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A fixed point theorem for continua which are hereditarily divisible by points

by

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1. Introduction. It has been conjectured (1) that a continuum which is hereditarily divisible by points (that is, a continuum each of whose non-degenerate subcontinua has a cutpoint) has the fixed point property. The main result of this paper (2) is a special case of this conjecture. Specifically, it will be shown that if H is a continuum which is hereditarily divisible by points and $\tau(H) \neq \infty$, then H has the fixed point property. By $\tau(H)$ we denote the degree of non-local connectedness of H defined by Charatonik in [1] (3). This result generalizes the well known theorem (see [5] and others) that trees have the fixed point property, since (as is observed below) a continuum H is a tree if and only if H is hereditarily divisible by points and $\tau(H) = 0$. In the course of proving the main theorem we also prove a fixed point theorem (4) which is the generalization to the non-metric setting of a theorem of Young [7].

2. Preliminaries. This section is devoted to a number of preliminary results which will be needed in the proof of the fixed point theorem mentioned in the introduction. The main theorems of the section are generalizations of theorems due to Charatonik [1] and Young [7].

2.1. Degree of non-local connectedness. In [1] Charatonik defines the degree of non-local connectedness $\tau(H)$ of a hereditarily unicoherent metric continuum H and proves a number of properties of $\tau(H)$. All of the main results of his paper generalize to the non-metric setting

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(1) The conjecture is due to Knaster.
(2) Theorem 3.27 below.
(3) Numbers in square brackets refer to the bibliography at the end of the paper.
(4) Theorem 2.2.18.

with only minor changes in the proofs. For the sake of brevity we therefore simply state these results in the non-metric setting and for the proofs refer the reader to [1]. In what follows we will understand a continuum to be a compact connected Hausdorff space.

DEFINITION 2.1.1. A continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

Remark 2.1.2. It follows immediately from the definition that if H is a subcontinuum of a hereditarily unicoherent continuum, then H is hereditarily unicoherent.

LEMMA 2.1.3. *A continuum H is hereditarily unicoherent if and only if given any set $X \subset H$, there is a unique subcontinuum $I(X)$ of H which is irreducible with respect to containing X.*

DEFINITION 2.1.4. If H is a hereditarily unicoherent continuum and $X \subset H$, we let $I(X)$ denote the continuum of lemma 2.1.3.

DEFINITION 2.1.5. If X is a topological space, then $N(X)$ denotes the points at which X fails to be locally connected.

PROPOSITION 2.1.6. *If H_1 and H_2 are hereditarily unicoherent continua and $H_1 \subset H_2$, then $N(H_1) \subset N(H_2)$.*

DEFINITION 2.1.7. If H is a hereditarily unicoherent continuum, then we define $J(H) = I(N(H))$.

DEFINITION 2.1.8. Let H be a hereditarily unicoherent continuum. For each ordinal α we define $J^\alpha(H)$ as follows:

$$J^0(H) = H\,.$$

If $\alpha \neq 0$, then

$$J^\alpha(H) = \begin{cases} J(J^\beta(H)) & \text{if } \alpha = \beta+1\,, \\ \bigcap_{\beta<\alpha} J^\beta(H) & \text{if } \alpha = \lim_{\beta<\alpha} \beta\,. \end{cases}$$

LEMMA 2.1.9. If $\beta < \alpha$, then $J^\alpha(H) \subset J^\beta(H)$.

Remark 2.1.10. It is clear from 2.1.9 that eventually $J^\alpha(H)$ is constant.

DEFINITION 2.1.11. If H is a hereditarily unicoherent continuum, we define $\tau(H)$, the *degree of non-local connectedness of H*, as follows:

$$\tau(H) = \begin{cases} \min\{\alpha\colon J^{\alpha+1}(H) = \emptyset\} & \text{if } \{\alpha\colon J^{\alpha+1}(H) = \emptyset\} \neq \emptyset\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

PROPOSITION 2.1.12. *If H_1 and H_2 are hereditarily unicoherent continua and $H_1 \subset H_2$, then $\tau(H_1) \leqslant \tau(H_2)$.*

PROPOSITION 2.1.13. *If f is a continuous map from a hereditarily unicoherent continuum H onto a hereditarily unicoherent continuum $f(H)$, then $J^\alpha(f(H)) \subset f(J^\alpha(H))$ for every ordinal α.*

THEOREM 2.1.14. *If f, H and $f(H)$ are as in 2.1.13, then $\tau(f(H)) \leqslant \tau(H)$.*

2.2. A fixed point theorem. In [7] Young proves the following theorem: If X is an arcwise connected metric space with the property that the union of any countable nest of arcs in X is contained in an arc in X, then X has the fixed point property. The object of this section will be to generalize this theorem to the non-metric setting with "arc" being replaced by "topological chain" (to be defined below). In [2] Harris proves this result under the further assumption that X be compact. We begin with some definitions.

DEFINITION 2.2.1. A *topological chain* is a continuum with exactly two non-cutpoints (in particular an arc is a topological chain). Two points in a topological space X are said to be *joined by a topological chain* in X if there is some topological chain in X of which x and y are the two non-cutpoints.

DEFINITION 2.2.2. A space X is said to be *topologically chained* if any two points in X can be joined by a topological chain in X (in particular an arcwise connected space is topologically chained).

DEFINITION 2.2.3. Suppose that X is a topological chain with non-cutpoints x_0 and x_1. We then set $x \leqslant y$ in X if x separates x_0 from y (i.e. if x_0 and y lie in different components of $X \setminus \{x\}$) or if $x_0 = x$ or $x = y$.

The following is a folk theorem:

THEOREM 2.2.4. *If X is a topological chain, then $\leqslant$ is a total order on X and the order topology generated by $\leqslant$ on X is identical with the original topology.*

Remark 2.2.5. It easily follows from 2.2.4 that if X is a topological chain, then since X is compact and connected it must be order complete and order dense in itself under $\leqslant$.

DEFINITION 2.2.6. A continuum X is said to be a *tree* if and only if any two distinct points in X are separated (see 2.2.3) by some third point in X.

It is clear from theorem 2.2.4 and definition 2.2.6 that if X is a topological chain, then X is a tree. In fact the following characterization of topological chains is possible:

LEMMA 2.2.7. *A space X is a topological chain if and only if it is a tree with exactly two non-cutpoints.*

Proof. This is clear from the preceding remark and definitions 2.2.1 and 2.2.6. ∎

We now recall two theorems proved by Ward in [5].

THEOREM 2.2.8. *If X is a tree, then X is topologically chained.*

THEOREM 2.2.9. *A continuum X is a tree if and only if it is locally connected and hereditarily unicoherent.*

COROLLARY 2.2.10. *A hereditarily unicoherent continuum X is a tree if and only if $\tau(X) = 0$.*

Proof. This is immediate from the above theorem and definition 2.1.11. ■

DEFINITION 2.2.11. Let X be a topological chain with non-cutpoints x_0 and x_1. Then for each x, y in X we define

$$[x, y] = \{z \in X\colon x \leqslant z \leqslant y\}, \qquad (x, y) = \{z \in X\colon x \leqslant z \leqslant y \text{ and } x \neq z \neq y\}.$$

(Note that in particular $X = [x_0, x_1]$).

Theorem 2.2.4 implies the following two lemmas whose proofs are not included. The proofs are identical to the proofs of the analogous results for real arcs.

LEMMA 2.2.12. *If X is a topological chain, then every subcontinuum of X is a topological chain (or a point). In fact the subcontinua of X are exactly the "intervals" $[x, y]$ where $x, y \in X$.*

LEMMA 2.2.13. *If X is a topological chain, then the open sets of X are exactly those sets which are disjoint unions of "open intervals" $(x, y) \subset X$ (we think of $[x_0, x)$ and $(x, x_1]$ as open intervals if x_0 and x_1 are the non-cutpoints of X).*

LEMMA 2.2.14. *If X is a topologically chained Hausdorff space with the property that the union of any nest of topological chains in X is contained in a topological chain in X, then given any two points x and y in X, there is a unique topological chain in X joining x and y.*

Proof. Suppose that X is as above and that the theorem does not hold for X. Let $[x, y]_1$ and $[x, y]_2$ be two distinct topological chains in X joining x and y. Then $[x, y]_1 \setminus [x, y]_2$ is open in $[x, y]_1$ and thus contains an "open interval" (z, w) where $z, w \in [x, y]_1 \cap [x, y]_2$. Let $[z, w]_1$ and $[z, w]_2$ be the subintervals of $[x, y]_1$ and $[x, y]_2$ respectively joining z and w. Then $[z, w]_1 \cap [z, w]_2 = \{z, w\}$ by the choice of (z, w). Now let x_α be an increasing (in the sense of $\leqslant$) net in $[z, w]_1$ converging to w and let x_β be an increasing net in $[z, w]_2$ converging to w. Then given any α and β, $[z, x_\alpha] \cup [z, x_\beta]$ is a topological chain. Let $\mathcal{N}$ be a maximal nest of these topological chains. Then by hypothesis $\bigcup \mathcal{N}$ is contained in a topological chain in X. Thus $\mathrm{Cl}(\bigcup \mathcal{N})$ is contained in a topological chain in X (in fact the same topological chain which contains $\bigcup \mathcal{N}$). But clearly $\mathrm{Cl}(\bigcup \mathcal{N}) = [z, w]_1 \cup [z, w]_2$. Thus we can infer from 2.2.12 that $[z, w]_1 \cup [z, w]_2$ is a topological chain. But it is not difficult to see that $[z, w]_1 \cup [z, w]_2$ has no cutpoints. This contradiction establishes the lemma. ■

DEFINITION 2.2.15. If X is as in 2.2.14 and $x, y \in X$ then we let $[x, y]$ denote the unique topological chain in X joining x and y.

LEMMA 2.2.16. *If X is as in 2.2.14, then the intersection of any two topological chains in X is connected.*

Proof. Suppose that C_1 and C_2 are topological chains in X and that $C_1 \cap C_2 \neq \emptyset$. If $z, w \in C_1 \cap C_2$, then by lemma 2.2.12. C_1 and C_2 both contain topological chains joining z and w. By lemma 2.2.14 these must be identical, i.e. $C_1 \cap C_2$ contains the unique topological chain in X joining z and w. Clearly then $C_1 \cap C_2$ is connected. ■

A well known theorem due to R. L. Moore states that a locally connected, metric continuum is arcwise connected. This implies that the continuous image of an arc in a Hausdorff space is arcwise connected. While Moore's theorem cannot be generalized to the non-metric setting (5), the next theorem proved by Harris in [2] shows that the above-mentioned corollary to Moore's theorem does generalize.

THEOREM 2.2.17. *The continuous image of a topological chain in a Hausdorff space is topologically chained.*

THEOREM 2.2.18. *If X is a topologically chained Hausdorff space with the property that the union of any nest of topological chains in X is contained in a topological chain in X, then X has the fixed point property.*

Proof. Let X be as above and suppose that $f\colon X \to X$ is continuous. We can define a partial order $\leqslant^*$ on X as follows: Let 0 be some distinguished point. Then we set $x \leqslant^* y$ in X if x lies on the unique topological chain in X joining 0 and y. It is not difficult to see that $\leqslant^*$ is a genuine partial order on X. Moreover, any chain (in the sense of $\leqslant^*$) in X has a supremum in X (this follows from the hypotheses on X and the fact that topological chains are order complete under $\leqslant$). Let M be a maximal chain (in the sense of $\leqslant^*$) in X with the property that $x \leqslant^* f(x)$ for every $x \in M$. Since $0 \leqslant^* f(0)$ we can always find such an M. Let $x_0 = \sup M$. We wish first to show that $x_0 \in M$. First observe that the predecessors of x_0 and $f(x_0)$ under $\leqslant^*$ are the topological chains $[0, x_0]$ and $[0, f(x_0)]$ respectively. If $x_0 \nleqslant^* f(x_0)$, then $x_0 \notin [0, f(x_0)]$ and so $[0, x_0] \cap [0, f(x_0)]$ is a proper subinterval (recall lemmas 2.2.16 and 2.2.12) of $[0, x_0]$. That is $[0, x_0] \cap [0, f(x_0)] = [0, x]$ where $x \in [0, x_0]$ and $x \neq x_0$. We now consider two cases. First suppose that $x \neq f(x_0)$. Then we can choose a neighborhood U of $f(x_0)$ such that $x \notin U$. Let V be a neighborhood of x_0 such that $f(V) \subset U$. Now $V \cap [x, x_0]$ is open in $[x, x_0]$ and therefore contains an open interval $(z, x_0]$ in $[x, x_0]$. By our choice of x_0 there must be an $x_1 \in (z, x_0] \cap M$;

(5) See [3].

i.e. there is an $x_1 \in (z, x_0]$ such that $x_1 \leqslant^* f(x_1)$. Now $[x, x_1] \cap [x, f(x_0)] = \{x\}$ by our choice of x. Therefore, the fact that $[x, f(x_1)] \cap [x, f(x_0)]$ is connected allows us to infer that $[x, f(x_1)] \cap [x, f(x_0)] = \{x\}$. We can now conclude that $[x, f(x_1)] \cup [x, f(x_0)]$ is a topological chain. Thus $[x, f(x_1)] \cup [x, f(x_0)]$ is the unique topological chain in X joining $f(x_1)$ and $f(x_0)$. By theorem 2.2.17 $f([x_1, x_0])$ is topologically chained and must therefore contain $[x, f(x_1)] \cup [x, f(x_0)]$ (by the uniqueness of $[x, f(x_1)] \cup \cup [x, f(x_0)]$). Thus $x \in f([x_1, x_0])$. But $[x_1, x_0] \subset (x, x_0] \cap V$ and so we have $x \in f(V)$ contradicting the fact that $f(V) \subset U$. This concludes the consideration of the case $x \neq f(x_0)$. Now suppose $x = f(x_0)$. Then choose disjoint neighborhoods V and U of x_0 and $f(x_0)$ respectively with $f(V) \subset U$. A similar argument to the above can be produced to find $x_1 \in V \cap M$ such that $[x_1, x_0] \subset V$ but with the property that $x_1 \in f([x_1, x_0])$ contradicting our choice of V and U. Therefore we must have $x_0 \leqslant^* f(x_0)$, i.e. $x_0 \in M$. If $x_0 \neq f(x_0)$, then we can choose $x_0 \leqslant^* z \leqslant^* f(x_0)$ with $x_0 \neq z \neq f(x_0)$. By the maximality of x_0 in M we may conclude that $f([x_0, z]) \cap [z, f(x_0)] = \{f(x_0)\}$. One can now argue as above to show that $z \leqslant^* f(z)$, contradicting the choice of x_0. Therefore we must have $x_0 = f(x_0)$, i.e. x_0 is a fixed point for f. ∎

3. The fixed point theorem. In this section we prove the fixed point theorem mentioned in the introduction. In fact we will prove a slightly stronger result. Namely, that if H is a hereditarily unicoherent continuum and $\tau(H) \neq \infty$, then H is hereditarily divisible by points and H has the fixed point property.

DEFINITION 3.1. A continuum H is said to be *hereditarily divisible by points* if every non-degenerate subcontinuum of H (including H itself) has a cutpoint.

The following theorem is well known [6]. A partial converse is given below which allows us to classify those hereditarily unicoherent continua H with $\tau(H) \neq \infty$.

THEOREM 3.2. *If H is a continuum which is hereditarily divisible by points, then H is hereditarily unicoherent.*

Before proving the converse to theorem 3.2 for those H with $\tau(H) \neq \infty$, a few preliminaries are needed. In [6] Whyburn proves the following theorem for metric trees (dendrites). The proof in the non-metric case is the same and is omitted here.

THEOREM 3.3. *If T is a tree and $x \in T$, then x has arbitrarily small connected neighborhoods whose boundaries are finite.*

[6] The author has been unable to find a reference for this result. But in any case it is not difficult to prove.

COROLLARY 3.4. *If H is a hereditarily unicoherent continuum and $x \in H \setminus \mathrm{Cl}(N(H))$ (recall definition 2.1.5), then x has arbitrarily small connected neighborhoods with finite boundary.*

Proof. Let x and H be as above and let V be a connected neighborhood of x such that $\mathrm{Cl}(V) \subset H \setminus \mathrm{Cl}(N(H))$. By remark 2.1.2 $\mathrm{Cl}(V)$ is a hereditarily unicoherent continuum. By proposition 2.1.6 $\mathrm{Cl}(V)$ is locally connected. Therefore $\mathrm{Cl}(V)$ is a tree by theorem 2.2.9.

Theorem 3.3 then implies that x has small connected neighborhoods in $\mathrm{Cl}(V)$ with finite boundary. But $\mathrm{Cl}(V)$ contains an open set about x (namely V). Therefore H contains small connected neighborhoods of x with finite boundary. ∎

THEOREM 3.5. *If H is a hereditarily unicoherent continuum and $\tau(H) \neq \infty$, then H is hereditarily divisible by points.*

Proof. Let H_1 be a non-degenerate subcontinuum of H. By proposition 2.1.12 $\tau(H_1) \leqslant \tau(H)$ and thus $\tau(H_1) \neq \infty$. This implies by the definition of τ that $J(H_1)$ is properly contained in H_1, i.e. $H_1 \setminus I(N(H_1)) \neq \emptyset$. But $\mathrm{Cl}(N(H_1)) \subset I(N(H_1))$. Thus $H_1 \setminus \mathrm{Cl}(N(H_1)) \neq \emptyset$. By corollary 3.4 we can find a connected open set V such that $\mathrm{Cl}(V) \subset H_1 \setminus \mathrm{Cl}(N(H_1))$ and V has finite boundary. As was observed in the proof of 3.4, $\mathrm{Cl}(V)$ is a tree. Therefore $\mathrm{Cl}(V)$ has a cutpoint p. Since V is connected and has finite boundary and H_1 is hereditarily unicoherent, it is easily seen that p must be a cutpoint of H_1. ∎

LEMMA 3.6. *If H is a hereditarily unicoherent continuum such that $\tau(H) \neq \infty$ and if $f\colon H \to H$ is a fixed point free map, then H contains a subcontinuum H_1 (such that $\tau(H_1) \neq \infty$) and such that $f|_{H_1}$ maps H_1 onto H_1.*

Proof. Let f and H be as above and let M be a maximal nest of subcontinua H_m of H such that $f(H_m) \subset H_m$ (such a nest always exists since $f(H) \subset H$). It is easy to show that setting $H_1 = \bigcap M$ we get the desired result ($\tau(H_1) \neq \infty$ by proposition 2.1.12). ∎

By virtue of the previous lemma we need only consider surjective maps in establishing the fixed point property for the hereditary unicoherent continua H with $\tau(H) \neq \infty$. Therefore for the remainder of this section we will make the standing assumption that H is a fixed hereditarily unicoherent continuum, that $\tau(H) = \alpha_0$, a fixed ordinal and that $f\colon H \to H$ is a fixed surjective map.

LEMMA 3.7. *$J^{\alpha_0}(H)$ is a tree.*

Proof. We first show that $J^{\alpha_0}(H) \neq \emptyset$. Recall that

$$\alpha_0 = \tau(H) = \min\{\beta\colon J^{\beta+1}(H) = \emptyset\}.$$

If $\alpha_0 = \beta+1$ for some β, then certainly $J^{\alpha_0}(H) \neq \emptyset$.

If $\alpha_0 = \lim_{\beta<\alpha_0} \beta$, then $J^{\alpha_0}(H) = \bigcap_{\beta<\alpha_0} J^{\beta}(H)$, and $J^{\beta}(H) \neq \emptyset$ for every $\beta < \alpha_0$. Thus since each $J^{\beta}(H)$ is compact $J^{\alpha_0}(H) \neq \emptyset$. Now $J^{\alpha_0}(H)$ is a continuum and therefore by remark 2.1.2 $J^{\alpha_0}(H)$ is a hereditarily unicoherent continuum. Finally, observe that we must have $J^{\alpha_0+1}(H) = \emptyset$ by the definition of α_0. Therefore

$$(*) \qquad \emptyset = J^{\alpha_0+1}(H) = J\big(J^{\alpha_0}(H)\big) = I\Big(N\big(J^{\alpha_0}(H)\big)\Big).$$

$(*)$ implies in particular that $N\big(J^{\alpha_0}(H)\big) = \emptyset$, i.e. that $J^{\alpha_0}(H)$ is locally connected. The lemma now follows from theorem 2.2.9. ■

LEMMA 3.8. $J^{\alpha_0}(H) \subset f\big(J^{\alpha_0}(H)\big)$.

Proof. By proposition 2.1.13 $J^{\alpha_0}\big(f(H)\big) \subset f\big(J^{\alpha_0}(H)\big)$. But since f is surjective $f(H) = H$. ■

LEMMA 3.9. *$f\big(J^{\alpha_0}(H)\big)$ is topologically chained.*

Proof. Since $f\big(J^{\alpha_0}(H)\big)$ is a subcontinuum of H, it is hereditarily unicoherent. Since $J^{\alpha_0}(H)$ is locally connected (theorem 2.2.9), $f\big(J^{\alpha_0}(H)\big)$ is locally connected. Thus by theorem 2.2.9 $f\big(J^{\alpha_0}(H)\big)$ is a tree. Theorem 2.2.8 now implies the desired result. ■

DEFINITION 3.10. For the remainder of this section we let x_0 denote a particular point in $J^{\alpha_0}(H)$ and let A denote the set of all points in H which can be joined to x_0 by a topological chain in H.

LEMMA 3.11. $f(A) \subset A$.

Proof: Let $x \in A$. Then there is a topological chain C in H joining x and x_0. By theorem 2.2.17 $f(C)$ is topologically chained. Therefore there is a topological chain C_1 in H joining $f(x)$ and $f(x_0)$. Now lemma 3.9 states that $f\big(J^{\alpha_0}(H)\big)$ is topologically chained and lemma 3.8 implies that x_0 and $f(x_0)$ are in $f\big(J^{\alpha_0}(H)\big)$. Therefore there is a topological chain C_2 in H joining x_0 and $f(x_0)$. Since H is hereditarily unicoherent, $C_1 \cap C_2$ is connected. Therefore it is clear that $C_1 \cup C_2$ contains a topological chain joining x_0 and $f(x)$, i.e. $f(x) \in A$. ■

Remark 3.12. Certainly A is topologically chained and Hausdorff. Thus in order to show that f has a fixed point in A it will suffice to show that the union of any nest of topological chains in A is contained in a topological chain in A (applying theorem 2.2.18). Note that the full power of 2.2.18 is needed here since in general we cannot expect A to be compact (in fact it is possible to construct an example where A is dense in H).

DEFINITION 3.13. Given $a, b \in A$, we let $[a, b]$ denote the unique topological chain in A joining a and b ($[a, b]$ is unique since $[a, b] = I(\{a, b\})$ in H).

We now make another series of standing assumptions. For the rest of this section $\{[a_\alpha, b_\alpha]\}_{\alpha\in\Gamma}$ will denote a nest of topological chains in A with the a_α's being the "left" endpoints always (we can define *left* as follows: Fix some interval $[a_0, b_0]$ in the nest and call a_0 its left endpoint. If $[a_\alpha, b_\alpha]$ is another interval in the nest and $[a_0, b_0] \subset [a_\alpha, b_\alpha]$, then a_α is the left endpoint if $a_0 \in [a_\alpha, b_0]$. If $[a_\alpha, b_\alpha] \subset [a_0, b_0]$, then a_α is the left endpoint if $a_\alpha \in [a_0, b_\alpha]$). Now since the $[a_\alpha, b_\alpha]$'s are nested it is clear that if x is any point in $\bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$, then

$$(*) \qquad \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\} = \bigcup\{[a_\alpha, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}$$
$$= \bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\} \cup \bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}.$$

DEFINITION 3.14. For the remainder of this section we will let C denote $\mathrm{Cl}(\bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\})$. If $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$ we set $R(x) = \mathrm{Cl}(\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\})$ and $L(x) = \mathrm{Cl}(\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\})$ where all three of the above closures are taken relative to H.

Remark 3.15. From $(*)$ above and the fact that the $[a_\alpha, b_\alpha]$'s are nested it is clear that $C = L(x) \cup R(x)$ for any $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$. Note that C, $L(x)$ and $R(x)$ are all subcontinua of H.

LEMMA 3.16. *If*

$$x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$$

and for some $\alpha \in \Gamma$ $x \in [a_\alpha, b_\alpha]\setminus\{a_\alpha, b_\alpha\}$, then there is a $\gamma \in \Gamma$ such that $[a_\gamma, b_\gamma]$ contains an open set relative to C about x.

Proof. Suppose the lemma is not true. Then given any neighborhood U (relative to C) of x and any $[a_\gamma, b_\gamma]$ such that $x \in [a_\gamma, b_\gamma]\setminus\{a_\gamma, b_\gamma\}$ we can find an interval $[a_\alpha, b_\alpha]$ in the nest such that $[a_\gamma, b_\gamma] \subset [a_\alpha, b_\alpha]$ and

$$(1) \qquad ([a_\alpha, b_\alpha]\setminus[a_\gamma, b_\gamma]) \cap U \neq \emptyset.$$

Then without loss of generality we may assume that given any neighborhood U of x relative to $L(x)$ and given any γ such that $x \in [a_\gamma, x]\setminus\{a_\gamma\}$, there is an α such that $[a_\gamma, x] \subset [a_\alpha, x]$ and

$$(2) \qquad ([a_\alpha, x]\setminus[a_\gamma, x]) \cap U \neq \emptyset.$$

Now by theorem 3.5 $L(x)$ must have a cutpoint p. Thus we can find non-degenerate subcontinua E and F of $L(x)$ such that $E \cup F = L(x)$ and $E \cap F = \{p\}$. We suppose (without loss of generality) that $x \in E$. Certainly $x \neq p$ (the connected set $\bigcup\{[a_\alpha, x]\setminus\{x\}\colon x \in [a_\alpha, b_\alpha]\}$ is dense in $L(x)$), so we can find a neighborhood U of x such that $\mathrm{Cl}(U) \cap F = \emptyset$. Now let y be any point of $\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\}$. Then there must be a γ such that $x \in [a_\gamma, b_\gamma]$ and $y \in [a_\gamma, x]$. By our hypothesis about $L(x)$ we can then find an interval $[a_\alpha, b_\alpha]$ containing $[a_\gamma, b_\gamma]$ and such that (2) is satisfied. Let $z \in ([a_\alpha, x]\setminus[a_\gamma, x]) \cap U$. Then $y \in [z, x]$. Moreover since $U \cap F = \emptyset$ we must have $z \in E$. Now E is a continuum containing x and z. Therefore E must contain $I(\{x, z\}) = [z, x]$. But this implies that $y \in E$. Since y was arbitrary in $\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\}$ we have shown that

$$\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\} \subset E\,. \tag{3}$$

Since E is closed in H, this implies that

$$\mathrm{Cl}\big(\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\}\big) \subset E\,; \tag{4}$$

i.e. $L(x) \subset E$, contradicting the assumption that F was nondegenerate. ■

Lemma 3.17. *If* $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$ *and* $x \in [a_\alpha, b_\alpha]\setminus\{a_\alpha, b_\alpha\}$ *for some* $\alpha \in \Gamma$, *then* x *is a cutpoint of* C.

Proof. Let x be as above. The last lemma implies that $[a_\alpha, b_\alpha]$ contains an open set relative to C about x. Thus x has small neighborhood with two point boundaries. Moreover x is a cutpoint of $[a_\alpha, b_\alpha]$. We can now infer, as in the proof of 3.6, that x is a cutpoint of C. ■

Corollary 3.18. *If* x *is as in* 3.17, *then* $L(x) \cap R(x) = \{x\}$.

Proof. Since x cuts C, there must be two non-degenerate subcontinua, E and F, of C such that $E \cup F = C$ and $E \cap F = \{x\}$. Now $L(x) \cup R(x) = C$ and x cuts neither $L(x)$ nor $R(x)$ (this was observed in the proof of 3.16). Therefore the two continua E and F must be $L(x)$ and $R(x)$. ■

Lemma 3.19. *Suppose that*

$$x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}\,,$$

$$y \in L(x)\setminus\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\} \quad \textit{and} \quad z \in R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}\,.$$

Then C *is irreducible between* z *and* y (*i.e.* $C = I(\{z, y\})$ *in* H).

Proof. This follows immediately from corollary 3.18. ■

We now state four lemmas about $R(x)$ which are analogous to the last four lemmas about C. The proofs are strictly analogous to the proofs of the last four lemmas and are therefore omitted here.

Lemma 3.20. *If* $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$ *and* $y \in [x, b_\alpha]\setminus\{x, b_\alpha\}$ *for some* α *such that* $x \in [a_\alpha, b_\alpha]$, *then there is a* γ *such that* $x \in [a_\gamma, b_\gamma]$ *and* $[x, b_\gamma]$ *contains an open set relative to* $R(x)$ *about* y.

Lemma 3.21. *If* y *is as in* 3.20, *then* y *is a cutpoint of* $R(x)$.

Corollary 3.22. *If* y *is as in* 3.20, *then* $[x, y] \cup R(y) = R(x)$ *and* $[x, y] \cap R(y) = \{y\}$.

Lemma 3.23. *If*

$$x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\} \quad \textit{and} \quad z \in R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}\,,$$

then $R(x)$ *is irreducible between* x *and* z (*i.e.* $R(x) = I(\{x, z\})$).

Lemma 3.24. *If* $R(x)$ *is locally connected at some*

$$z \in R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}\,,$$

then $R(x)$ *is a topological chain.*

Proof. Suppose that z is as above. We will show that the set $R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}$ consists of the single point z. For suppose that $y \neq z$ is another point of this set. Since $R(x)$ is locally connected at z, we can choose a connected neighborhood (in $R(x)$) U of z such that $y \notin \mathrm{Cl}(U)$. Since $z \in R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}$ there must be some

$$w \in \bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\} \cap U\,.$$

By corollary 3.22 $R(x) = [x, w] \cup R(w)$ (if $w \notin [x, b_\alpha]\setminus\{x, b_\alpha\}$ for some α, then $R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\} = \emptyset$). By lemma 3.23 $R(w) = I(\{w, z\})$. Since $w, z \in U$ and U is connected, we must have $I(\{w, z\}) \subset \mathrm{Cl}(U)$. That is, $R(w) \subset \mathrm{Cl}(U)$. Therefore

$$R(x) = [x, w] \cup R(w) \subset [x, w] \cup \mathrm{Cl}(U)\,.$$

But clearly $y \notin [x, w]$ and $y \notin \mathrm{Cl}(U)$. This contradiction shows that $R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\} = \{z\}$. This implies by lemma 3.21 that $R(x)$ has at most two non-cutpoints, namely x and z. But every continuum has at least two non-cutpoints. Therefore $R(x)$ has exactly two non-cutpoints and is a topological chain. ■

Lemma 3.25. *If* $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$, *then either* $L(x)$ *or* $R(x)$ *is a topological chain* (*or a point*).

Proof. Suppose that $x \in \bigcup\{[a_\alpha, b_\alpha]\colon \alpha \in \Gamma\}$ and neither $L(x)$ nor $R(x)$ is a topological chain. Then by lemma 3.24 and its dual we can find

$$(*)\qquad \begin{aligned} z &\in \big(L(x)\setminus\bigcup\{[a_\alpha, x]\colon x \in [a_\alpha, b_\alpha]\}\big) \cap N(C)\,,\\ w &\in \big(R(x)\setminus\bigcup\{[x, b_\alpha]\colon x \in [a_\alpha, b_\alpha]\}\big) \cap N(C)\,. \end{aligned}$$

Therefore $z, w \in N(C)$. Thus $I(\{z, w\}) \subset I\big(N(C)\big) = J(C)$. But $(*)$ and lemma 3.19 imply that $I(\{z, w\}) = C$. Therefore $C \subset J(C)$, i.e. $C = J(C)$. But then $C = J^\gamma(C)$ for every ordinal number γ, i.e. $\tau(C) = \infty$. But by proposition 2.1.12

$$\tau(C) \leqslant \tau(H) = a_0 \neq \infty\,.$$

This contradiction establishes the lemma. ■

By virtue of the last lemma we may assume without loss of generality that $R(x)$ is a topological chain for some $x \in \bigcup\{[a_\alpha, b_\alpha]:\ \alpha \in \Gamma\}$. That is, there is some $y_0 \in C$ such that the topological chain $[x, y_0] = R(x)$. If we set $L(y_0) = \mathrm{Cl}(\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\})$, corollary 3.22 implies that

$$C = L(x) \cup R(x) = L(x) \cup [x, y_0] = L(y_0)\,.$$

We now show that $L(y_0)$ is a topological chain. This will suffice to establish the desired fixed point theorem.

LEMMA 3.26. *$L(y_0)$ is a topological chain.*

Proof. For suppose not. Then by the dual of lemma 3.24 $L(y_0)$ fails to be locally connected at every point of the set $L(y_0)\setminus\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}$. Let

$$(*)\qquad \beta = \sup\{\gamma:\ L(y_0)\setminus\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\} \subset J^\gamma(H)\}\,.$$

Then it easily follows that

$$(1)\qquad L(y_0)\setminus\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\} \subset J^\beta(H)$$

and

$$(2)\qquad L(y_0)\setminus\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\} \not\subset J^{\beta+1}(H)\,.$$

So $J^\beta(H)$ is locally connected at some point $z_0 \in L(y_0)\setminus\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}$.

CLAIM 1. $J^\beta(H) \cap (\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}) = \emptyset$.

Proof of claim. For suppose not and let

$$(**)\qquad x \in J^\beta(H) \cap (\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\})\,.$$

Since $J^\beta(H)$ is a continuum we must have

$$(3)\qquad I(\{z_0, x\}) \subset J^\beta(H)\,.$$

Now by the dual of lemma 3.23,

$$(4)\qquad I(\{z_0, x\}) = L(x)\,.$$

And by the dual of corollary 3.22,

$$(5)\qquad L(y_0) = L(x) \cup [x, y_0]$$

and

$$(6)\qquad L(x) \cap [x, y_0] = \{x\}\,.$$

Thus since $L(y_0)$ fails to be locally connected at z_0, $L(x)$ must also fail to be locally connected at z_0. But (3) and (4) imply that $L(x) \subset J^\beta(H)$. Therefore, proposition 2.1.6 implies that $J^\beta(H)$ fails to be locally connected at z_0. This contradiction establishes the claim.

Now clearly $y_0 \in A$ (see 3.10 for the definition of A and x_0). So let $[x_0, y_0]$ be the unique topological chain in H joining x_0 and y_0.

Lemma 2.1.9 states that $J^{\alpha_0}(H) \subset J^\beta(H)$. Therefore, $x_0 \in J^\beta(H)$. This implies that $\{x_0, y_0\} \subset J^\beta(H) \cup L(y_0)$. But $J^\beta(H) \cup L(y_0)$ is connected (both $J^\beta(H)$ and $L(y_0)$ are connected and both sets contain z_0). Therefore,

$$(7)\qquad [x_0, y_0] = I(\{x_0, y_0\}) \subset J^\beta(H) \cup L(y_0)\,.$$

CLAIM 2. $[x_0, y_0] \cap L(y_0) = L(y_0)$.

Proof of claim. Suppose that $x \in \bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}$. We wish to show that $[x, y_0] \subset L(y_0) \cap [x_0, y_0]$. For suppose that this is not the case. Then for some $z \in [x, y_0]$, $z \notin [x_0, y_0] \cap L(y_0)$. By the dual of corollary 3.22

$$(8)\qquad L(y_0) = L(z) \cup [z, y_0]$$

and

$$(9)\qquad L(z) \cap [z, y_0] = \{z\}\,.$$

Now $L(y_0) \cap [x_0, y_0]$ is a continuum by the hereditary unicoherence of H. Therefore, since $z \notin L(y_0) \cap [x_0, y_0]$, (8) and (9) allow us to infer that

$$(10)\qquad L(y_0) \cap [x_0, y_0] \subset [z, y_0]\,.$$

(10) and (7) now imply

$$(11)\qquad [x_0, y_0] \subset [z, y_0] \cup J^\beta(H)\,.$$

But claim 1 implies that $[z, y_0] \cap J^\beta(H) = \emptyset$.

Therefore, $[z, y_0] \cup J^\beta(H)$ is disconnected. Then, since $[x_0, y_0] \cap [z, y_0] \neq \emptyset$, we must have $[x_0, y_0] \subset [z, y_0]$. But this implies that $[x_0, y_0] \cap J^\beta(H) = \emptyset$, a contradiction $(x_0 \in J^\beta(H))$. Therefore, we must have

$$(12)\qquad [x, y_0] \subset L(y_0) \cap [x_0, y_0]$$

for every $x \in \bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}$. But then

$$(13)\qquad L(y_0) = \mathrm{Cl}(\bigcup\{[a_\alpha, y_0]:\ \alpha \in \Gamma\}) \subset L(y_0) \cap [x_0, y_0]$$

i.e. $L(y_0) = L(y_0) \cap [x_0, y_0]$.

Claim 2 implies that $L(y_0)$, being a subcontinuum of the topological chain $[x_0, y_0]$, must be a topological chain by lemma 2.2.12. ∎

The desired fixed point theorem is now clear.

THEOREM 3.27. *If H is a hereditarily unicoherent continuum such that $\tau(H) \neq \infty$, then H has the fixed point property.*

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On the topology of curves I

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By a *curve* we mean a 1-dimensional compact connected metric space. Thus all curves are non-degenerate continua. Although a theory of curves had been established at the early stage of set-theoretical topology by Karl Menger and P. S. Urysohn, some set-theoretical aspects of the theory seem to be far from being explored. Among them are various cardinality problems concerning topological structure of curves and their subsets. For non-compact subsets a classification relevant to connectivity properties had been elaborated by A. D. Taĭmanov [5]. A countable ordinal which we call the non-connectivity index of a space (see § 1) indicates the level on which quasi-components become components of a given point. Solving a problem raised by P. S. Novikov (see § 3) we show that there exists a plane G_δ-set whose non-connectivity indexes are arbitrarily high. This is done by constructing a subset of a pseudo-arc, and we use a result of Howard Cook [2] to prove that an uncountable compact bundle of pseudo-arcs is embedable in a pseudo-arc itself (see § 2). On the other hand, it is shown (see § 4) that non-connectivity indexes of a subset of a rational curve are bounded by a countable ordinal. The results of the present paper were partially announced in [4].

§ 1. Non-connectivity indexes. Let us recall that the quasi-component $Q(X, x)$ of a topological space X at a point $x \in X$ is the intersection of all closed-open subsets of X that contain x. We write $Q^0(X, x) = X$, and we use a transfinite induction to define $Q^\alpha(X, x)$ for each ordinal α, namely

$$Q^{\alpha+1}(X, x) = Q\big(Q^\alpha(X, x), x\big)$$

and

$$Q^\lambda(X, x) = \bigcap_{\alpha<\lambda} Q^\alpha(X, x)$$

for limit λ. The set $Q^\alpha(X, x)$ is said to be the *quasi-component of order* α of the space X at the point x. Observe that $Q^\alpha(X, x)$ is a closed subset of X, and therefore the decreasing transfinite sequence

$$Q^0(X, x) \supset Q^1(X, x) \supset \ldots \supset Q^\alpha(X, x) \supset \ldots$$