On the singularity of Mazurkiewicz in absolute neighborhood retracts

by

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1. Introduction. According to Borsuk [6], a compact metric space which cannot be expressed as a finite or countable union of compact absolute retracts of arbitrarily small diameter has the *singularity of Mazurkiewicz*. In [6], Borsuk raises the following questions:

1. *Suppose X and Y are compact metric absolute neighborhood retracts. If X has the singularity of Mazurkiewicz, then does $X\times Y$ also have the singularity of Mazurkiewicz?*

2. *If a polyhedron is represented as a cartesian product, is every factor free from the singularity of Mazurkiewicz?*

The purpose of this paper is to give negative solutions to both of these questions. Our solution consists of the following: We give an example of an upper semicontinuous decomposition G of the 3-sphere S^3 into a null sequence of arcs and points such that if X is the associated decomposition space, X has the singularity of Mazurkiewicz. By results of [7] or [9], $X\times S^1$ is homeomorphic to $S^3\times S^1$. By a theorem of Borsuk's [6], X is a compact absolute neighborhood retract. Hence X is a compact metric absolute neighborhood retract with the singularity of Mazurkiewicz. $X\times S^1$ is a polyhedron, and no polyhedron has the singularity of Mazurkiewicz [6]. Further, X is a factor of the polyhedron $S^3\times S^1$. Indeed, the triangulable manifold $S^3\times S^1$ can be factored into a product of compact absolute neighborhood retracts, one of which has the singularity of Mazurkiewicz.

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In this paper, by "retract" we shall always understand a retract of compact metric spaces. We use the abbreviations "AR" and "ANR" for "absolute retract" and "absolute neighborhood retract", respectively.

If M is a manifold with boundary, then Bd M and Int M denote the boundary and interior, respectively, of M.

2. Antoine's necklaces. In the construction of the decomposition, we use sets similar to the standard Antoine's necklaces in S^3, and which we shall also call "*Antoine's necklaces*". In Section 3, for each positive integer r, we shall construct such a set. In this section, we describe the construction, notation for the construction, and certain auxiliary sets.

Suppose r is a positive integer (fixed in this section). Suppose Σ_r is a polyhedral solid torus in S^3, and suppose that $\{T_{r1}, T_{r2}, \ldots, T_{rm_{r0}}\}$ is a chain of linked polyhedral unknotted solid tori in $\operatorname{Int}\Sigma_r$ circling Σ_r exactly once; see Figure 1. We suppose that if $i = 1, 2, \ldots$, or m_{r0}, T_{ri} has

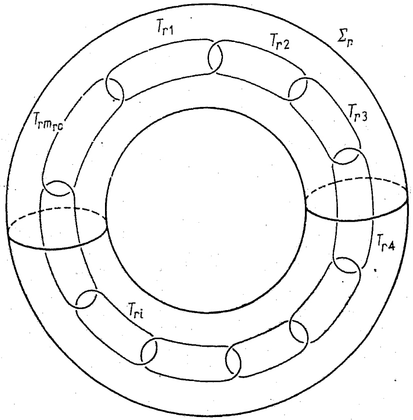

Fig. 1

diameter less than one. If $i = 1, 2, \ldots$, or m_{r0}, let $\{T_{ri1}, T_{ri2}, \ldots, T_{rim_{ri}}\}$ be a chain of linked polyhedral unknotted solid tori in $\operatorname{Int} T_{ri}$ circling T_{ri} exactly twice; see Figure 2. We suppose that if $j = 1, 2, \ldots$, or m_{ri}, $\operatorname{diam} T_{rij} < \frac{1}{2}$. If $i = 1, 2, \ldots$, or m_{r0}, and $j = 1, 2, \ldots$, or m_{ri}, then let $\{T_{rij1}, T_{rij2}, \ldots, T_{rijm_{rij}}\}$ be a chain of linked polyhedral unknotted solid tori in $\operatorname{Int} T_{rij}$, each of diameter less than $\frac{1}{4}$, circling T_{rij} exactly once. Let this process be continued, with subsequent chains circling exactly once, and let $M_{r1}, M_{r2}, M_{r3}, \ldots$ denote $\bigcup_{i=1}^{m_{r0}} T_{ri}$, $\bigcup_{i=1}^{m_{ri}} \bigcup_{j=1}^{m_{rij}} T_{rij}$, $\bigcup_{i=1}^{m_{r0}} \bigcup_{j=1}^{m_{ri}} \bigcup_{k=1}^{m_{rij}} T_{rijk}, \ldots$, respectively.

Let N_r denote $\bigcap_{i=0}^{\infty} M_i$; N_r is an *Antoine's necklace of type A circling* Σ_r. Note that $N_r \subset \operatorname{Int}\Sigma_r$.

In the construction of N_r, $T_{ri}, T_{r2}, \ldots$, and T_{rm_0} are the solid tori of the first stage of the construction of N_r, the solid tori T_{rij}, where $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant m_i$, are the solid tori of the second stage of the construction of N_r, and so on.

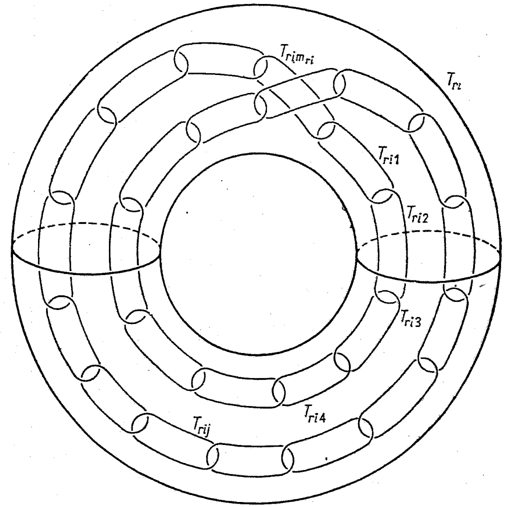

Fig. 2

If n is a positive integer, then a is a stage n index in the construction of N_r if and only if there exist integers $i_1, i_2, \ldots$, and i_n such that $1 \leqslant i_1 \leqslant m_{r0}$, $1 \leqslant i_2 \leqslant m_{ri_1}$, ..., and $1 \leqslant i_n \leqslant m_{ri_1i_2\ldots i_{n-1}}$. The statement that a is an index (in the construction of N_r) means that for some positive integer n, a is a stage n index.

It is easy to see that if x is any point of $\operatorname{Int}\Sigma$, we may construct N_r so that $x \in N_r$.

Now we shall describe certain arcs associated with N_r. Suppose $i = 1, 2, \ldots$, or m_{r0}. Consider the first stage torus T_{ri} and the second stage tori $T_{ri1}, T_{ri2}, \ldots$, and $T_{rim_{ri}}$ lying in T_{ri}. It is well known that if $j = 1, 2, \ldots$, or m_{ri}, there is an arc a_{rij} lying in $\operatorname{Int} T_{rij}$ and containing $N_r \cap T_{rij}$.

We shall construct arcs $b_{ri1}, b_{ri2}, \ldots$, and $b_{ri(m_{ri}-1)}$ so that $(\bigcup_{j=1}^{m_{ri}} a_{rij}) \cup (\bigcup_{j=1}^{m_{ri}-1} b_{rij})$ is an arc A_{ri} with certain properties. We regard T_{ri} as a copy of $D^2 \times S^1$ where D^2 is a disc and S^1 is a circle. The copies of $D^2 \times \{t\}$, where $t \in S^1$, will be called *cross-sections* of T_{ri}. We give S^1 an orientation, "clockwise", and use the induced orientation on the family of all cross sections of T_{ri}. We assume that $a_{ri1}, a_{ri2}, \ldots$, and $a_{rim_{ri}}$ are constructed so that if $1 \leqslant j \leqslant m_{ri}$, we may label the endpoints of a_{rij} by x_j and y_j in such a way that if we start at x_1 and go clockwise through the cross-sections of T_{ri}, these points occur in the order $x_1 y_1 x_2 y_2 \ldots x_{m_{ri}} y_{m_{ri}} x_1$, circling through T_{ri} twice. Then if $1 \leqslant j \leqslant m_{ri}$, b_{rij} is to be an arc in $\operatorname{Int} T_{ri}$ from y_j to x_{j+1}, intersecting precisely those cross-sections of T_{ri} that are encountered in going from y_j to x_{j+1} in the clockwise direction. We suppose the construction done so that the union of all the cross-sections intersecting b_{rij} is a 3-cell. Further, it is to be true that $(\bigcup_{i=1}^{m_{ri}} a_{rij}) \cup (\bigcup_{i=1}^{m_{ri}-1} b_{rij})$ is an arc A_{ri}. Then $N_r \cap T_{ri} \subset A_{ri}$.

3. Construction of the decomposition. Let Σ_0 be a polyhedral solid torus in S^3. Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of $\operatorname{Int} \Sigma_0$.

Let J_1 be a polygonal simple closed curve in $\operatorname{Int} \Sigma_0$, circling Σ_0 exactly once, and containing x_1. Let Σ_1 be a polyhedral tubular neighborhood of J_1 lying in $\operatorname{Int} \Sigma_0$.

Let N_1 be an Antoine's necklace of type A circling Σ_1 (and hence lying in $\operatorname{Int} \Sigma_0$ and circling Σ_0) such that (1) $x_1 \in N_1$ and (2) each of the first stage solid tori used in describing N_1 has diameter at most 1.

If $i = 1, 2, \ldots$, or m_{10}, there is an arc A_{1i} lying in $\operatorname{Int} T_{1i}$, containing $N_1 \cap T_{1i}$, and constructed as described in Section 2.

The arcs $A_{11}, A_{12}, \ldots$, and $A_{1m_{10}}$ are mutually disjoint, and each has diameter less than 1. Let A_1 denote $\bigcup_{i=1}^{m_{10}} A_{1i}$.

Let r_2 be the least positive integer q such that $x_q \notin A_1$. Let J_2 be a polygonal simple closed curve in $\operatorname{Int} \Sigma_0$, circling Σ_0 exactly once, containing x_{r_2}, and disjoint from A_1. Let Σ_2 be a polyhedral tubular neighborhood of J_2 lying in $\operatorname{Int} \Sigma_0$ and disjoint from A_1.

Let N_2 be an Antoine's necklace of type A circling Σ_2 such that (1) $x_{r_2} \in N_2$ and (2) each of the first stage solid tori used in describing N_2 has diameter at most $\frac{1}{2}$.

If $i = 1, 2, \ldots$, or m_{20}, there is an arc A_{2i} lying in $\operatorname{Int} T_{2i}$, containing $N_2 \cap T_{2i}$, and constructed as described in Section 2. The arcs $A_{21}, A_{22}, \ldots$, and $A_{2m_{20}}$ are mutually disjoint, and each has diameter less than $\frac{1}{2}$. Let A_2 denote $\bigcup_{i=1}^{m_{20}} A_{2i}$. Note that if $i = 1, 2, \ldots$, or m_{10} and $j = 1, 2, \ldots$, or m_{20}, A_{1i} and A_{2j} are disjoint.

Let this process be continued. There results a sequence $N_1, N_2, N_3, \ldots$ of Antoine's necklaces of type A in $\operatorname{Int} \Sigma_0$, each circling Σ_0, and a sequence $A_{11}, A_{12}, \ldots, A_{1m_{10}}, A_{21}, \ldots, A_{2m_{20}}, \ldots$ of mutually disjoint arcs in $\operatorname{Int} \Sigma_0$, such that for each positive integer n, the following hold:

(1) $N_n \subset A_{n1} \cup A_{n2} \cup \ldots \cup A_{nm_{n0}}$.

(2) $x_n \in \bigcup_{i=1}^{n} (A_{i1} \cup A_{i2} \cup \ldots \cup A_{im_{i0}})$.

(3) If $j = 1, 2, \ldots$, or m_{n0}, then $(\operatorname{diam} A_{nj}) < 1/2^n$.

Let $\mathfrak{a}$ denote the collection $\{A_{11}, A_{12}, \ldots, A_{1m_{10}}, A_{21}, A_{22}, \ldots, A_{2m_{20}}, \ldots\}$. Then $\mathfrak{a}$ is a *null collection*, i.e., for each positive number ε, at most finitely many sets of $\mathfrak{a}$ have diameters greater than ε.

Let G denote the collection consisting of the arcs of the family $\mathfrak{a}$ together with the singleton subsets of $S^3 - \bigcup\{A\colon A \in \mathfrak{a}\}$. Since $\mathfrak{a}$ is a null collection, it follows that G is an upper semicontinuous decomposition of S^3.

Throughout the remainder of the paper, we shall let X denote the decomposition space associated with G, and we shall let P denote the projection map from S^3 onto X.

Each nondegenerate element of G lies in $\operatorname{Int} \Sigma_0$, and thus $P[\operatorname{Int} \Sigma_0]$ is open in the associated decomposition space. Further, since $\{x_1, x_2, x_3, \ldots\}$ is dense in $\operatorname{Int} \Sigma_0$, it follows that if U is any open set in the decomposition space intersecting $P[\Sigma_0]$, then for some arc A of $\mathfrak{a}$, $P[A] \subset U$.

An open set W in S^3 is *saturated* if and only if W is a union of elements of G.

4. Preliminary lemmas. If T is a solid torus, then D is a *meridional disc* in T if and only if D is a disc in T such that $\operatorname{Bd} D \subset \operatorname{Bd} T$, $\operatorname{Bd} D \nsim 0$ on $\operatorname{Bd} T$, and $\operatorname{Int} D \subset \operatorname{Int} T$.

LEMMA 1. *Suppose k is a positive integer, $i = 1, 2, \ldots$, or m_{k0}, and D is a polyhedral meridional disc in T_{ki}. Then there is a subarc B_{ki} of A_{ki} such that* (1) *the endpoints of B_{ki} lie on D and* $\operatorname{Int} B_{ki}$ *misses D, and* (2) *the two ends of B_{ki} abut on D from opposite sides.*

Proof. Let T^* be the universal covering space of T_{ki}, and let φ be the projection map from T^* onto T_{ki}. Let D_1 be a copy of D in T^*. It is easily seen that D_1 intersects $\varphi^{-1}[N_k]$. Let p_1 be a point of $D_1 \cap \varphi^{-1}[N_k]$, and let j be an integer such that $\varphi(p_1) \in T_{kij}$. Let T_j^* be the copy of T_{kij} in T^* containing p_1.

Let $T_1^*, T_2^*, \ldots, T_{j-1}^*, T_{j+1}^* \ldots$, and $T_{m_{ki}}^*$ be copies in T^* of $T_{ki1}, T_{ki2}, \ldots, T_{ki(j-1)}, T_{ki(j+1)}, \ldots$, and $T_{kim_{ki}}$, respectively, so that $\{T_1^*, T_2^*, \ldots, T_j^*, \ldots, T_{m_{ki}}^*\}$ forms a (linear) chain. Let $T_1^{**}, T_2^{**}, \ldots$, and $T_{m_{ki}}^{**}$ be copies in T^* of $T_{ki1}, T_{ki2}, \ldots$, and $T_{kim_{ki}}$, respectively, so that (1) $\{T_1^{**}, T_2^{**}, \ldots, T_{m_{ki}}^{**}\}$ forms a (linear) chain and (2) T_1^{**} links $T_{m_{ki}}^{**}$.

Let D_2 and D_3 be copies of D in T^* such that D_2 is adjacent to D_1, D_3 is adjacent to D_2, D_2 separates D_1 and D_3 in T^*, and D_3 intersects T_j^{**}. Let p_3 be the point of D_3 such that $\varphi(p_3) = \varphi(p_1)$.

Finally, let A^* and A^{**} denote the copies, in T^*, of A_{ki} containing p_1 and p_3, respectively. We shall establish the following:

PROPOSITION 1. *Every point of* $\varphi^{-1}[N_k]$ *in* $\bigcup_{r=1}^{m_{ki}} (T_r^* \cup T_r^{**})$ *lies either in* A^* *or in* A^{**}.

Proof. First consider $\varphi^{-1}[N_k] \cap (\bigcup_{r=1}^{m_i} T_r^*)$. Clearly, if $r = 1, 2, \ldots,$ or m_i, the subarc a_{kir} of A_{ki} (lying in T_{kir} and containing $N_k \cap T_{kir}$) lifts to an arc a_r^* lying in T_r^*. Let x_r^* and y_r^* be the points of a_r^* so that $\varphi(x_r^*)$ and $\varphi(y_r^*)$ are the endpoints x_{kr} and y_{kr}, respectively, of a_{kir}.

We may regard T^* as $D^2 \times E^1$ where D^2 is a disc and E^1 is the real line. We may suppose that if $t \in E^1$, $\varphi[D^2 \times \{t\}]$ is a cross-section of T_{ki}. Further, we suppose that the positive direction on E^1 corresponds to the clockwise orientation on S^1. Suppose $1 \leqslant r < m_{ki}$. It then follows from the construction of b_{kir} that there is a copy b_r^* of b_{kir} in T^* with endpoints x_r^* and y_r^*. Clearly then, A^* is $(\bigcup_{r=1}^{m_i} a_r^*) \cup (\bigcup_{r=1}^{m_i-1} b_r^*)$. Thus each point of $\varphi^{-1}[N_k] \cap (\bigcup_{r=1}^{m_i} T_r^*)$ lies in A^*, and by a similar argument, $\varphi^{-1}[N_k] \cap (\bigcup_{r=1}^{m_i} T_r^{**}) \subset A^{**}$. This establishes Proposition 1.

It is easily seen that there is a point p_2 of $D_2 \cap \varphi^{-1}[N_k]$ lying in $\bigcup_{r=1}^{m_i} (T_r^* \cup T_r^{**})$. Thus $p_2 \in A^*$ or $p_2 \in A^{**}$. Hence one of A^* and A^{**} intersects adjacent ones of D_1, D_2, and D_3. Thus there is a subarc B^* of A^* or of A^{**} with its endpoints on adjacent ones of D_1, D_2, and D_3 and so that $\operatorname{Int} B^*$ misses $D_1 \cup D_2 \cup D_3$. Let B_{ki} denote $\varphi[B^*]$. It is clear that B_{ki} satisfies the conclusion of Lemma 1.

Suppose that M is a polyhedral 2-manifold in S^3, Δ is a polyhedral singular disc in S^3 such that $\operatorname{Bd}\Delta$ misses M, and M and Δ are in relative general position. Let Δ_0 be a 2-simplex, and let f be a piecewise linear map from Δ_0 onto Δ such that at each point of $f^{-1}[\Delta \cap M]$, f is locally a homeomorphism. It follows that each component of $f^{-1}[\Delta \cap M]$ is a simple closed curve. The statement that γ is a *curve of intersection* of Δ with M means that for some component γ_0 of $f^{-1}[\Delta \cap M]$, $\gamma = f[\gamma_0]$.

LEMMA 2. *Suppose that k is a positive integer, $i = 1, 2, \ldots,$ or m_{k0}, and U is a saturated open set in S^3 containing a singular disc Δ such that $\operatorname{Bd}\Delta \subset T_{k(i+1)}$ and $\operatorname{Bd}\Delta \not\sim 0$ in $T_{k(i+1)}$. Then U contains a loop γ such that $\gamma \subset T_{ki}$ and $\gamma \not\sim 0$ in T_{ki}.*

Proof. We may suppose that Δ is a polyhedral singular disc, in general position relative to $\operatorname{Bd} T_{ki}$. If there exists a curve of intersection γ of Δ with $\operatorname{Bd} T_{ki}$ such that $\gamma \not\sim 0$ in T_{ki}, then the Lemma is established. Hence we shall suppose that each such curve of intersection is homotopic to 0 in T_{ki}.

If every curve of intersection of Δ with $\operatorname{Bd} T_{ki}$ is homotopic to 0 on $\operatorname{Bd} T_{ki}$, it would follow that T_{ki} and $T_{k(i+1)}$ are not linked, a contradiction. Thus for some curve of intersection δ of Δ with $\operatorname{Bd} T_{ki}$, $\delta \not\sim 0$ on $\operatorname{Bd} T_{ki}$. It follows that there exists a curve of intersection λ of Δ with $\operatorname{Bd} T_{ki}$ such that (1) $\lambda \not\sim 0$ on $\operatorname{Bd} T_{ki}$ but (2) if Λ is the subdisc of Δ bounded by λ and λ' is any curve of intersection of Λ with $\operatorname{Bd} T_{ki}$ lying in $\operatorname{Int}\Lambda$, then $\lambda' \sim 0$ on $\operatorname{Bd} T_{ki}$.

Let T'_{ki} be a polyhedral solid torus in $\operatorname{Int} T_{ki}$, concentric with T_{ki}, and such that $A_{ki} \cap (\bigcup_{r=1}^{m_{ki}} T_{kir}) \subset \operatorname{Int} T'_{ki}$. For each curve of intersection λ' of Λ with $\operatorname{Bd} T_{ki}$ such that $\lambda' \sim 0$ on $\operatorname{Bd} T_{ki}$, replace the subdisc of Λ bounded by λ' by a singular disc on $\operatorname{Bd} T_{ki}$, and deform this new singular disc slightly into $(\operatorname{Int} T_{ki}) - T'_{ki}$. This yields a singular disc Λ' with $\operatorname{Bd}\Lambda' = \lambda$, $\operatorname{Int}\Lambda' \subset \operatorname{Int} T_{ki}$, and $\Lambda' \cap T'_{ki} \subset \Lambda$.

By the loop theorem [10, 12], there is a polygonal disc D in T_{ki} such that $\operatorname{Bd} D \subset \operatorname{Bd} T_{ki}$, $\operatorname{Bd} D \not\sim 0$ on $\operatorname{Bd} T_{ki}$, $\operatorname{Int} D \subset \operatorname{Int} T_{ki}$, and D lies in a small neighborhood of Λ'. Indeed, we may assume that $D \cap T'_{ki} \subset U$. We suppose D and $\operatorname{Bd} T'_{ki}$ to be in relative general position.

Since D is a meridional disc in T_{ki}, it follows that D contains a punctured disc D_0 such that $\operatorname{Bd} D_0 \subset \operatorname{Bd} T'_{ki}$, $\operatorname{Int} D_0 \subset \operatorname{Int} T'_{ki}$, one boundary curve μ_0 of D_0 is not homotopic to 0 on $\operatorname{Bd} T'_{ki}$, and every other boundary curve is homotopic to 0 on $\operatorname{Bd} T'_{ki}$. Note that $D_0 \subset U$. Now we may construct a polyhedral meridional disc F in T_{ki} by (1) attaching to D_0 an annulus in $T_{ki} - \operatorname{Int} T'_{ki}$ having μ_0 as one boundary curve and having as its other a simple closed curve μ on $\operatorname{Bd} T_{ki}$ such that $\mu \not\sim 0$ on $\operatorname{Bd} T_{ki}$, and (2) capping every other boundary curve of D_0 with a disc lying, except for its boundary, in $(\operatorname{Int} T_{ki}) - T'_{ki}$. We suppose that F is constructed so that $F \cap T'_{ki} = D_0$.

By Lemma 1, there is a subarc B_{ki} of A_{ki} such that (1) the endpoints of B_{ki} lie on F and $\operatorname{Int} B_{ki}$ misses F, and (2) the two ends of B_{ki} abut on F from opposite sides. Clearly the endpoints of B_{ki} lie in D_0. Hence $D_0 \cup B_{ki}$ contains a loop γ such that $\gamma \not\sim 0$ in T_{ki}.

Since $D_0 \subset U$, A_{ki} intersects U. Since U is saturated, $A_{ki} \subset U$. Hence $\gamma \subset U$. Clearly $\gamma \subset T_{ki}$. This establishes Lemma 2.

LEMMA 3. *Suppose k and n are positive integers, U is an open set, T_{ka} is a stage n torus in the construction of N_k, and if $i = 1, 2, \ldots,$ or m_{ka}, T_{kai} contains a polygonal simple closed curve γ_i such that $\gamma_i \not\sim 0$ in T_{kai},*

$\gamma_i \subset U$, and $\gamma_i \sim 0$ in U. Then T_{ka} contains a polygonal simple closed curve γ such that $\gamma \subset U$ and $\gamma \nsim 0$ in T_{ka}.

Proof. If $i = 1, 2, \ldots,$ or m_{ka}, we shall assume that γ_i bounds a polyhedral singular disc Δ_i in general position relative to $\operatorname{Bd} T_{ai}$.

Suppose there is an integer j, $1 \leqslant j \leqslant m_{ka}$, such that some curve of intersection of Δ_j and $\operatorname{Bd} T_{ki}$ is not homotopic to 0 in T_{ka}. If we let γ be such a curve, then γ satisfies the conclusion of Lemma 3. Thus we may assume that if $1 \leqslant j \leqslant m_{ka}$, each curve of intersection of Δ_j with $\operatorname{Bd} T_{ka}$ is homotopic to 0 in T_{ka}.

Let T^* be the universal covering space of T_{ki}, and let φ be the projection from T^* onto T_{ka}. Let $\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m_{ka}}^*$, and γ_1^{**} be copies in T^* of $\gamma_1, \gamma_2, \ldots, \gamma_{m_{ka}}$, and γ_1, respectively, so that $\{\gamma_1^*, \gamma_2^*, \ldots, \gamma_{m_{ka}}^*, \gamma_1^{**}\}$ forms a (linear) chain of loops.

If $j = 1, 2, \ldots,$ or m_{ka}, let f_j be a piecewise linear map from a standard 2-simplex Δ_0 onto Δ_j such that $f_j | \operatorname{Bd}\Delta_0$ is a homeomorphism onto γ_j. Some component of $\Delta_0 - f_j^{-1}[\operatorname{Bd} T_{ai}]$ contains $\operatorname{Bd}\Delta_0$; let Λ_{j0} denote this component. Let Λ_j denote $f_j[\Lambda_{j0}]$. Then Λ_j is a polyhedral singular punctured disc with γ_j as one boundary curve and such that every other boundary curve of Λ_j lies on $\operatorname{Bd} T_{ai}$ and is homotopic to 0 there.

If $j = 1, 2, \ldots,$ or m_{ka}, there exists a singular punctured disc Λ_j^* in T^* and a piecewise linear map g_j from Λ_{j0} onto Λ_j^* such that $\varphi g_j = f_j | \Lambda_{j0}$ and $g_j | \operatorname{Bd}\Delta_0$ is a homeomorphism from $\operatorname{Bd}\Delta_0$ onto γ_j^*. This may be seen as follows: Since each boundary curve of Λ_j other than γ_j is homotopic to 0 in T_{ka}, there is an extension h_j of $f_j | \Lambda_{j0}$ to all of Δ_0, so that $h_j[\Delta_0] \subset T_{ka}$. There is a map q_j from Δ_0 into T^* such that $\varphi q_j = h_j$. Then let g_j denote $q_j | \Lambda_{j0}$; we may assume g_j piecewise linear.

By a similar argument, there is a singular punctured disc Λ_1^{**} in T^* and a piecewise linear map g_1' from Λ_{j0} onto Λ_1^{**} such that $\varphi g_1' = f_1 | \Lambda_{j0}$ and $g_1' | \operatorname{Bd}\Delta_0$ takes $\operatorname{Bd}\Delta_0$ homeomorphically onto γ_1^{**}. For the following, let $\gamma_{m_{ka}+1}^*$, denote γ_1^{**}, and let $\Lambda_{m_{ka}+1}^*$ denote Λ_1^{**}.

Proposition 2. *If $1 \leqslant j < m_{ka}+1$, γ_j^* intersects Λ_{j+1}^*, and if $1 < j \leqslant m_{ka}+1$, γ_j^* intersects Λ_{j-1}^*.*

Proof. We establish only the first assertion; the second follows by an analogous argument.

Suppose $1 \leqslant j < m_{ka}+1$. If γ_j^* does not intersect Λ_{j+1}^*, then let D be a cross-sectional disc in T^* such that D misses γ_j^*. Then $D \cup \operatorname{Bd} T^*$ is simply connected. Then since each boundary curve of Λ_{j+1}^* distinct from γ_{j+1}^* lies on $\operatorname{Bd} T^*$, there is an extension d_j of g_j to all of Δ_0 so that d_j sends $\Delta_0 - \operatorname{Int}\Lambda_{j0}$ into $D \cup \operatorname{Bd} T^*$. Thus γ_{j+1}^* bounds a singular disc in T^* missing γ_j^*. This is a contradiction since γ_j^* and γ_{j+1}^* are linked in T^*. Hence γ_j^* intersects Λ_{j+1}^*. This establishes Proposition 2.

Let x be a point of γ_1^*, and (1) if $n = 2$, let y be a double translate of x belonging to γ_1^{**}, and (2) if $n \neq 2$, let y be a translate of x belonging to γ_1^*. (Recall that, in each first stage solid torus, the chain of second stage solid tori circles twice, but for every other n, the chain of $(n+1)$-st stage solid tori in a stage n solid torus circles only once.) It is easy to see that $\Lambda_1^* \cup \Lambda_2^* \cup \ldots \cup \Lambda_{mk_a}^* \cup \Lambda_1^{**}$ contains a path β from x to y such that $\varphi[\beta]$ is a loop in T_{ai} circling T_{ai} once (if $n \neq 2$) or twice (if $n = 2$). Since each of $\Lambda_1, \Lambda_2, \ldots,$ and $\Lambda_{m_{ka}}$ lies in U, $\varphi[\beta] \subset U$. A slight adjustment of $\varphi[\beta]$ yields a polygonal simple closed curve γ such that $\gamma \subset U$, $\gamma \subset T_{ai}$, and $\gamma \nsim 0$ in T_{ai}. This establishes Lemma 3.

Lemma 4. *Suppose that $U_0, U_1, U_2, \ldots$ is a sequence of open sets in S^3 such that for each i, $U_{i+1} \subset U_i$ and each loop in U_{i+1} is homotopic to 0 in U_i. Suppose V is an open set, $V \subset \bigcap_{i=0}^{\infty} U_i$, and for some integers k and j, $A_{kj} \subset V$. Then there is a polygonal simple closed curve γ in $U_0 \cap T_{kj}$ such that $\gamma \nsim 0$ in T_{kj}.*

Proof. Now $N_k \cap T_{kj} \subset A_{kj}$, and since A_{kj} lies in V, there is a positive integer n such that each stage n torus in the construction of N_k lying in T_{kj} lies in V.

Now consider the set U_n. Since $V \subset U_n$, then each stage n torus in the construction of N_k lying in T_{kj} lies in U_n. Consider any stage $(n-1)$ solid torus T_{ka} in the construction of N_k lying in T_{kj}. Then $T_{ka1}, T_{ka2}, \ldots,$ and $T_{kam_{ka}}$ are the stage n solid tori in T_{ka}. If $r = 1, 2, \ldots,$ or m_{ka}, let γ_{kar} be a polygonal simple closed curve in T_{kar} such that $\gamma_{kar} \nsim 0$ in T_{kar}. Since $T_{kar} \subset U_n$, $\gamma_{kar} \subset U_n$. Then $\gamma_{kar} \sim 0$ in U_{n-1}.

By Lemma 3, there is a polygonal simple closed curve γ_{ka} in $T_{ka} \cap U_{n-1}$ such that $\gamma_{ka} \nsim 0$ in T_{ka}. Thus, if T_{ka} is *any* stage $(n-1)$ solid torus lying in T_{kj}, then there is a polygonal simple closed curve γ_{ka} such that $\gamma_{ka} \subset T_{ka} \cap U_{n-1}$ and $\gamma_{ka} \nsim 0$ in T_{ka}. Hence the argument above may be repeated, using any stage $(n-2)$ solid torus $T_{k\beta}$ and the stage $(n-1)$ solid tori (in the construction of N_k) that lie in $T_{k\beta}$.

After at most n repetitions of this argument, we obtain a polygonal simple closed curve γ lying in some one of $U_0, U_1, U_2, \ldots,$ and hence in U_0, such that $\gamma \subset T_{kj}$ and $\gamma \nsim 0$ in T_{kj}.

Lemma 5. *Suppose that $U_0, U_1, U_2, \ldots$ is a sequence of saturated open sets in S^3 such that for each i, $U_{i+1} \subset U_i$ and each loop in U_{i+1} is homotopic to 0 in U_i. Suppose V is an open set in S^3 such that $V \subset \bigcap_{i=0}^{\infty} U_i$ and for some integers k and j, $A_{kj} \subset V$. Then there is a loop γ in $U_0 \cap \Sigma_k$ such that $\gamma \nsim 0$ in Σ_k.*

Proof. By Lemma 4, there is a loop γ_j in U_{m_k} such that $\gamma_j \subset T_{kj}$ and $\gamma_j \nsim 0$ in T_{kj}. Now γ_j bounds in $U_{(m_k-1)}$, and hence, by Lemma 2, there is a loop γ_{j+1} in $U_{(m_k-1)}$, such that $\gamma_{j+1} \subset T_{k(j+1)}$ and $\gamma_{j+1} \nsim 0$ in

$T_{k(j+1)}$. We apply Lemma 2 repeatedly, using the sets U_{m_k}, $U_{(m_k-1)}$, $U_{(m_k-2)}, \ldots, U_1$, and U_0, and going around the chain T_{kj}, $T_{k(j+1)}, \ldots$ $\ldots, T_{km_{k_0}}, T_{k1}, T_{k2}, \ldots$, and $T_{k(j-1)}$.

There results, for each r such that $1 \leqslant r \leqslant m_{k_0}$, a loop γ_r such that $\gamma_r \subset T_{kr}$, $\gamma_r \nsim 0$ in T_{kr}, and γ_r lies in some one of $U_0, U_1, U_2, \ldots$, and hence $\gamma_r \subset U_0$. Then by Lemma 3, there is a loop γ in $\Sigma_k \subset U_0$ such that $\gamma \nsim 0$ in Σ_k.

5. Additional preliminary results. The following lemma is a consequence of ([11], Theorem 4) and the fact that each AR is simply connected.

LEMMA 6. *Suppose M is a compact absolute retract in an LC^1 locally compact metric space, and suppose that U is an open set containing M. Then there is an open set V such that $M \subset V \subset U$ and each loop in V is homotopic to 0 in U.*

By repeated application of Lemma 6, we may establish the following result.

LEMMA 7. *Suppose M is a compact absolute retract in an LC^1 locally compact metric space, and suppose U_0 is an open set containing M. Then there is a sequence $U_0, U_1, U_2, \ldots$ of open sets such that for each i, $U_{i+1} \subset U_i$ and each loop in U_{i+1} is homotopic to 0 in U_i.*

The following lemma is just a restatement of Corollary 12.14 of Chapter V of [6]; it is also established in [4].

LEMMA 8. *If R is an ANR, H is an upper semicontinuous decomposition of R into absolute retracts, and the associated decomposition space S has finite dimension, then S is an ANR.*

LEMMA 9. *Suppose S is a metric space, G is an upper semicontinuous decomposition of S into compact absolute retracts, and S' is the associated decomposition space with π the projection map from S onto S'. Suppose that U and V are open sets in S' such that $V \subset U$ and each loop in V is homotopic to 0 in U. Then each loop in $\pi^{-1}[V]$ is homotopic to 0 in $\pi^{-1}[U]$.*

Proof. Suppose γ is a loop in $\pi^{-1}[V]$. Then $\pi\gamma$ is a loop in V and thus $\pi\gamma \sim 0$ in U. Let F be a map from a disc D into U such that $F|\mathrm{Bd}\,D = \pi\gamma$. Then $\pi^{-1}[F[D]]$ is a compact set in $\pi^{-1}[U]$, and is a union of elements of G.

For each point x of $F[D]$, let W_x be an open set such that $\pi^{-1}[x] \subset W_x \subset \pi^{-1}[U]$ and each loop in W_x is homotopic to 0 in $\pi^{-1}[U]$; such an open set W_x exists since each element of G is a CAR. We further assume that each such W_x is a union of elements of G.

By compactness of $\pi^{-1}[F[D]]$, there is a finite subset $\{x_1, x_2, \ldots, x_r\}$ of $F[D]$ such that $\{W_{x_1}, W_{x_2}, \ldots, W_{x_r}\}$ covers $\pi^{-1}[F[D]]$. If $1 \leqslant i \leqslant r$, we denote W_{x_i} by W_i; then $\{W_1, W_2, \ldots, W_r\}$ is an open cover W of $\pi^{-1}[F[D]]$. Note that if $x \in F[D]$, $\pi^{-1}[x]$ lies in some set of W. It follows that $\{\pi[W_1], \pi[W_2], \ldots, \pi[W_r]\}$ is an open cover W' of $F[D]$. Let T be a triangulation of D such that $\{F[\sigma]\colon \sigma \in T\}$ refines W'.

We now construct a certain singular disc in $\pi^{-1}[U]$. We shall do this by "lifting" $F[D]$ into $\pi^{-1}[U]$. If v is a vertex of T in Int D, let v' be a point of $\pi^{-1}[F(v)]$. If v is a vertex of T on Bd D, let v' be $\gamma(v)$. If σ is a 1-simplex $\langle v_0 v_1 \rangle$ of T not in Bd D, let Δ and Δ' be the 2-simplexes of T so that $\sigma = \Delta \cap \Delta'$. Select open sets W_i and W_j of W so that $F[\Delta] \subset \pi[W_i]$ and $F[\Delta'] \subset \pi[W_j]$. Then v_0' and v_1' belong to the same component of $W_i \cap W_j$, and let σ' be an arc in this component of $W_i \cap W_j$ from v_0' to v_1'. If σ is a 1-simplex of T on Bd D, let σ' denote $\gamma[\sigma]$.

Suppose Δ is a 2-simplex of T, with 1-dimensional faces σ_1, σ_2, and σ_3. Let W_k be a set of W so that $F[\Delta] \subset \pi[W_k]$. In W_k, we constructed arcs σ_1', σ_2', and σ_3' so that $\sigma_1' \cup \sigma_2' \cup \sigma_3'$ is a loop μ. Now $\mu \sim 0$ in $\pi^{-1}[U]$ by construction of W_i. Let Δ' be a singular disc in $\pi^{-1}[U]$ bounded by μ.

It is clear that $\bigcup\{\Delta'\colon \Delta \in T\}$ is a singular disc in $\pi^{-1}[U]$, and that this singular disc has boundary γ. Hence $\gamma \sim 0$ in $\pi^{-1}[U]$.

6. The main result.

THEOREM 1. *The space X described in Section 2 is a compact absolute neighborhood retract with the singularity of Mazurkiewicz but such that $X \times S^1$ is homeomorphic to $S^3 \times S^1$.*

To proof Theorem 1 we first establish two lemmas:

LEMMA 10. *$X \times S^1$ is homeomorphic to $S^3 \times S^1$.*

Proof. It follows from Theorem 5 of [9] (and from [7]) that if G is a monotone decomposition of E^3 into countably many arcs and points, and W is the associated decomposition space, then $W \times E^1$ is homeomorphic to E^4. With no essential change in the proof, an analogous result could be established for S^3 (in place of E^3).

In the proof of Theorem 5 of [9], the E^1-factor of the product $E^3 \times E^1$ is divided into a sequence $\ldots I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots$ of closed intervals of equal length and, corresponding to each interval I_j, certain homeomorphisms h_j of the product $E^3 \times I_j$ are defined. Now the homeomorphisms corresponding to different intervals are constructed by "copying" those for one interval, so that for any j, there is an order-preserving translation g_j from I_0 onto I_j such that if $x \in E^3$, $t \in I_0$, and $h_0(x, t) = (x', t')$, then $h_j(x, g_j(t)) = (x', t' + g_j(t))$. It follows that such homeomorphisms can be constructed for products of the form $E^3 \times S^1$ and $S^3 \times S^1$, since, in the case of S^1, the factors repeat cyclically.

Thus by a simple modification of the proof of Theorem 5 of [9], we have the following result: If G is a monotone decomposition of S^3 into countably many arcs and points, and W is the associated decomposition space, then $W \times S^1$ is homeomorphic to $S^3 \times S^1$. Hence Lemma 10 follows.

Corollary. *X has dimension* 3.

Proof. It is known that if S is a compact metric space of finite dimension, then $\dim(S\times S^1)=1+\dim S$. Thus $\dim X=3$.

Let Ω_0 denote a polyhedral solid torus in S^3 such that $\Sigma_0\subset \operatorname{Int}\Omega_0$, and Σ_0 and Ω_0 are concentric.

Lemma 11. *There exists no* AR M *in* X *such that* (1) $M\subset P[\operatorname{Int}\Omega_0]$, *and* (2) *if* $\mathrm{i}(M)$ *denotes the* (*topological*) *interior* (*in* X) *of* $M\cap P[\operatorname{Int}\Sigma_0]$, *then for some integers* k *and* j, $P[A_{kj}]\subset \mathrm{i}(M)$.

Proof. Suppose there is such an AR M. By Lemma 8, X is an ANR, so X is LC^1. Thus by Lemma 7, there exists a sequence of open sets in X, $P[\operatorname{Int}\Omega_0]$, $W_0, W_1, W_2, W_3, \ldots$ such that for each i, $M\subset W_{i+1}\subset W_i$, each loop in W_{i+1} is homotopic to 0 in W_i, $W_0\subset P[\operatorname{Int}\Omega_0]$, and each loop in W_0 is homotopic to 0 in $P[\operatorname{Int}\Omega_0]$. Further, the interior (in X) $\mathrm{i}(M)$ of $M\cap P[\operatorname{Int}\Sigma_0]$ has the property that for each i, $\mathrm{i}(M)\subset W_i$. Since $P[\operatorname{Int}\Sigma_0]$ is open in X, $\mathrm{i}(M)$ is open in X.

Let V denote $P^{-1}[\mathrm{i}(M)]$, and for each i, let U_i denote $P^{-1}[W_i]$. Then by Lemma 9, for each i, each loop in U_{i+1} is homotopic to 0 in U_i, and each loop in U_0 is homotopic to 0 in $\operatorname{Int}\Omega_0$. Since for some integers k and j, $P[A_{kj}]\subset \mathrm{i}(M)$, then $A_{kj}\subset V$. By Lemma 5, there is a loop γ in $\Sigma_k\cap U_0$ such that $\gamma\not\sim 0$ in Σ_k. From the construction of Σ_k and Ω_0, it follows that $\gamma\not\sim 0$ in Ω_0. However, each loop in U_0 is homotopic to 0 in $\operatorname{Int}\Omega_0$. This is a contradiction, and Lemma 11 is established.

We return to the proof of the theorem.

Proof of Theorem 1. By the Corollary to Lemma 10, X is finite dimensional. Hence by Lemma 8, X is an ANR.

The referee pointed out the following simple proof that X is an ANR. X is a retract of $X\times S^1$. But $X\times S^1$ is an ANR since, by Lemma 10, $X\times S^1$ is homeomorphic to $S^3\times S^1$.

Now we shall show that X has the singularity of Mazurkiewicz. We suppose X metrized.

There is a positive number ε such that any set in X of diameter less than ε and intersecting $P[\Sigma_0]$ must lie in $P[\operatorname{Int}\Omega_0]$. Suppose X is covered by at most countably many absolute retracts, each of diameter less than ε. Let C be the family of those that intersect $P[\Sigma_0]$; each set of C lies in $P[\operatorname{Int}\Omega_0]$.

Let C' be $\{c\cap P[\operatorname{Int}\Sigma_0]\colon c\in C\}$; C' covers $P[\operatorname{Int}\Sigma_0]$ and is countable. By the Baire theorem, some set of C' has nonvoid interior relative to the locally compact metric space $P[\operatorname{Int}\Sigma_0]$. Let M be a set of C such that $M\cap P[\operatorname{Int}\Sigma_0]$ has nonvoid interior in $P[\operatorname{Int}\Sigma_0]$. Since $P[\operatorname{Int}\Sigma_0]$ is open in X, then $M\cap P[\operatorname{Int}\Sigma_0]$ has nonvoid interior $\mathrm{i}(M)$ in X.

By the construction of G, each open subset of $P[\operatorname{Int}\Sigma_0]$ contains, for some integers k and j, $P[A_{kj}]$. Thus for some k and j, $P[A_{kj}]\subset \mathrm{i}(M)$.

Thus if X is covered by at most countably many absolute retracts, each of diameter less than ε, there exists an AR M in X such (1) $M\subset P[\operatorname{Int}\Omega_0]$ and (2) for some k and j, $P[A_{kj}]\subset \mathrm{i}(M)$ where $\mathrm{i}(M)$ is the interior (in X) of $M\cap P[\operatorname{Int}\Sigma_0]$. This contradicts Lemma 11, and thus X has the singularity of Mazurkiewicz. This establishes Theorem 1.

7. Local properties of X.

The methods of this paper are closely related to those used in other papers studying local properties of decomposition spaces, especially [1]; see [2] and [3] also. In this section we consider further the local structure of X.

The argument of Lemma 11 can be used to establish the following result.

Lemma 12. *If* $x\in P[\Sigma_0]$, *there is no compact, locally connected, simply connected neighborhood of* x (*in* X) *lying in* $P[\operatorname{Int}\Omega]$.

The conclusions of Lemmas 6 and 7 hold if, in the hypotheses of those lemmas, "compact AR" is replaced by "compact, locally connected, simply connected set". For Lemma 6, this follows from ([11], Theorem 4).

Theorem 2. *It is not true that each point of* X *has arbitrarily small compact, locally connected, simply connected neighborhoods.*

The following may be established by a modification of the argument given in this paper.

Lemma 13. *If* $x\in P[\Sigma_0]$, *there is no simply connected open neighborhood of* x *lying in* $P[\operatorname{Int}\Omega_0]$.

We shall say that a topological space is *strongly locally simply connected* provided each point of the space has arbitrarily small simply connected open neighborhoods. Thus we have the following result:

Theorem 3. *The space* X *is not strongly locally simply connected.*

In proving Lemma 13, we use the following consequence of Corollary 5.4 of [5].

Lemma 14. *If* U *is a simply connected open set in* X, *then* $P^{-1}[U]$ *is simply connected.*

The proof of Lemma 13 is essentially the same as that of Lemma 11, except that, in place of the sequence $U_0, U_1, U_2, \ldots$ of open sets, we can use a single open set.

A topological space is *locally peripherally spherical* provided each point of the space has arbitrarily small neighborhoods whose (topological) boundaries are (topological) 2-spheres.

Theorem 4. *X is not locally peripherally spherical.*

Theorem 4 follows from Lemmas 12 and 15. For a proof of Lemma 15, see [1].

LEMMA 15. *If W is a compact neighborhood of a point of a simply connected metric space such that the topological boundary of W is a 2-sphere, then W is compact, locally connected, and simply connected.*

Thus we have proved that the space X described in Section 2 is a 3-dimensional ANR but (1) X is not strongly locally simply connected, (2) X is not locally peripherally spherical, and (3) it is not true that X has arbitrarily small compact, locally connected, simply connected neighborhoods.

8. Remarks.

1. By representing S^3 as the union of two solid tori Σ_0 and Σ_0', and carrying out the construction of Section 2 in each solid torus, we obtain a 3-dimensional totally non-euclidean space Y such that $Y \times S^1$ is homeomorphic to $S \times S^1$.

2. For each of the results mentioned above, there is a corresponding result obtained by decomposing E^3.

3. According to Borsuk [6], a topological property is *multiplicative* provided that for every two spaces X_1 and X_2 with the property, their product $X_1 \times X_2$ has that property. Borsuk raises the following question [6]: *Is the singularity of Mazurkiewicz multiplicative?*

Kwun [8] established the following theorem: *Suppose m and n are positive integers, α and β are arcs in E^m and E^n, respectively, and A and B denote the spaces obtained by collapsing α and β, respectively, to points. Then $A \times B$ is homeomorphic to E^{n+m}.*

One may conjecture that Kwun's result holds in the case of upper semicontinuous decompositions of euclidean spaces into at most countably many arcs. If this conjecture is true, then the construction of Section 2, applied to E^3, would yield a space Z with the singularity of Mazurkiewicz but such that $Z \times Z$ is homeomorphic to E^6. It seems plausible to conjecture that, for locally compact metric spaces, the singularity of Mazurkiewicz is not multiplicative.

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