure*).

Proof. Given $B \leqslant C$, then by Lemma 5 C belongs to every end $\mathcal{E}$ to which B belongs. Applying this to $\mathcal{E} = \mathcal{B}_x$ we conclude $B \subseteq C$. Conversely, given $B \nleqslant C$, there exists Z by (iv) of Lemma 2 such that $Z \leqslant B$ and Z fails to meet C. By Lemma 4, Z and C belong to no $\mathcal{B}_x$ in common. That is, $Z \cap C = \emptyset$. Moreover, $Z \subseteq B$ since $Z \leqslant B$. So $Z \subseteq B - C$. Z is non-empty since it belongs to $\mathcal{B}$. So $B \nsubseteq C$. Therefore (i) holds.

(ii) follows from (i) through Lemma 4 and B_1.

(iii) follows from (i), B_1, F(i) of Lemma 3, and Proposition 3 in Chapter 8 of [9].

To prove (iv), let $A \Subset B$ in $\mathcal{B}$ and consider any x in $\bar{A}$. The latter condition means according to (ii) that A meets every member of $\mathcal{B}_x$. Hence B belongs to $\mathcal{B}_x$ by B_1 and E(ii). That is, x belongs to B. So $\bar{A} \subseteq B$.

Regularity of X under $\mathcal{C}$ follows from (iv), B_1, and E(i). Therefore, since B_2 implies that singletons are closed, (v) holds.

To prove (vi) we need only show $\bar{C}^\circ \subseteq C$ for every member C of $\mathcal{B}$ since the reverse inclusion holds for all open C. Consider any member B of $\mathcal{B}$ with $B \subseteq \bar{C}$. We must show $B \subseteq C$. For any member Z of $\mathcal{B}$ with $Z \subseteq B$ we have $Z \subseteq \bar{C}$. Hence Z, being open, meets C. Therefore $B \subseteq C$ by (iv) of Lemma 2 and (i) of Theorem 2.

COROLLARY 2 (a). *A regulated space $(X, \mathcal{B}, \Subset)$ consists of a non-empty regular Hausdorff space X, a base $\mathcal{B}$ of non-empty regularly open subsets of X, and a binary relation $\Subset$ on $\mathcal{B}$ such that for A, B, C in $\mathcal{B}$*

(i) *If* $A \Subset B \subseteq C$ *then* $A \Subset C$.

(ii) *If* $A \Subset B$ *then* $\bar{A} \subseteq B$.

(iii) *Given x in X and B in $\mathcal{B}_x$, there exists A in $\mathcal{B}_x$ with $A \Subset B$.*

DEFINITION 4. A regulated space $(X, \mathcal{B}, \Subset)$ is *weakly regulated* if $A \Subset B$ is just $\bar{A} \subseteq B$. $(X, \mathcal{B}, \Subset)$ is *topologically regulated* if it is weakly regulated and $\mathcal{B}$ consists of all non-empty regularly open subsets of X.

THEOREM 3. *A weakly regulated space corresponds to any pair $(X, \mathcal{B})$ where X is a non-empty regular Hausdorff space and $\mathcal{B}$ is a base of non-empty regularly open subsets. A topologically regulated space corresponds to any non-empty regular Hausdorff space.*

Proof. Given $(X, \mathcal{B})$ as characterized above, introduce the weak regulator $\bar{A} \subseteq B$. Then Definition 1 yields (ii) of Theorem 2. A_1, A_3, B_1, and B_2 are easily verified. In particular, for $\mathcal{B}$ consisting of all non-empty regularly open subsets of X, the latter statement in Theorem 3 clearly holds.

DEFINITION 5. A regulated space $(X, \mathcal{B}, \Subset)$ is *complete* if every end in $\mathcal{B}$ has a non-empty intersection in X. That is, every end is of the form $\mathcal{B}_x$ for some x in X. (See Lemma 6.)

4. Representation of abstract regulated bases.

THEOREM 4. *Given an abstract regulated base $(\mathbf{B}, \leqslant)$, there exists a unique complete regulated space $(Y, \mathcal{B}, \Subset)$ such that $(\mathbf{B}, \leqslant)$ is isomorphic to $(\mathcal{B}, \Subset)$.*

Proof. Let Y be the set of all ends in $\mathbf{B}$. Define a mapping φ of $\mathbf{B}$ into the power set of Y as follows. For each b in $\mathbf{B}$, let $\varphi(b)$ be the set U_b of all ends to which b belongs. By A_4 each U_b is non-empty. Let $\mathcal{B}$ be the range of φ. Then Lemma 5 gives $b \leqslant c$ if and only if $U_b \subseteq U_c$. So $b = c$ if and only if $U_b = U_c$. That is, φ maps $\mathbf{B}$ one-one onto $\mathcal{B}$. Therefore, defining $U_b \Subset U_c$ to be $b \lll c$, we have $(\mathcal{B}, \Subset)$ isomorphic to $(\mathbf{B}, \lll)$. Now for $\mathbf{E}$ any end in $\mathbf{B}$ the set $\mathcal{B}_{\mathbf{E}}$ of all members of $\mathcal{B}$ to which $\mathbf{E}$ belongs is the φ-image of $\mathbf{E}$. Hence B_1 and B_2 follow by isomorphism. So $(Y, \mathcal{B}, \Subset)$ is a regulated space. Moreover, isomorphism implies that every end $\mathcal{E}$ in $(\mathcal{B}, \Subset)$ is the φ-image $\mathcal{B}_{\mathbf{E}}$ of some end $\mathbf{E}$ in $(\mathbf{B}, \lll)$. So $(Y, \mathcal{B}, \Subset)$ is complete.

To prove uniqueness let $(X, \mathcal{A}, \lll)$ and $(Y, \mathcal{B}, \Subset)$ be complete regulated spaces with $(\mathcal{A}, \lll)$ isomorphic to $(\mathcal{B}, \Subset)$. Such an isomorphism matches all ends $\mathcal{A}_x$ in $\mathcal{A}$ biuniquely with all ends $\mathcal{B}_y$ in $\mathcal{B}$, thereby inducing a one-one correspondence between X and Y which is an isomorphism between $(X, \mathcal{A}, \lll)$ and $(Y, \mathcal{B}, \Subset)$.

DEFINITION 6. A member b of an abstract regulated base $(\mathbf{B}, \lll)$ is *bounded* if given any subset $\mathbf{A}$ of $\mathbf{B}$ with the property that it intersects every end $\mathbf{E}$ whose members all meet b there exists a finite subset $\mathbf{D}$ of $\mathbf{A}$ with that property.

DEFINITION 7. A subset $\mathbf{D}$ of a partially ordered set $\mathbf{C}$ is a *base* for $\mathbf{C}$ if given c in $\mathbf{C}$ there exists d in $\mathbf{D}$ with $d \leqslant c$.

COROLLARY 4(a). *The following table gives the equivalence between properties of an abstract regulated base $(\mathbf{B}, \lll)$ and corresponding properties of the complete regulated space $(Y, \mathcal{B}, \Subset)$ representing $(\mathbf{B}, \lll)$;*

(1) *If b belongs to every end whose members all meet a then $a \lll b$.*	(1) *$(Y, \mathcal{B}, \Subset)$ is weakly regulated.* (See Definition 4 and Theorem 3.)

(2) *Every non-empty subset* $\boldsymbol{A}$ *of* $\boldsymbol{B}$ *has a supremum in* $(\boldsymbol{B}, \leqslant)$. *Every member of* $\boldsymbol{B}$ *which meets* $\sup \boldsymbol{A}$ *meets some member of* $\boldsymbol{A}$.	(2) $\mathcal{B}$ *consists of all non-empty regularly open subsets of* Y.
(3) *Given any subset* $\boldsymbol{A}$ *of* $\boldsymbol{B}$ *with the property that* $\boldsymbol{A}$ *intersects every end* $\boldsymbol{E}$ *in* $\boldsymbol{B}$ *there exists a finite subset of* $\boldsymbol{A}$ *with that property.*	(3) Y *is compact.*
(4) *A member* b *of* $\boldsymbol{B}$ *is bounded.*	(4) *The member* U_b *of* $\mathcal{B}$ *corresponding to* b *has compact closure.*
(5) *Every end in* $\boldsymbol{B}$ *has a bounded member.*	(5) Y *is locally compact.*
(6) *Every end in* $\boldsymbol{B}$ *has a countable base.*	(6) *The First Axiom of Countability holds in* Y.
(7) $\boldsymbol{B}$ *has a countable base.*	(7) *The Second Axiom of Countability holds in* Y.
(8) *Given subsets* $\boldsymbol{A}, \boldsymbol{C}$ *of* $\boldsymbol{B}$ *such that* $\boldsymbol{E} \cap (\boldsymbol{A} \cup \boldsymbol{C}) \neq \emptyset$ *for every end* $\boldsymbol{E}$ *in* $\boldsymbol{B}$, *there exist subsets* $\boldsymbol{D}, \boldsymbol{F}$ *of* $\boldsymbol{B}$ *such that for every end* $\boldsymbol{E}$ *either* $\boldsymbol{E} \cap \boldsymbol{D} = \emptyset$ *and* $\boldsymbol{E} \cap \boldsymbol{A} \neq \emptyset$ *or* $\boldsymbol{E} \cap \boldsymbol{F} = \emptyset$ *and* $\boldsymbol{E} \cap \boldsymbol{C} \neq \emptyset$.	(8) Y *is normal.*
(9) $(\boldsymbol{B}, \leqslant)$ *is a Stone base.* (See Definition 3 and Theorem 1.)	(9) *The space is weakly regulated and every member of* $\mathcal{B}$ *is both open and closed.*

COROLLARY 4(b). $(\boldsymbol{B}, \leqslant)$ *is a Stone base* (Definition 3 and Theorem 1) *if and only if there exists a Boolean algebra* $(\boldsymbol{A}, \leqslant)$ *with a base in* $\boldsymbol{A} - [0]$ *isomorphic to* $(\boldsymbol{B}, \leqslant)$.

COROLLARY 4(c). *If* $(\boldsymbol{B}, \leqslant)$ *is an abstract regulated base, then* $(\boldsymbol{B}, \leqslant)$ *under Definition 2 is a Stone base.*

5. Completion of regulated spaces.

THEOREM 5. *Given a regulated space* $(X, \mathcal{A}, \leqslant)$, *there exists a unique complete regulated space* $(Y, \mathcal{B}, \Subset)$ *which admits an injection* $f\colon X \to Y$ *for which the mapping* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *defined by* $\varphi(A) = \overline{f(A)}^{\circ}$ *is an isomorphism between* $(\mathcal{A}, \leqslant)$ *and* $(\mathcal{B}, \Subset)$.

Proof. Apply Theorem 4 to the regulated base $(\mathcal{A}, \leqslant)$ to get $(Y, \mathcal{B}, \Subset)$ with an isomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$. For each point x in X the end $\mathcal{A}_x$ is carried into an end in $\mathcal{B}$ which must be of the form $\mathcal{B}_y$ since Y is complete. Define $f(x) = y$. Clearly f injects X into Y.

Let $B = \varphi(A)$. Then for $y = f(x)$ we have the equivalence of the following conditions: $y \in f(A)$, $x \in A$, $A \in \mathcal{A}_x$, $B \in \mathcal{B}_y$, $y \in B$, $y \in B \cap f(X)$. Therefore $f(A) = B \cap f(X)$. So $f(X)$ intersects every basic open set B since $f(A) \neq \emptyset$ because $A \neq \emptyset$. That is, $\overline{f(X)} = Y$. Hence $\overline{B \cap f(X)} = \overline{B}$ since B is open. So $\overline{f(A)}^{\circ} = \overline{B \cap f(X)}^{\circ} = \overline{B}^{\circ} = B$ since B is regularly open by (vi) of Theorem 2.

6. Compact spaces.

THEOREM 6. *The following are equivalent:*

(i) X *is a compact Hausdorff space.*

(ii) X *is a normal Hausdorff space such that* $(X, \mathcal{B}, \Subset)$ *is complete for every regulated base* $(\mathcal{B}, \Subset)$ *compatible with the topology in* X.

(iii) X *is a normal Hausdorff space which is complete as a topologically regulated space.* (See Definition 4.)

Proof. Let $(X, \mathcal{B}, \Subset)$ be a compact regulated space. As is well known every compact Hausdorff space is normal. Consider any end $\mathcal{E}$ in $(\mathcal{B}, \Subset)$. By Lemma 3 and (i) of Theorem 2, $\mathcal{E}$ is a filter in the power set of X. Since X is compact, $\mathcal{E}$ has a cluster point x in X. Thus every member of $\mathcal{E}$ meets every member of $\mathcal{B}_x$. So $\mathcal{E} = \mathcal{B}_x$ by (i) of Lemma 6. Hence $(X, \mathcal{B}, \Subset)$ is complete. So (i) implies (ii).

(ii) implies (iii) *a fortiori*.

Given (iii) for the topologically regulated space $(X, \mathcal{B}, \Subset)$ consider any ultrafilter $\mathfrak{U}$ in the power set of X. We must show that $\mathfrak{U}$ converges to some point x in X, that is, $\mathcal{B}_x \subseteq \mathfrak{U}$. Let $\mathcal{E}$ consist of all members of $\mathcal{B}$ which contain closed members of $\mathfrak{U}$. Thus $\mathcal{E} \subseteq \mathfrak{U}$ and $\mathcal{E}$ is closed under finite intersection. Normality implies that for each A in $\mathcal{E}$ there exists B in $\mathcal{E}$ with $\overline{B} \subseteq A$. So E(i) holds for $\mathcal{E}$. To verify E(ii), let A meet every member of $\mathcal{E}$ and $\overline{A} \subseteq B$. By normality, $\overline{A}$ meets every closed member of $\mathfrak{U}$. Also by normality there exists D in $\mathcal{B}$ with $\overline{A} \subseteq D$ and $\overline{D} \subseteq B$. Thus D meets every member of $\mathfrak{U}$, hence D belongs to $\mathfrak{U}$. So $\overline{D}$ belongs to $\mathfrak{U}$ and therefore B belongs to $\mathcal{E}$. So $\mathcal{E}$ is an end. Since the space is complete, $\mathcal{E} = \mathcal{B}_x$ for some x in X. So (iii) implies (i).

7. Some special cases.

I. Given a metric space X and a base $\mathcal{A}$ of interiors of closed balls, define $S \Subset T$ as in the *Introduction*. Then $(X, \mathcal{A}, \Subset)$ is a regulated space whose completion is the metric completion.

II. Given a completely regular Hausdorff space X, let $\mathcal{A}$ consist of all subsets A of X which are interiors of z-sets [5], that is, $A = (f^{-1}0)^\circ$ for some continuous real-valued function f on X. Define $A \Subset B$ in $\mathcal{A}$ to mean that A is functionally separated from $X - B$. That is, there exists a continuous real-valued f on X with $A \subseteq f^{-1}(0)$ and $X - B \subseteq f^{-1}(1)$. Then $(X, \mathcal{A}, \Subset)$ is a regulated space whose completion is the Stone–Čech compactification.

III. Given a proximity space X in the sense of [3], let $\mathcal{A}$ consist of all non-empty regularly open subsets. Define $A \Subset B$ to mean that A is remote from $X - B$. Then $(X, \mathcal{A}, \Subset)$ is a regulated space whose completion is the Smirnov compactification [10].

IV. Given a local proximity space in the sense of [7], let $\mathcal{A}$ consist of all bounded non-empty regularly open subsets. Define the regulator as in III. Then $(X, \mathcal{A}, \Subset)$ is a regulated space whose completion is the local compactification constructed in [7].

V. Let $(\boldsymbol{B}, \leqslant, \lll)$ be a compingent algebra in the sense of [13]. Then $(\boldsymbol{B}, \lll)$ is an abstract regulated base whose induced partial ordering (Definition 2) coincides with that of the compingent algebra. Moreover, the representation given by Theorem 4 is just the compact space constructed by de Vries [13].

8. Product spaces. For each σ in a non-empty indexing set Σ, let $(X_\sigma, \mathcal{B}_\sigma, \Subset_\sigma)$ be a regulated space. For notational simplicity we assume $X_\sigma \in \mathcal{B}_\sigma$ and $A \Subset_\sigma X_\sigma$ for all A belonging to $\mathcal{B}_\sigma$. Let $X = \times_\sigma X_\sigma$. Let $\mathcal{B}$ consist of all product sets $\times_\sigma B_\sigma$ where $B_\sigma \in \mathcal{B}_\sigma$ for all σ and $B_\sigma = X_\sigma$ for all but finitely many σ. For $A = \times_\sigma A_\sigma$ and $B = \times_\sigma B_\sigma$ in $\mathcal{B}$ define $A \Subset B$ to be $A_\sigma \Subset_\sigma B_\sigma$ for all σ. A routine verification of the axioms shows that $(X, \mathcal{B}, \Subset)$ is a regulated space. The topology induced by the base $\mathcal{B}$ is the product topology since $\mathcal{B}_\sigma$ is a base for the topology in X_σ. For each σ in Σ, let $\mathcal{E}_\sigma$ be an end in $(\mathcal{B}_\sigma, \Subset_\sigma)$. Let $\mathcal{E}$ consist of all product sets $E = \times_\sigma E_\sigma$ where $E_\sigma \in \mathcal{E}_\sigma$ for all σ and $E_\sigma = X_\sigma$ for all but finitely many σ. Then $\mathcal{E}$ is easily seen to be an end in $(\mathcal{B}, \Subset)$. Conversely, every end $\mathcal{E}$ in $(\mathcal{B}, \Subset)$ is of this form. That is, the canonical projection of X onto X_σ carries $\mathcal{E}$ into an end $\mathcal{E}_\sigma$. Clearly $\mathcal{E}$ has a non-empty intersection if and only if each $\mathcal{E}_\sigma$ does. So $(X, \mathcal{B}, \Subset)$ is complete if and only if each $(X_\sigma, \mathcal{B}_\sigma, \Subset_\sigma)$ is complete. Moreover, by considering the imbedding of the product X in the product of the completions $\overline{X_\sigma}$ we conclude that the completion of the product is the product of the completions. Finally we note that a product of weakly regulated spaces is weakly regulated.

Note added in proof: J. R. Porter has pointed out that "normal" in Theorem 6 can by replaced by "completely regular".

References

[1] P. S. Alexandroff, *On bicompact extensions of topological spaces*, Mat. Sborn. N.S. 5 (47) (1939), pp. 403–423.

[2] Á. Császár, *Foundations of General Topology*, New York 1963.

[3] V. A. Efremovich, *The geometry of proximity*, Mat. Sborn. N.S. 31 (73) (1952), pp. 189–200.

[4] H. Freudenthal, *Neuaufbau der Endentheorie*, Ann. Math. 43 (1942), pp. 261–279.

[5] L. Gillman, M. Jerison, *Rings of Continuous Functions*, Princeton 1962.

[6] S. Leader, *On duality in proximity spaces*, Proc. Amer. Math. Soc. 13 (1962), pp. 518–523.

[7] — *Local proximity spaces*, Math. Annalen 169 (1967), pp. 275–281.

[8] K. Menger, *Topology without points*, Rice Inst. Pamphlet 27 (1940), pp. 80–107.

[9] H. L. Royden, *Real Analysis*, New York 1968.

[10] Yu. M. Smirnov, *On proximity spaces*, Mat. Sborn. N.S. 31 (73) (1952), pp. 543–574.

[11] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), pp. 375–481.

[12] W. J. Thron, *Topological Structures*, New York 1966.

[13] H. de Vries, *Compact Spaces and Compactifications*, Assen 1963.

RUTGERS UNIVERSITY
THE STATE UNIVERSITY OF NEW JERSEY
New Brunswick, N. J.

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